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APPROXIMATION FOR THE SOLUTIONS OF LORENZ SYSTEM WITH SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. Using the systems of difference equations from [1] and [2] as approximation for the solutions of the Lorenz system of differential equations we obtain three new systems of differential equations whose locally behavior is close to the behavior of Lorenz system. By computer simulations we give examples where locally behavior of the systems is analyzed by comparing of behavior for the different time step. By increasing the time step can be seen that behavior of systems is farther away from the behavior of Lorenz system.

1. Introduction

Lorenz system is a system of three differential equations who depends of three parameters. Its explicit solutions are not known. Behavior of the Lorenz system as nonlinear autonomous dynamic system is studied in the mathematical literature, like in the papers [3], [4] and [5].

The use of power series is one of the oldest methods for examining differential equations. It is used for numerical calculations and for theoretical results. In the mathematical literature there are numerous papers concerned with such a use of power series, as example [6], [7] and [8].

In [1] and [2] we have used power series combined with difference equations to find local approximations to the solutions of the Lorenz system of differential equations:

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= x(r-z)-y \\ \dot{z} &= xy-bz \end{aligned} \tag{1}$$

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with parameters σ, r, b. For initial values $a_0 = x(0), b_0 = y(0), c_0 = z(0)$ by [9], [10], [11] we assumie that the solutions of the system (1) are expanded as Maclaurin series,

$$\begin{aligned} x(t) &= a_0 + a_1 t + a_2 \frac{t^2}{2!} + \cdots + a_n \frac{t^n}{n!} + \cdots \\ y(t) &= b_0 + b_1 t + b_2 \frac{t^2}{2!} + \cdots + b_n \frac{t^n}{n!} + \cdots \\ z(t) &= c_0 + c_1 t + c_2 \frac{t^2}{2!} + \cdots + c_n \frac{t^n}{n!} + \cdots \end{aligned} \tag{2}$$

By consecutive differentiation of (1), for the coefficients a_n, b_n, c_n from (2), we obtain the following system of difference equations:

$$\begin{aligned} a_n &= \sigma(b_{n-1} - a_{n-1}) \\ b_n = r a_{n-1} - b_{n-1} &- \sum_{i=0}^{n-1} \binom{n-1}{i} a_i c_{n-i-1} \\ c_n = -b c_{n-1} &+ \sum_{i=0}^{n-1} \binom{n-1}{i} a_i b_{n-i-1} \end{aligned} \tag{3}$$

In [1] and [2] of (3) with some mathematical transformations and forgetting some parts from a_n, b_n, c_n with the initial values $a_0 = x(0), b_0 = y(0), c_0 = z(0) = 0$ and parameters σ, r, b, it is obtained the system of difference equations for $a_n(\approx), b_n(\approx), c_n(\approx)$:

$$\begin{aligned} a_n(\approx) = {} & \sigma(b_{n-1}(\approx) - a_{n-1}(\approx)) + A\sigma(b_{n-2}(\approx) - a_{n-2}(\approx)) \\ & -B\sigma(b_{n-3}(\approx) - a_{n-3}(\approx)) + C\sigma(b_{n-4}(\approx) - a_{n-4}(\approx)) \\ & -D\sigma(b_{n-5}(\approx) - a_{n-5}(\approx)) - A a_{n-1}(\approx) + B a_{n-2}(\approx) \\ & -C a_{n-3}(\approx) + D a_{n-4}(\approx), \qquad n > 7 \\ b_n(\approx) = {} & (r a_{n-1}(\approx) - b_{n-1}(\approx)) + A(r a_{n-2}(\approx) - b_{n-2}(\approx)) \\ & -B(r a_{n-3}(\approx) - b_{n-3}(\approx)) + C(r a_{n-4}(\approx) - b_{n-4}(\approx)) \\ & -D(r a_{n-5}(\approx) - b_{n-5}(\approx)) - A b_{n-1}(\approx) + B b_{n-2}(\approx) \\ & -C b_{n-3}(\approx) + D b_{n-4}(\approx), \qquad n > 6 \\ c_n(\approx) = {} & -A c_{n-1}(\approx) + B c_{n-2}(\approx) - C c_{n-3}(\approx) + D c_{n-4}(\approx), \qquad n > 5 \end{aligned} \tag{4}$$

