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A SOLUTION OF ONE OLD PROBLEM

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Abstract. In this paper the following problem

$$1+\frac{2^1}{1}+\frac{2^2}{1\cdot 3}+\frac{2^3}{1\cdot 3\cdot 5}+\cdots+\frac{2^n}{1\cdot 3\cdot 5\cdots(2n-1)}+\cdots=e\Big(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt\Big)$$

is proved. As a consequence, the following problem

$$\frac{3}{2}e\sqrt{\pi}+\frac{3}{2}e\int_0^1 e^{-1/t^2}dt=\Big(\frac{1\cdot 2^1}{1}+\frac{2\cdot 2^2}{1\cdot 3}+\frac{3\cdot 2^3}{1\cdot 3\cdot 5}+\frac{4\cdot 2^4}{1\cdot 3\cdot 5\cdot 7}+\cdots\Big)+\frac{1}{2}$$

is proved also.

1. Formulation of the problem

More than of 20 years ago, K.Trenčevski, has worked on topic of *fractional derivatives* and as a consequence of the identity $(e^x)^{(1/2)}=e^x$, for $x=1$ he obtained the following numerical expansion

$$1+\sum_{s=1}^{\infty}\frac{2^s}{(2s-1)!!}=e\Big(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt\Big). \tag{1.1}$$

This problem has not appeared until now in the literature and it is a subject of our consideration. However, applying some known results obtained by D.S.Mitrinović, J.D.Kečkić [1] and M.R.Spiegel [2], this identity finally is proved by the author Ž.Tomovski.

2. Solution of the problem

Let $F(z)=\int_z^{\infty}\frac{e^{-t}}{t^p}dt$, where $p>0$ and $Rez>0$. In [2], p.288-289, the following identity was proved

$$F(z)=e^{-z}\Big[\frac{1}{z^p}-\frac{p}{z^{p+1}}+\frac{p(p+1)}{z^{p+2}}-\cdots+(-1)^n\frac{p(p+1)(p+2)\cdots(p+n-1)}{z^{p+n}}\Big]$$

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$$+(-1)^{n+1}p(p+1)(p+2)\cdots(p+n)\int_z^\infty \frac{e^{-t}}{t^{p+n+1}}dt. \tag{2.1}$$

Let us evaluate the integral $I=\int_0^1 e^{-1/t^2}dt$. By putting $u=1/t$ and applying the identity (2.1), we obtain

$$I=\frac{1}{2}\int_1^\infty e^{-u}u^{3/2}du=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{3}{2}+\frac{3\cdot 5}{2^2}-\cdots+(-1)^n\frac{(2n+1)!!}{2^n}\Big)+$$

$$+(-1)^{n+1}\frac{(2n+3)!!}{2^{n+1}}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du\Big] \tag{2.2}$$

On the other hand, by Poisson's integral, we obtain

$$\sqrt{\pi}=2\int_0^\infty e^{-u^2}du=2\int_0^1 e^{-u^2}du+2\int_1^\infty e^{-u^2}du.$$

It is obvious that $\int_0^1 e^{-t^2}dt=\sum_{s=0}^\infty \frac{(-1)^s}{s!}\frac{1}{2s+1}$. From (2.1), we get

$$\int_1^\infty e^{-u^2}du=\frac{1}{2}\int_1^\infty e^{-t}t^{-1/2}dt=$$

$$=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\cdots+(-1)^n\frac{(2n-1)!!}{2^n}\Big)+(-1)^{n+1}\frac{(2n+1)!!}{2^{n+1}}\int_1^\infty \frac{e^{-u}}{u^{n+3/2}}du\Big].$$

Integrating by parts, we obtain

$$\int_1^\infty \frac{e^{-u}}{u^{n+3/2}}du=e^{-1}-\frac{2n+3}{2}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du.$$

Thus,

$$\int_1^\infty e^{-u^2}du=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\cdots+(-1)^{n+1}\frac{(2n+1)!!}{2^{n+1}}\Big)+$$

$$+(-1)^{n+2}\frac{(2n+3)!!}{2^{n+2}}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du\Big].$$

Hence,

$$\sqrt{\pi}=2\sum_{s=0}^\infty \frac{(-1)^s}{s!}\frac{1}{2s+1}+e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\cdots+(-1)^{n+1}\frac{(2n+1)!!}{2^{n+1}}\Big)+$$

$$+(-1)^{n+2}\frac{(2n+3)!!}{2^{n+2}}\int\limits_1^\infty \frac{e^{-u}}{u^{n+5/2}}du. \tag{2.3}$$

