INEQUALITIES OF DUNKL-WILLIAMS AND MERCER IN QUASI 2-NORMED SPACE

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Abstract. C. Park [3] introduced the term of quasi 2-normed space, and further he has also proved few properties of quasi 2-norm. M. Kir and M. Acikgoz [4] gave the procedure for completing the quasi 2-normed space. Families of quasi-norms generated by quasi 2-norm are considered in [2] and are also proven few statements according to that ones. The inequalities of Dunkl-Williams, Mercer, Pečarić-Rajić and the sharp parallelepiped inequalities are fundamental in the theory of a 2-normed spaces. In quasi 2-normed spaces are proven, [1] and[2], the analogous inequalities of sharp inequalities and inequalities of Pečarić-Rajić type. In this paper will be considered inequalities, which are analogies to Dunkl-Williams and Mercer inequalities in quasi 2-normed spaces.

1. INTRODUCTION

S. Gähler (1965) gave the term of 2-norm ([11]). One of the axioms of 2-norm is the parallelepiped inequality, which is basic one in the theory of 2-normed spaces. Precisely this inequality, analogous as in normed spaces, C. Park has replaced by a new condition, and thus he actually obtained the following definition of quasi 2-normed space:

Definition 1 ([3]). Let $L$ be a real vector space and $\dim L \geq 2$. Quasi 2-norm is real function $\| \cdot, \cdot \|: L \times L \to [0, \infty)$ such that:

a) $\| x, y \| \geq 0$, for all $x, y \in L$ and $\| x, y \| = 0$ iff the set $\{x, y\}$ is linearly dependent;

b) $\| x, y \| = \| y, x \|$, for all $x, y \in L$;

c) $\| \alpha x, y \| = |\alpha| \cdot \| x, y \|$, for all $x, y \in L$ and for each $\alpha \in \mathbb{R}$, and

d) it exists a constant $C \geq 1$ so that $\| x + y, z \| \leq C (\| x, z \| + \| y, z \|)$, holds for all $x, y, z \in L$.

An ordered pair $(L, \| \cdot, \cdot \|)$ is called as quasi 2-normed space. The smallest possible $C$ such that it satisfies the condition d) is called as modulus of concavity of quasi 2-norm $\| \cdot, \cdot \|$. 

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Further, M. Kir and M. Acikgoz [4] have given few examples of trivial quasi 2-normed spaces and have also considered the question about completing a quasi 2-normed space. In [2] is proven the following Lemma which is one of the basic while proving important inequalities in quasi 2-normed spaces.

**Lemma 1.** If \( L \) is a quasi 2-normed space with modulus of concavity \( C \geq 1 \), then
\[
\| \sum_{i=1}^{n} x_i, z \| \leq C^{1+[\log_2(n-1)]} \| \sum_{i=1}^{n} x_i, z \|.
\]
(1)
holds for each \( n>1 \) and for all \( z, x_1, x_2, \ldots, x_n \in L \). 

Further, C. Park gave a characterization of quasi 2-normed space, i.e. proved the following theorem.

**Theorem 1 ([3]).** Let \( (L, \| \cdot \|) \) be a quasi 2-normed space. It exists \( p, \ 0 < p \leq 1 \) and an equivalent quasi 2-norm \( \| \cdot, \cdot \| \) over \( L \) so that
\[
\| x + y, z \| \leq \| x, z \|^p + \| y, z \|^p,
\]
(2)
holds for all \( x, y, z \in L \). 

**Definition 2 ([3]).** The quasi 2-norm defined in Theorem 1 is called a \( (2, p) \)-norm, and the quasi 2-normed space \( L \) is called a \( (2, p) \)-normed space.

2. **OUR RESULTS**

**Theorem 2.** Let \( L \) be a quasi 2-normed space with modulus of concavity \( C \geq 1 \), and \( V(z) \) be the subspace generated by vector \( z \). The following inequality
\[
\| x \| - \| y \| = \| x \| - \| y \| \leq 4C \| x - y \| + (C - 1) \max \{ \| x, z \|, \| y, z \| \},
\]
(3)
holds true for each \( z \in L \setminus \{0\} \) and for all \( x, y \in L \setminus V(z) \).