where $A = 1 + \sigma + b, B = \sigma r - a_0^2, C = \sigma a_0 b_0, D = -\sigma^2 b_0^2$. The coefficients $a_p(\approx), b_q(\approx), c_s(\approx)$ for $p \in \{1,2,3,4,5,6,7\}, q \in \{1,2,3,4,5,6\}, s \in \{1,2,3,4,5\}$ are calculated direct of (3) as the exact values $a_p(\approx) = a_p = x^{(p)}(0)$,
$b_q(\approx) = b_q = y^{(q)}(0), c_s(\approx) = c_s = z^{(s)}(0)$ and they are:

$$
\begin{aligned}
&a_1 = \sigma(b_0 - a_0), && b_1 = ra_0 - b_0, && c_1 = a_0 b_0,\\
&a_2 = \sigma(b_1 - a_1), && b_2 = ra_1 - b_1 - a_0 c_1, && c_2 = -bc_1 + a_0 b_1 + a_1 b_0\\
&a_3 = \sigma(b_2 - a_2), && b_3 = ra_2 - b_2 - \textstyle\sum_{i=0}^{1}\binom{2}{i} a_i c_{2-i}, && c_3 = -bc_2 + \displaystyle\sum_{i=0}^{2}\binom{2}{i} a_i b_{2-i}\\
&a_4 = \sigma(b_3 - a_3), && b_4 = ra_3 - b_3 - \textstyle\sum_{i=0}^{2}\binom{3}{i} a_i c_{3-i}, && c_4 = -bc_3 + \displaystyle\sum_{i=0}^{3}\binom{3}{i} a_i b_{3-i},\\
&a_5 = \sigma(b_4 - a_4), && b_5 = ra_4 - b_4 - \textstyle\sum_{i=0}^{3}\binom{4}{i} a_i c_{4-i}, && c_5 = -bc_4 + \displaystyle\sum_{i=0}^{4}\binom{4}{i} a_i b_{4-i}\\
&a_6 = \sigma(b_5 - a_5), && b_6 = ra_5 - b_5 - \textstyle\sum_{i=0}^{4}\binom{5}{i} a_i c_{5-i} && \qquad (5)\\
&a_7 = \sigma(b_6 - a_6)
\end{aligned}
$$

2. Systems of differential equations

In this section, we will present systems of differential equations which approximated the solutions of Lorenz system.

For the initial values

$$
\begin{aligned}
x(0) = a_0, x^{(1)}(0) = a_1, x^{(2)}(0) = a_2, x^{(3)}(0) = a_3, x^{(4)}(0) = a_4,\\
y(0) = b_0, y^{(1)}(0) = b_1, y^{(2)}(0) = b_2, y^{(3)}(0) = b_3, y^{(4)}(0) = b_4,\\
z(0) = c_0 = 0, z^{(1)}(0) = c_1, z^{(2)}(0) = c_2, z^{(3)}(0) = c_3
\end{aligned} \tag{6}
$$

calculated directly from (5), the system of difference equations (4) is transformed in the following system of differential equations:

$$
\begin{aligned}
x^{(5)} &= \sigma(y^{(4)} - x^{(4)}) + A\sigma(y^{(3)} - x^{(3)}) - B\sigma(y^{(2)} - x^{(2)}) + C\sigma(y^{(1)} - x^{(1)})\\
&\quad -D\sigma(y - x) - Ax^{(4)} + Bx^{(3)} - Cx^{(2)} + Dx^{(1)}\\
y^{(5)} &= (rx^{(4)} - y^{(4)}) + A(rx^{(3)} - y^{(3)}) - B(rx^{(2)} - y^{(2)}) + C(rx^{(1)} - y^{(1)})\\
&\quad -D(rx - y) - Ay^{(4)} + By^{(3)} - Cy^{(2)} + Dy^{(1)}\\
z^{(4)} &= -Az^{(3)} + Bz^{(2)} - Cz^{(1)} + Dz
\end{aligned} \tag{7}
$$