According to (2.2) and (2.3), we obtain

$$e\Big(\sqrt{\pi}+\int\limits_0^1 e^{-1/t^2}dt\Big)=e\Big(2\sum_{s=0}^\infty\frac{(-1)^s}{s!}\frac{1}{2s+1}+e^{-1}$$

$$-e^{-1}\sum_{k=0}^n(-1)^k\frac{(2k+1)!!}{2^{k+1}}+(-1)^{n+2}\frac{(2n+3)!!}{2^{n+2}}\int\limits_1^\infty\frac{e^{-u}}{u^{n+5/2}}du$$

$$+e^{-1}\cdot\sum_{k=0}^n(-1)^k\frac{(2k+1)!!}{2^{k+1}}+(-1)^{n+1}\frac{(2n+3)!!}{2^{n+2}}\int\limits_1^\infty\frac{e^{-u}}{u^{n+5/2}}du\Big)$$

$$=1+2\sum_{s=0}^\infty\frac{1}{s!}\cdot\sum_{s=0}^\infty\frac{(-1)^s}{s!}\frac{1}{2s+1}$$

Let $a_s=\frac{1}{s!}$, $b_s=\frac{(-1)^s}{s!}\frac{1}{2s+1}$. Then,

$$\Big(\sum_{s=0}^\infty a_s\Big)\Big(\sum_{s=0}^\infty b_s\Big)=\sum_{s=0}^\infty c_s,$$

where

$$c_s=\sum_{k=0}^s b_k a_{s-k}=\sum_{k=0}^s\frac{(-1)^k}{k!}\frac{1}{2k+1}\frac{1}{(s-k)!}=\frac{1}{s!}\sum_{k=0}^s\frac{(-1)^k}{2k+1}\frac{s!}{k!(s-k)!}=$$

$$\frac{1}{s!}\sum_{k=0}^s\frac{(-1)^k}{2k+1}\binom{s}{k}.$$

In [1] p.145-146, D.S.Mitrinović and J.D.Kečkić, for all $a,b\in\mathbb{R}$, verified the following identity

$$\sum_{k=0}^n(-1)^k\binom{n}{k}\frac{1}{ak+b}=\frac{n!a^n}{b(a+b)(2a+b)\cdots(na+b)}.$$

Specially, if $a=2$ and $b=1$, we obtain

$$\sum_{k=0}^n(-1)^k\binom{n}{k}\frac{1}{2k+1}=\frac{n!2^n}{(2n+1)!!}.$$

Finally, $c_s=\frac{2^s}{(2s+1)!!}$, i.e.

$$e\Big(\sqrt{\pi}+\int\limits_0^1 e^{-1/t^2}dt\Big)=1+\sum_{s=0}^\infty\frac{2^{s+1}}{(2s+1)!!}.$$

3. Solution of another close problem.

If we consider the equality $(xe^x)^{(1/2)} = e^x\Big(x+\frac{1}{2}\Big)$ for $x=1$, we obtain the following numerical identity

$$\frac{3}{2}e\sqrt{\pi}+\frac{3}{2}e\int_0^1 e^{-1/t^2}dt=\Big(\frac{1\cdot 2^1}{1}+\frac{2\cdot 2^2}{1\cdot 3}+\frac{3\cdot 2^3}{1\cdot 3\cdot 5}+\frac{4\cdot 2^4}{1\cdot 3\cdot 5\cdot 7}+\cdots\Big)+\frac{1}{2}.$$

The solution of this problem is a consequence of the previous problem. Indeed, applying the identity (1.1) we obtain

$$\frac{3}{2}e\sqrt{\pi}+\frac{3}{2}e\int_0^1 e^{-1/t^2}dt=\frac{3}{2}e\Big(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt\Big)=\frac{3}{2}\Big(1+\sum_{s=1}^{\infty}\frac{2^s}{(2s-1)!!}\Big).$$

It is sufficient to show that

$$1+\frac{3}{2}\sum_{s=1}^{\infty}\frac{2^s}{(2s-1)!!}=\sum_{s=1}^{\infty}\frac{s\cdot 2^s}{(2s-1)!!},$$

i.e.

$$\sum_{s=1}^{\infty}\frac{s\cdot 2^s}{(2s-1)!!}-\frac{3}{2}\sum_{s=1}^{\infty}\frac{2^s}{(2s-1)!!}=1.$$

Let $a_s=\frac{2^s}{(2s-1)!!}$. Then,

$$\sum_{s=1}^{\infty}\frac{s\cdot 2^s}{(2s-1)!!}-\sum_{s=1}^{\infty}\frac{3\cdot 2^{s-1}}{(2s-1)!!}=\sum_{s=1}^{\infty}\frac{(2s-3)\cdot 2^{s-1}}{(2s-1)!!}=$$

$$=\sum_{s=1}^{\infty}\Big[\frac{2^{s-1}}{(2s-3)!!}-\frac{2^s}{(2s-1)!!}\Big]=\sum_{s=1}^{\infty}(a_{s-1}-a_s)=a_0=\frac{2^0}{(2\cdot 0-1)!!}=1,$$

because $(-1)!!=1$ and $\lim_{s\to\infty}\frac{2^s}{(2s-1)!!}=0$.

References

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[2] SPI M.R.Spiegel, *Theory and Problems of Complex Variables*, Singapore, 1988.

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