**Proof.** Let \( z \in L \setminus \{0\} \) and \( x, y \in L \setminus V(z) \). Since definition 1 we get that
\[
\| x, z \| \cdot \| y \| \leq \| x, z \| \cdot \| y \| = \| x, z \| + \| y, z \| + \| x, y \| = \| x, z \| + \| y, z \| = \| x - y, z \|.
\]
(4)
Further, once again definition 1 implies that
\[
\| y, z \| \leq \| y - z, z \| + \| y, z \| \text{ and } \| x, z \| \leq \| x - y, z \| + \| y, z \|.
\]
Therefore,
\[
\| y, z \| = \| y, z \| \leq \| y - z, z \| + (C - 1) \| y, z \| \leq \| x - y, z \| + (C - 1) \max \{ \| x, z \|, \| y, z \| \}
\]
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and
\[ \| x, z \| - \| y, z \| \leq C \| x - y, z \| + (C - 1) \| y, z \| \]
\[ \leq C \| x - y, z \| + (C - 1) \max \{ \| x, z \| , \| y, z \| \}, \]
i.e. the inequality
\[ \| y, z \| - \| x, z \| \leq C \| x - y, z \| + (C - 1) \max \{ \| x, z \| , \| y, z \| \}. \]
holds true. The inequalities (4) and (5) imply the inequality
\[ \| x, z \| \cdot \left( \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|} \right) \cdot \| z \| \leq 2C \| x - y, z \| + (C - 1) \max \{ \| x, z \| , \| y, z \| \} \]
(6)
The following inequality can be proven analogously
\[ \| y, z \| \cdot \left( \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|} \right) \cdot \| z \| \leq 2C \| x - y, z \| + (C - 1) \max \{ \| x, z \| , \| y, z \| \}. \]
(7)
Finally, if we add the inequalities (6) and (7) and so obtained inequality we divide by \( \| x \| + \| y \| > 0 \) we get the inequality (3). ■

**Theorem 3.** Let \( L \) be a \((2, p)\)–normed space, \( 0 < p \leq 1 \), and \( V(z) \) be a subspace generated by vector \( z \). Then
\[ \| x, z \| - \| y, z \| \leq 2 \| x - y, z \| + \| y, z \| \| x, z \| ^p \]
\[ \leq 2 \| x - y, z \| + \| y, z \| \max \{ \| x, z \| , \| y, z \| \} \]
for each \( z \in L \setminus \{0\} \) and for all \( x, y \in L \setminus V(z) \).

**Proof.** Definition 2, i.e. the properties of \((2, p)\)–norm imply that each \( z \in L \setminus \{0\} \) and all \( x, y \in L \setminus V(z) \) satisfy the following
\[ \| x, z \| ^p \cdot \left( \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|} \right) \cdot \| z \| ^p \leq 2 \| x - y, z \| ^p + \| y, z \| - \| x, z \| ^p \]
\[ \leq 2 \| x - y, z \| ^p + \| y, z \| - \| x, z \| ^p \]
and
\[ \| y, z \| ^p \cdot \left( \frac{x}{\| x, z \|} - \frac{y}{\| y, z \|} \right) \cdot \| z \| ^p \leq 2 \| y - x, z \| ^p + \| x, z \| - \| y, z \| ^p \]
\[ \leq 2 \| y - x, z \| ^p + \| x, z \| - \| y, z \| ^p \]
Finally, if we add the inequalities (9) and (10) and the so obtained inequality we divide by \( \| x \| ^p + \| y \| ^p > 0 \), we get the inequality (8). ■

**Remark 1.** The inequalities (3) and (8) are actually inequalities of Dunkl–Williams type in quasi-normed and \( p \)–normed space, \( 0 < p \leq 1 \), respectively.