By differentiation of the system (7), we obtain a new system of differential equations

$$
\begin{aligned}
x^{(6)} &= \sigma(y^{(5)} - x^{(5)}) + A\sigma(y^{(4)} - x^{(4)}) - B\sigma(y^{(3)} - x^{(3)}) + C\sigma(y^{(2)} - x^{(2)})\\
&\quad -D\sigma(y^{(1)} - x^{(1)}) - Ax^{(5)} + Bx^{(4)} - Cx^{(3)} + Dx^{(2)}\\
y^{(6)} &= (rx^{(5)} - y^{(5)}) + A(rx^{(4)} - y^{(4)}) - B(rx^{(3)} - y^{(3)}) + C(rx^{(2)} - y^{(2)})
\end{aligned} \tag{8}
$$

$$-D(rx^{(1)} - y^{(1)}) - Ay^{(5)} + By^{(4)} - Cy^{(3)} + Dy^{(2)}$$
$$z^{(5)} = -Az^{(4)} + Bz^{(3)} - Cz^{(2)} + Dz^{(1)}$$

with the initial values

$$\begin{gathered} x(0) = a_0, x^{(1)}(0) = a_1, x^{(2)}(0) = a_2, x^{(3)}(0) = a_3, x^{(4)}(0) = a_4, x^{(5)} = a_5, \\ y(0) = b_0, y^{(1)}(0) = b_1, y^{(2)}(0) = b_2, y^{(3)}(0) = b_3, y^{(4)}(0) = b_4, y^{(5)} = b_5, \\ z(0) = c_0 = 0, z^{(1)}(0) = c_1, z^{(2)}(0) = c_2, z^{(3)}(0) = c_3, z^{(4)} = c_4 \end{gathered} \tag{9}$$

calculated directly from (5).

If we differentiate and of the system (8), then we will obtain a new system of differential equations

$$\begin{aligned} x^{(7)} &= \sigma(y^{(6)} - x^{(6)}) + A\sigma(y^{(5)} - x^{(5)}) - B\sigma(y^{(4)} - x^{(4)}) + C\sigma(y^{(3)} - x^{(3)}) \\ &\quad -D\sigma(y^{(2)} - x^{(2)}) - Ax^{(6)} + Bx^{(5)} - Cx^{(4)} + Dx^{(3)} \\ y^{(7)} &= (rx^{(6)} - y^{(6)}) + A(rx^{(5)} - y^{(5)}) - B(rx^{(4)} - y^{(4)}) + C(rx^{(3)} - y^{(3)}) \\ &\quad -D(rx^{(2)} - y^{(2)}) - Ay^{(6)} + By^{(5)} - Cy^{(4)} + Dy^{(3)} \\ z^{(6)} &= -Az^{(5)} + Bz^{(4)} - Cz^{(3)} + Dz^{(2)} \end{aligned} \tag{10}$$

with the initial values

$$\begin{gathered} x(0) = a_0, x^{(1)}(0) = a_1, x^{(2)}(0) = a_2, x^{(3)}(0) = a_3, x^{(4)}(0) = a_4, x^{(5)} = a_5, \\ x^{(6)} = a_6, \\ y(0) = b_0, y^{(1)}(0) = b_1, y^{(2)}(0) = b_2, y^{(3)}(0) = b_3, y^{(4)}(0) = b_4, \\ y^{(5)}(0) = b_5, y^{(6)}(0) = b_6, \\ z(0) = c_0 = 0, z^{(1)}(0) = c_1, z^{(2)}(0) = c_2, z^{(3)}(0) = c_3, z^{(4)}(0) = c_4, z^{(5)}(0) = c_5 \end{gathered} \tag{11}$$

calculated directly from (5).

