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FREE BASIC PROCESS ALGEBRA

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ABSTRACT. A basic process algebra is an algebra with two binary operations $+, \cdot$ and a set of constants A, satisfying the laws (BPA1) - (BPA5) as given in the text. We present a description of free basic process algebras by using suitable descriptions of free semigroups and free semilattices.

The description of free basic process algebras is important since in the theories of process algebras, many more complex structures are built over the basic process algebras, and in the applications of process algebras one usually works with free ones.

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1. Preliminaries

Basic process algebra is called an algebra with two binary operations $+$, $\cdot$, a set of constants A (operations of arity 0), satisfying the following identities:

(BPA1)	$x + y = y + x$
(BPA2)	$(x + y) + z = x + (y + z)$
(BPA3)	$x + x = x$
(BPA4)	$(xy)z = x(yz)$
(BPA5)	$(x + y)z = xz + yz$

In fact, it is a semilattice with respect to $+$, it is a semigroup with respect to $\cdot$, and the right-distributive law holds for $\cdot$ with respect to $+$.

Each basic process algebra is parametrized by the set A, called a set of atomic actions. Choosing different sets A we obtain different basic process algebras. The importance of the set of constants A arises from the fact that free basic process algebras with empty base are used in the applications. Nevertheless, we consider nonempty bases too, since then the free basic process algebras can be used for suitable definitions of the processes obtained by the so called recursive specifications [1]. Namely, if F_{BPA} is a free basic process algebra with base $B \supseteq \{X_1, \ldots, X_n\} (n \geq 1)$, a process defined by a finite recursive specification can be considered as an element of the quotient algebra $P = F_{BPA}/\theta$, where θ is a congruence on F_{BPA} generated by a set of relations

$\{(X_i, s_i(Y_{i_1}, \dots, Y_{i_k}))|\ i = 1, \dots, n\}$ where s_i are terms in the signature $+, \cdot, A$ depending on the variables $Y_{i_1}, \dots, Y_{i_k}$ $(Y_j \in \{X_1, \dots, X_n\})$.

We will present a description of free basic process algebras by using a convenient description of free semigroups and free semilattices.

In what follows we take B to be a given set and by $\mathbb{N}$ we denote the set of nonnegative integers. Firstly, we present suitable descriptions of a free semigroup B^+ with base B and a free semilattice $SL(B)$ with base B. Although these free objects are well known here we are presenting them since they are used in our construction of a free basic process algebra.

Free semigroup

We define a sequence of sets B_i and a set B^+ by:

$$B_0 = B, \quad B_{i+1} = B_i \cup \{xy|\ x \in B,\ y \in B_i\},\ i \in \mathbb{N}, \quad B^+ = \cup\ (B_i|\ i \in \mathbb{N}).$$

Here uv denotes the concatenation of the strings u, v of elements of B.

Proposition 1. *a)* $xy = uv \in B^+,\ x, u \in B \implies x = u,\ y = v \in B^+.$
b) $x \in B^+ \setminus B \implies x = uv,$ *for some* $u \in B,\ v \in B^+.$ □

A length d is said to be a map $d : B^+ \longrightarrow \mathbb{N}$ defined by:

$$b \in B \implies d(b) = 1, \quad x \in B_{i+1} \setminus B_i \implies d(x) = i + 2,\ i \in \mathbb{N}.$$

The operation $\cdot$ on B^+ is the usual one, i.e. $x \cdot y = xy$ for each $x, y \in B^+$. Then $(B^+, \cdot)$ is a free semigroup with base B.

Free semilattice

Let $SL(B) = B \cup B'$, where B' is the set of all finite subsets of B with at least two elements. We define an operation $*$ on $SL(B)$ by

$$x * y = \begin{cases} x & x = y \in B \\ \{x\} \cup y & x \in B,\ y \in B' \\ x \cup \{y\} & x \in B',\ y \in B \\ x \cup y & x, y \in B' \\ \{x, y\} & x, y \in B,\ x \neq y \end{cases}$$

Then $(SL(B), *)$ is a free semilattice with base B.

2. Free basic process algebra

We are considering a basic process algebra with a fixed set A of constants, such that $A \cup B \neq \emptyset$.

Define a sequence of sets H_i as follows:

$$H_0 = (A \cup B)^+, \quad H_{i+1} = SL(H_i)^+,\ i \in \mathbb{N},$$

where X^+ stands for the universe of a free semigroup with base X and $SL(X)$ stands for the universe of a free semilattice with base X.