**Theorem 4.** Let \( L \) be a quasi \( 2 \)-normed space with modulus of concavity \( C \geq 1 \). The following statements are equivalent:
1) For each \( z \in L \setminus \{0\} \) and for all \( x, y \in L \setminus V(z) \)
\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| z \leq 2C \frac{\|x-y,z\|}{\|x\| + \|y\|} + (C-1) \max\{\|x\|,\|y\|\}. \tag{11}
\]
2) If \( x, y, z \in L \) are such that \( \|x, z\| \leq \|y, z\| = 1 \), holds then
\[
\left\| \frac{x+y}{2} \right\| z \leq C \left\| (1-t)x + ty, z \right\| + \frac{C-1}{2} \max\{1-t, t\}, \tag{12}
\]
for each \( t \in [0,1] \).

**Proof.** 1) \( \Rightarrow \) 2). Let assume that 1) is satisfied. Let \( x, y, z \in L \) be such that
\[
\|x, z\| \leq \|y, z\| = 1
\]
is satisfied. Clearly, for \( t = 0 \) and \( t = 1 \), the inequality (12) is satisfied. If \( t \in (0,1) \), then
1) implies
\[
\left\| \frac{x+y}{2} \right\| z = \frac{1-t}{2} \left\| x + y, z \right\|
\]
\[
= \frac{1-t}{2} \left( \|x, z\| + \|\frac{t}{1-t} y, z\| \right) \left\| \frac{x}{\|x\|} - \frac{t}{1-t} y, z \right\|
\]
\[
\leq \frac{1-t}{2} \left( \|x, z\| + \|\frac{t}{1-t} y, z\| \right) (2C \frac{\|x-y, z\|}{\|x\| + \|y\|} + (C-1) \max\{\|x\|,\|y\|\})
\]
\[
= C(1-t) \|x - \frac{t}{1-t} y, z\| + \frac{C-1(1-t)}{2} \max\{1, \frac{t}{1-t}\}
\]
\[
= C \left\| (1-t)x + ty, z \right\| + \frac{C-1}{2} \max\{1-t, t\},
\]
i.e. the inequality (12) holds true.

2) \( \Rightarrow \) 1). Let assume that 2) is satisfied. Further, let \( x \) and \( y \) be arbitrary non-null vectors at \( L \). Then for \( \frac{x}{\|x\|}, \frac{-y}{\|y\|} \in L \) it is true that
\[
\left\| \frac{x}{\|x\|}, z \right\| = \left\| \frac{-y}{\|y\|}, z \right\| = 1
\]
and if \( t = \frac{\|y, z\|}{\|x, z\| + \|y, z\|} \), then 2) implies that
\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|}, z \right\| = 2 \left\| \frac{x}{\|x\|} + \frac{-y}{\|y\|}, z \right\|
\]
\[
\leq 2C \left\{ (1-\frac{\|y, z\|}{\|x, z\| + \|y, z\|}) \frac{x}{\|x\|} + \frac{\|x\|}{\|x, z\| + \|y, z\|} \right\}
\]
\[
+ \frac{C-1}{2} \max\{1 - \frac{\|y, z\|}{\|x, z\| + \|y, z\|}, \frac{\|y, z\|}{\|x, z\| + \|y, z\|}\}
\]
\[
= 2C \frac{x-y, z}{\|x, z\| + \|y, z\|} + (C-1) \frac{\max\{\|x\|,\|y\|\}}{\|x, z\| + \|y, z\|}
\]
i.e. the inequality (11) holds true. ■

**Remark 2.** The inequality (11) is actually generalization of the inequality
\[
\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|}, z \right\| \leq 2\|x-y, z\| \frac{\|x, z\|}{\|x, z\| + \|y, z\|},
\]
which on 2-normed space, is satisfied if and only if the 2-norm is generated by 2-inner product ([10]). So, it is logically to be stated the following question:

*Does the inequality (11) in quasi 2-normed space with modulus of concavity $C \geq 1$ hold true if and only if it exists a function $f : L \times L \rightarrow \mathbb{R}$ so that $f(x, x, z) = \|x, z\|^2$.*

**References**


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