Normally, with further differentiation can be obtained larger number systems as systems of different equations (7), (8) and (10). Any future system of differential equations will be of a higher order and it will has larger number of exact initial values.

The systems (7), (8) and (10) are linear systems and their global behavior is far away from the behavior of the Lorenz system. But, we have the opinion, that local behavior of the systems (7), (8) and (10) for the small time step is close to the local behavior of the Lorenz system.

3. Computer Simulations for the systems of differential equations

In this section, we will give the local behavior for the systems of differential equations (7), (8) and (10) for different time step compared to their global behavior.

At this moment the question of what conditions would imply the convergence of these power series is open. For the systems of differential equations, for given parameters σ, r, b and initial values $a_0, b_0, c_0 = 0$, we have the Maclaurin series (2).

For given parameters σ, r, b the procedure for looking at the local behavior of the systems (7), (8) and (10) are as follows. Let T be a positive real number as the time step.

We take the solutions $x(t), y(t), z(t)$ for $t \in [0, T]$ of the systems (7), (8) and (10) obtained by the program Mathematica, for:

– the initial values (6) for the system (7);

– the initial values (9) for the system (8);

– the initial values (11) for the system (10).

Let $x_T(t) = x(t), y_T(t) = y(t), z_T(t) = z(t)$ for $t \in [0, T]$. Let $x_T(t), y_T(t), z_T(t)$ be defined for $t \in [0, kT]$. We extend them on $[0,(k+1)T]$ by defining them for $t \in [kT,(k+1)T]$ as follows. We take the solutions $x(t), y(t), z(t)$ for $t \in [0, T]$ of the systems (7), (8) and (10) obtained by the program Mathematica, for the initial values $a_0 = x(0) = x_T(kT)$, $b_0 = y(0) = y_T(kT), c_0 = z(0) = z_T(kT)$ and:

– a_p, b_q, c_s where $p, q \in \{1, 2, 3, 4\}, s \in \{1, 2, 3\}$ for the system (7) calculated from (5);

– a_p, b_q, c_s where $p, q \in \{1, 2, 3, 4, 5\}, s \in \{1, 2, 3, 4\}$ for the system (8) calculated from (5);

– a_p, b_q, c_s where $p, q \in \{1, 2, 3, 4, 5, 6\}, s \in \{1, 2, 3, 4, 5\}$ for the system (10) calculated from (5).

Then, we define: $x_T(t) = x(t-kT), y_T(t) = y(t-kT), z_T = z(t-kT)$.

In examples, by computer calculations, for given parameters and initial conditions, for different small values of time step T, we obtain functions $x_T(t), y_T(t), z_T(t)$ for the systems (7), (8) and (10). We compare these solutions with the solutions of the Lorenz system (1), for the same parameters and initial conditions, obtained by the program Mathematica for different time step T and we obtain that they are close.

Example : Parameters $\sigma = 5, r = 25, b = 0.8$ and the initial values $a_0 = 0, b_0 = 1, c_0 = 0$.

Figure 2 and figure 3 show that the local behavior of the system of differential equations (7) for two different time steps of the time interval $[0, 6]$ is close to the behavior of Lorenz system.

In Figure 4, the local behavior of the system of differential equations (7) of smaller time interval $[0, 1.2]$ moves away from the behavior of Lorenz system.

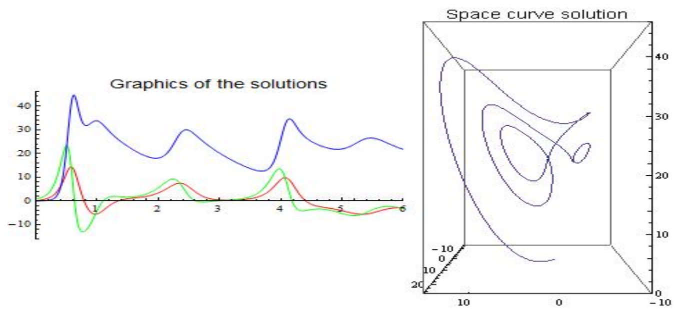

FIGURE 1. Results obtained by the program Mathematica for the system (1) of time interval $[0, 6]$

In Figure 5 the global behavior of the system of differential equations (7) of the interval $[0, 6]$ is not like the behavior of Lorenz system (fig.1), but its behavior characterizes the linear nature of the system of differential equations (7).