Obviously, $H_i \subseteq H_{i+1}$ for each i, and this implies the following:

Proposition 2. *$SL(H_i)$ is a subsemilattice of $SL(H_{i+1})$ and H_i is a subsemigroup of H_{i+1}, for each $i \in \mathbb{N}$.* □

Let $H = \cup\ (H_i|\ i \in \mathbb{N})\ (= \cup\ (SL(H_i)|\ i \in \mathbb{N}))$. By *Proposition 2* $(SL(H_i)|\ i \in \mathbb{N})$ is a chain of semilattices and $(H_i|\ i \in \mathbb{N})$ is a chain of semigroups, hence by $x \oplus y = x * y$, $x \otimes y = x \cdot y$ operations on H are defined such that $(H, \oplus)$ is a semilattice and $(H, \otimes)$ is a semigroup.

We define a map $D : H \longrightarrow \mathbb{N}$ as follows. If $x \in H_0$, then $D(x) = d(x)$, where d is the length as previously defined for $(A \cup B)^+$. Assume that $D(x)$ is defined for all $x \in H_i$. We put $D(x) = 1$ for $x \in SL(H_i) \setminus H_i$. If $x \in H_{i+1} \setminus SL(H_i)$ then, since H_i is a semigroup, there are uniquely determined $u_1, u_2, \ldots, u_t \in SL(H_i), t \geq 2$ such that $x = u_1 u_2 \ldots u_t$ and $u_i \in H_i \implies u_{i+1} \in SL(H_i) \setminus H_i$. Then we put $D(x) = D(u_1) + D(u_2) + \cdots + D(u_t)$.

Note that for $u, v \in H$ we have $D(uv) = D(u) + D(v)$.

Proposition 3. *If $u \in H \setminus H_0$ then there are uniquely determined $u_1, \ldots, u_{i-1} \in A \cup B,\ v_1, \ldots, v_p \in H,\ i \geq 1,\ p > 1$, such that*

(C) $\quad u = u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}$ *or* $u = u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w$

for a uniquely determined $w \in H$. (In both cases for the right-hand side of (C) we say that it is the canonical form of u. In what follows, for the sake of simplicity, the canonical form will be denoted by $u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w$, which means that we allow w to be the empty string.)

Proof We use induction on D. If $D(u) = 1$, then $u = \{v_1, \ldots, v_p\}$ for some $p > 1,\ v_1, \ldots, v_p \in H$, and u is in canonical form.

Let $D(u) > 1$. Then $u = u'u''$, where $D(u'), D(u'') < D(u)$.

If $u' \in H \setminus H_0$ then $u' = u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w'$, for some uniquely determined $u_1, \ldots, u_{i-1} \in A \cup B,\ v_1, \ldots, v_p, w' \in H,\ i \geq 1,\ p > 1$, and then $u = u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w$, where $w = w'u''$.

If $u' \in H_0$, then $u' = u_1 \ldots u_j$, for some $u_1, \ldots, u_j \in A \cup B,\ j \geq 1$, and $u'' \in H \setminus H_0$, so $u'' = u_{j+1} \ldots u_{i-1}\{v_1, \ldots, v_p\}w$, for some $u_{j+1}, \ldots, u_{i-1} \in A \cup B,\ v_1, \ldots, v_p, w \in H,\ i > j,\ p > 1$, and then (C) holds again. □

A hierarchy χ on H is said to be a map from H into $\mathbb{N}$ defined by:

$$x \in H_0 \Rightarrow \chi(x) = 0, \qquad x \in H \setminus H_0 \Rightarrow \chi(x) = 1 + \chi(v_1) + \cdots + \chi(v_p) + \chi(w)$$

where $u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w$ is the canonical form of x. Note that $\chi(x)$ is the total number of pairs of braces appearing in x and χ is well defined, since v_i and w have at least one pair of braces less than x.

Proposition 4. *If $u_1 \ldots u_{i-1}\{v_1, \ldots, v_p\}w$ is the canonical form of x, then*

$$\chi(v_j w) < \chi(x), \quad j = 1, \ldots, p.$$

Proof We show by induction on D that $\chi(uv) = \chi(u) + \chi(v)$ for each $u, v \in H$. The statement is clear for $u \in H_0$. Let $u'_1 \ldots u'_{i-1}\{v'_1, \ldots, v'_p\}w'$ be the canonical form of u. Then $\chi(uv) = 1 + \chi(v_1') + ... + \chi(v_p') + \chi(w'v) = 1 + \chi(v_1') + ... + \chi(v_p') + \chi(w') + \chi(v) = \chi(u) + \chi(v)$, since $D(w'v) < D(uv)$.

Now, $\chi(x) = 1 + \chi(v_1) + ... + \chi(v_p) + \chi(w) = 1 + \chi(v_1) + ... + \chi(v_{j-1}) + \chi(v_j w) + \chi(v_{j+1}) + ... + \chi(v_p) > \chi(v_j w)$. □

Denote by $\mathcal{V}$ the variety defined by the identities (BPA1) - (BPA4) in the signature $+, \cdot, A$. Note that $(H, \oplus, \otimes, A) \in \mathcal{V}$.

Proposition 5. *$(H, \oplus, \otimes, A)$ is a free algebra in $\mathcal{V}$ with base B.*

Proof Let $(G, +, \cdot, A) \in \mathcal{V}$ and $f : B \longrightarrow G$ a map. Then f can be extended to a homomorphism f^* from $(H, \oplus, \otimes, A)$ into $(G, +, \cdot, A)$ in the following way.

(*i*) $\chi(x) = 0,\ x \in A \implies f^*(x) = x$;
(*ii*) $\chi(x) = 0,\ x \in B \implies f^*(x) = f(x)$;
(*iii*) $\chi(x) = 0,\ x \in H_0 \setminus (A \cup B) \implies f^*(x) = f^*(x_1) \cdot ... \cdot f^*(x_k)$, where $x = x_1 \dots x_k$, $x_i \in A \cup B$.
(*iv*) $\chi(x) \geq 1 \implies f^*(x) = f^*(u_1) \cdot ... \cdot f^*(u_{i-1})(f^*(v_1) + ... + f^*(v_p)) \cdot f^*(w)$ where $u_1 \dots u_{i-1}\{v_1, \dots, v_p\}w$ is the canonical form of x.

By *Proposition 3* (and using an induction on D and χ) we have that f^* is well defined and a homomorphism as well. □

We define a map $R : H \longrightarrow H$, called a reduction, in the following way:

1) $\chi(u) = 0 \implies R(u) = u$
2) $\chi(u) \geq 1 \implies R(u) = u_1 \dots u_{i-1}(\{R(v_1 w), \dots, R(v_p w)\})$, where $u_1 \dots u_{i-1}\{v_1, \dots, v_p\}w$ is the canonical form of u. (If $R(v_1 w) = \cdots = R(v_p w)$ then $R(u) = u_1 \dots u_{i-1} R(v_1 w)$.)

Note that R is well defined by *Proposition 4.*

Proposition 6. *$R(R(u)) = R(u)$ for all $u \in H$.*

Proof If $u \in H \setminus H_0$ then u has a canonical form $u_1 \dots u_{i-1}\{v_1, \dots, v_p\}w$ and then, by using induction on χ, we have:

$$\begin{aligned} R(R(u)) &= R(u_1 \dots u_{i-1}\{R(v_1 w), \dots, R(v_p w)\}) = \\ &= u_1 \dots u_{i-1}\{R(R(v_1 w)), \dots, R(R(v_p w))\} = \\ &= u_1 \dots u_{i-1}\{R(v_1 w), \dots, R(v_p w)\} = R(u). \end{aligned}$$ □

Proposition 7. *$R(u \otimes v) = R(R(u) \otimes v) = R(u \otimes R(v))$ for each $u, v \in H$.*

Proof The statement is clear when $u \in H_0$, so let $u \in H \setminus H_0$ where $u_1 \dots u_{i-1}\{v_1, \dots, v_p\}w$ is its canonical form. Now we have

$$R(u \otimes v) = R(uv) = u_1 \dots u_{i-1}\{R(v_1 wv), \dots, R(v_p wv)\}$$

and, by using induction on χ,

$$R(R(u)\otimes v) = R(u_1\dots u_{i-1}\{R(v_1w),\dots,R(v_pw)\}v) =$$
$$= u_1\dots u_{i-1}\{R(R(v_1w)v),\dots,R(R(v_pw)v)\} =$$
$$= u_1\dots u_{i-1}\{R(R(v_1w)\otimes v),\dots,R(R(v_pw)\otimes v)\} =$$
$$= u_1\dots u_{i-1}\{R(v_1w\otimes v),\dots,R(v_pw\otimes v)\} = R(u\otimes v).$$