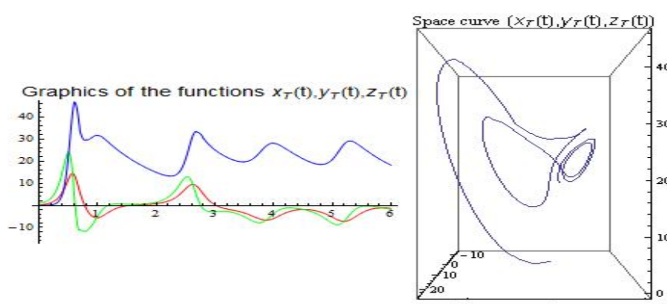

FIGURE 2. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.1$ for the system of differential equations (7) of the time interval $[0, 6]$

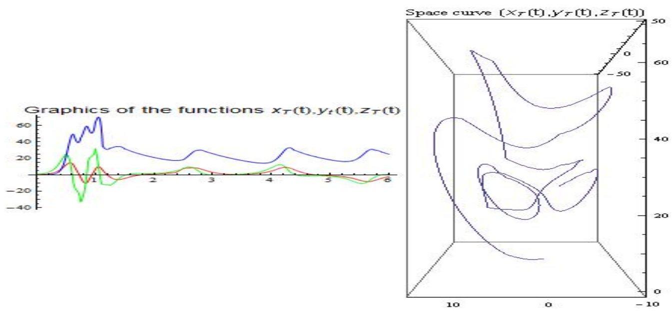

FIGURE 3. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.125$ for the system of differential equations (7) of the time interval $[0, 6]$

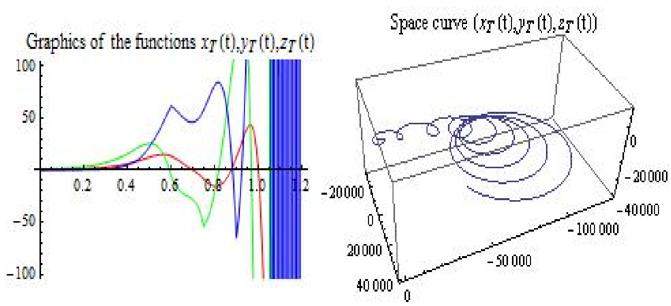

FIGURE 4. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.15$ for the system of differential equations (7) of the time interval $[0, 1.2]$

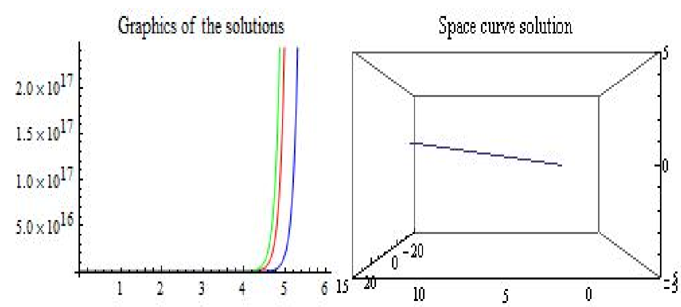

FIGURE 5. Results obtained by the program Mathematica for the system of differential equations (7) of the time interval $[0, 6]$

In figure 6 and figure 7, the local behavior of the system of differential equations (8) of the time interval $[0, 6]$ is close to the behavior of Lorenz system.

In figure 8, the local behavior of the system of differential equations (8) of smaller time interval $[0, 1.05]$ is all farther away from the behavior of Lorenz system.