Similarly, we have $R(u\otimes R(v)) = R(u\otimes v)$. □

As a consequence of *Proposition 7* we have:

$$R(u\otimes v) = R(R(u)\otimes R(v))$$

for all $u, v \in H$. Note that the definition of R immediately implies

$$R(u\oplus v) = R(u)\oplus R(v).$$

Let $F_{BPA} = R(H)$. Define operations $+$ and $\circ$ on F_{BPA} by:

$$u+v = u\oplus v, \qquad u\circ v = R(u\otimes v)$$

for all $u, v \in F_{BPA}$.

Theorem 1. *$(F_{BPA}, +, \circ, A)$ is a free basic process algebra with base B.*

Proof F_{BPA} is generated by B since by 1) we have $H_0 \subseteq F_{BPA}$ and by using induction on hierarchy χ by 2) we have that if $R(v_jw)$, for all j, is generated by B then $R(u)$ is generated by B as well.

F_{BPA} is a semilattice since R is an endomorphism on $(H, \oplus)$.

Let $u, v, w \in F_{BPA}$, i.e. $u = R(u)$, $v = R(v)$, $w = R(w)$. Then:

$$(u\circ v)\circ w = R(R(u\otimes v)\otimes R(w)) = R((u\otimes v)\otimes w)$$
$$= R(u\otimes(v\otimes w)) = R(u\otimes R(v\otimes w)) = u\circ(v\circ w),$$

i.e. F_{BPA} is a semigroup.

The right distributive law holds for $\circ$ with respect to $+$. Namely, for each $u, v, w \in F_{BPA}$ we have to consider the following cases.

If $D(u)$, $D(v) > 1$ or $u, v \in H_0$ then:

$$(u+v)\circ w = R(\{u,v\}w) = \{R(uw), R(vw)\} = (u\circ w)+(v\circ w),$$

for $R(uw) \neq R(vw)$. If $R(uw) = R(vw)$ the statement is clear.

If $D(u) = 1$, $u = \{u_1,\dots,u_p\}$, $D(v) > 1$ then:

$$(u+v)\circ w = R(\{u_1,\dots,u_p,v\}w) = \{R(u_1w),\dots,R(u_pw),R(vw)\}$$
$$= \{R(u_1w),\dots,R(u_pw)\} + R(vw) = R(\{u_1,\dots,u_p\}w) + R(vw)$$
$$= (u\circ w)+(v\circ w).$$

The cases $D(u) > 1$, $D(v) = 1$ and $D(u) = D(v) = 1$ are treated similarly.

So, $(F_{BPA}, +, \circ, A)$ is a basic process algebra.

Let $(P, +, \cdot, A)$ be any basic process algebra and $f : B \longrightarrow P$ a map. Then $(P, +, \cdot, A) \in \mathcal{V}$ and, in the same way as in the proof of *Proposition 5*, f can be extended to a homomorphism $f^* : H \longrightarrow P$.

By induction on the hierarchy χ we will show that $f^*(u) = f^*(R(u))$ for each $u \in H$. Let $u \in H\setminus H_0$, $u_1\dots u_{i-1}\{v_1,\dots,v_p\}w$ being its canonical form. Then:

$$f^*(u) = f^*(u_1)\cdot\ldots\cdot f^*(u_{i-1})(f^*(v_1)+\ldots+f^*(v_p))\cdot f^*(w) =$$

$$= f^*(u_1) \cdot \ldots \cdot f^*(u_{i-1})(f^*(v_1) \cdot f^*(w) + \ldots + f^*(v_p) \cdot f^*(w)) =$$
$$= f^*(u_1) \cdot \ldots \cdot f^*(u_{i-1})(f^*(v_1 w) + \ldots + f^*(v_p w)) =$$
$$= f^*(u_1) \cdot \ldots \cdot f^*(u_{i-1})(f^*(R(v_1 w)) + \ldots + f^*(R(v_p w))) =$$
$$= f^*(u_1 \ldots u_{i-1}\{R(v_1 w), \ldots, R(v_p w)\}) = f^*(R(u)).$$

Now we have that the restriction of f^* on F_{BPA} is a homomorphism from $(F_{BPA}, +, \circ, A)$ into $(P, +, \cdot, A)$, and it is an extension of f as well.

□

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