In figure 9 the global behavior of the system of differential equations (8) of the time interval $[0, 6]$ is far away from the behavior of Lorenz system (fig.1). In figure 9, the system of differential equations (8) reflects its linear nature.

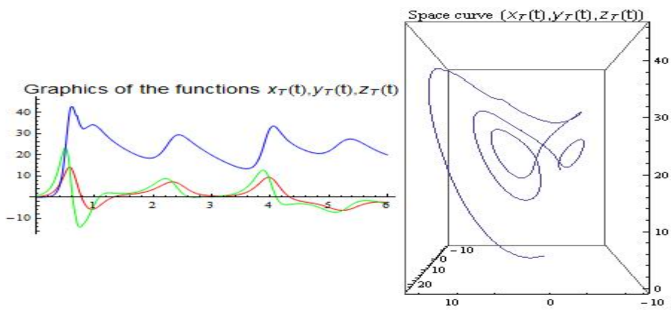

FIGURE 6. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.1$ for the system of differential equations (8) of the time interval $[0, 6]$

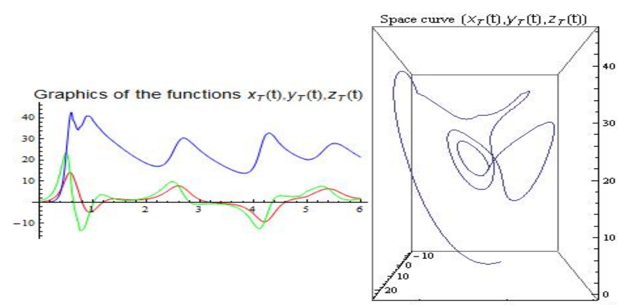

FIGURE 7. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.125$ for the system of differential equations (8) of the time interval $[0, 6]$

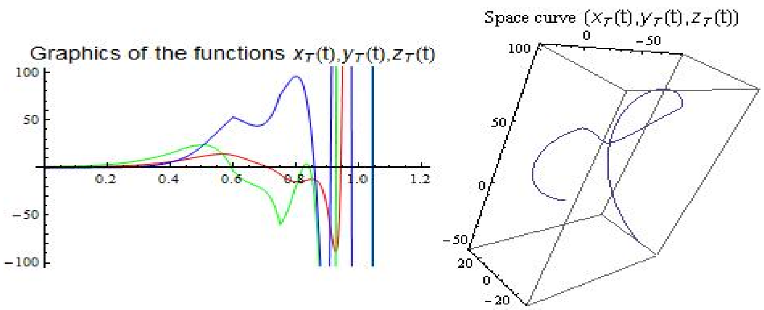

FIGURE 8. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.15$ for the system of differential equations (8) of the time interval $[0, 1.05]$

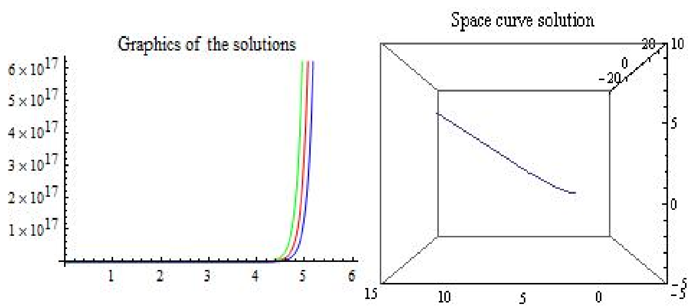

FIGURE 9. Results obtained by the program Mathematica for the system of differential equations (8) of the time interval $[0, 6]$

In figure 10 and figure 11 the local behavior of the system of differential equations (10) of the time interval $[0, 6]$ is close to the behavior of Lorenz system.

In figure 12, the local behavior of the system of differential equations (10) of smaller time interval $[0, 1]$ is farther away from the behavior of Lorenz system.

In figure 13, the global behavior of the system of differential equations (10) of the time interval $[0, 6]$ is far away from the behavior of Lorenz system (fig.1) ie in figure 13, the system of differential equations (10) reflects its linear nature.

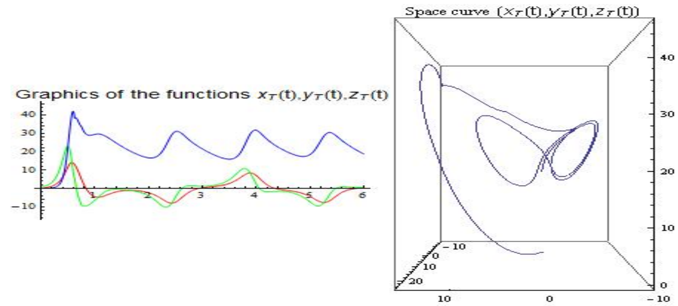

FIGURE 10. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.125$ for the system of differential equations (10) of the time interval $[0, 6]$

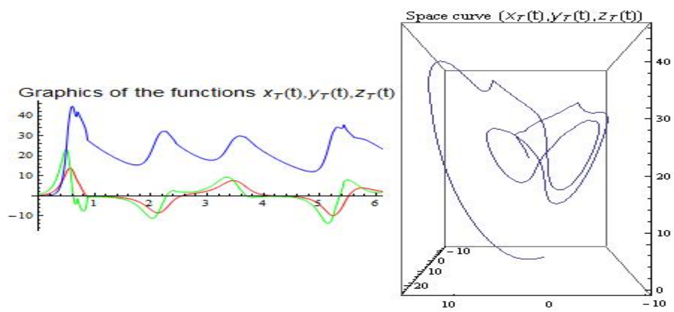

FIGURE 11. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.175$ for the system of differential equations (10) of the time interval $[0, 6]$

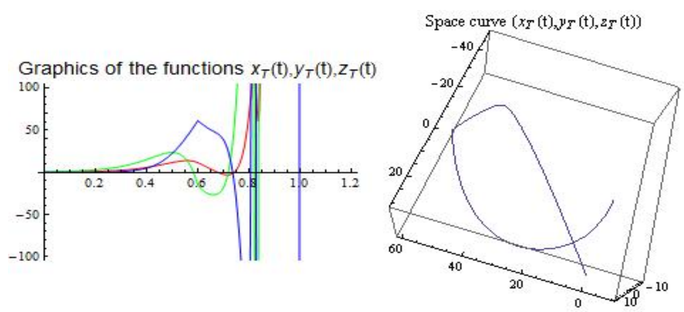

FIGURE 12. The solutions $x_T(t), y_T(t), z_T(t)$ where $T = 0.2$ for the system of differential equations (10) of the time interval $[0, 1]$

We conclude that: If the system of differential equations is of a higher order and incorporates more exact initial values then its local behavior will be closer to the behavior of Lorenz system for a larger time step. Normally,

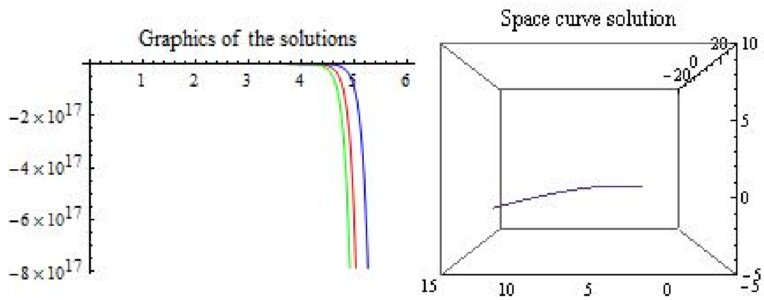

FIGURE 13. Results obtained by the program Mathematica for the system of differential equations (10) of the time interval $[0, 6]$

by increasing the time step the local behavior of these systems of differential equations are moving away from the behavior of Lorenz system. As expected, their global behavior reflects linearity of systems of differential equations.

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