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ON MONOASSOCIATIVE GROUPOIDS

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Abstract

The subject of this paper is the variety (denoted by *Mass*) of monoassociative groupoids, i.e. groupoids in which every cyclic subgroupoid is a subsemigroup. A description of free objects in *Mass* is given. Using a convenient definition of injective groupoids in *Mass*, it is shown that a groupoid $\underline{H}$ is free in *Mass* iff $\underline{H}$ is injective in *Mass* and the set of prime elements in $\underline{H}$ generates $\underline{H}$. (This property is named Bruck Theorem for *Mass*.) Neither of the classes *Massin* (injective objects in *Mass*) and *Massfr* (free objects in *Mass*) is hereditary. A characterization of free subgroupoids of a groupoid $\underline{H} \in$ *Massfr* is obtained. It is shown that every groupoid $\underline{H} \in$ *Massfr* with a two-element basis has a subgroupoid $\underline{Q} \in$ *Massfr* with an infinite basis.

1. Preliminaries

A groupoid is a pair $\underline{G} = (G, \cdot)$, where G is a nonempty set and " $\cdot$ " is a mapping $(x, y) \mapsto xy$, from G^2 into G. $\underline{G}$ is said to be *injective* iff:

$$(\forall x, y, u, v \in G)(xy = uv \Rightarrow (x, y) = (u, v)). \qquad (1.1)$$

An element $a \in G$ is *prime*[1] in $\underline{G}$ iff $a \notin GG$, where

$$GG = \{xy \mid x, y \in G\}. \qquad (1.2)$$

[1] The notions as subgroupoid, semigroup, variety of groupoids ... have usual meanings.

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The following statement is well known (for example; [1; L.1.5]).

Proposition 1.1(Bruck Theorem). A groupoid $\underline{F} = (F, \cdot)$ is absolutely free (i.e. free in the variety of groupoids) iff the following conditions hold:

a) $\underline{F}$ is injective.

b) The set B of primes in $\underline{F}$ is nonempty and generates $\underline{F}$.

Below we assume that $\underline{F}$ is a given absolutely free groupoid with the basis B. The *length* $|v|$, the *set* $P(v)$ *of parts* and the *content* $\mathrm{cn}(v)$ of an element $v \in F$, are defined as follows:

$$|b| = 1, \quad |tu| = |t| + |u|; \quad P(b) = \{b\}, \quad P(tu) = \{tu\} \cup P(t) \cup P(u);$$
$$\mathrm{cn}(b) = \{b\}, \quad \mathrm{cn}(tu) = \quad \mathrm{cn}(t) \cup \mathrm{cn}(u), \tag{1.3}$$

for any $b \in B$, $t, u \in F$.

We will also use an absolutely free groupoid $\underline{E} = (E, \cdot)$ with a one-element basis $\{e\}$, assuming that $F \cap E = \emptyset$. Elements of E will be denoted by $f, g, h, \ldots$ and will be called *(groupoid) powers.* It should be noted that (1.3) makes meaningful notions "the length $|f|$" and "the set $P(f)$ of parts" of an element $f \in E$.

If $\underline{G}$ is a groupoid, then each $f \in E$ induces a transformation $f^{\underline{G}} : G \to G$ defined by:

$$f^{\underline{G}}(x) = \varphi_x(f),$$

where $\varphi_x : E \to G$ is the homomorphism from $\underline{E}$ into $\underline{G}$ such that $\varphi_x(e) = x$. Therefore:

$$e^{\underline{G}}(x) = x, \quad (fh)^{\underline{G}}(x) = f^{\underline{G}}(x)\, h^{\underline{G}}(x), \tag{1.4}$$

for any $f, h \in E$, $x \in G$. (We will usually write $f(x)$ instead of $f^{\underline{G}}(x)$ when we work with a fixed groupoid $\underline{G}$.)

The following statement is clear.

Proposition 1.2. If $\underline{G}$ is a groupoid and $a \in G$, then $\{f(a) \mid f \in E\}$ is the subgroupid of $\underline{G}$ generated by a. □

In the following sections we will use a subset D of E defined as follows:

$$D = \{e^n \mid n \in \mathbf{N}\}, \tag{1.5}$$

where $\mathbf{N}$ is the set of positive integers and

$$e^1 = e, \quad e^{k+1} = e^k e. \tag{1.6}$$

The fact that $\underline{F}$ is injective implies that $\underline{F}$ has the following property:

$$(\forall t, u \in F,\ m, n \in \mathbf{N})(t^{m+1} = u^{n+1} \Rightarrow t = u \ \&\ m = n). \qquad (1.7)$$

If $\underline{G}$ is a groupoid, and $b \in G$ is such that

$$(\forall c \in G,\ n \in \mathbf{N})(n \geq 2 \Rightarrow b \neq c^n), \qquad (1.8)$$

then we say that b is a *base element* (or, shortly: a *base*) in $\underline{G}$.

2. Monoassociative groupoids

We say that a groupoid $\underline{G} = (G, \cdot)$ is *monoassociative* iff, for any $a \in G$, the subgroupoid $\underline{Q}$ of $\underline{G}$ generated by a is associative, i.e. a subsemigroup of $\underline{G}$. (The class of monoassociative groupoids will be usually denoted by *Mass*.)

The proofs of the following statements are obvious corollaries from the definition of *Mass*.

Proposition 2.1. $\underline{G} \in Mass$ iff for any $f \in E$ and $x \in G$, the following equation

$$f(x) = x^{|f|} \qquad (2.1)$$

holds in $\underline{G}$. □

Proposition 2.2. If $\underline{G} \in Mass$, then

$$x^m x^n = x^{m+n}, \quad (x^m)^n = x^{mn}, \qquad (2.2)$$

for any $x \in G$, $m, n \in \mathbf{N}$. □

Proposition 2.3. If $\underline{G}$ is a groupoid, then the following statements are equivalent:

(a) $\underline{G} \in Mass$.

(b) $\underline{G}$ is a union of subsemigroups of $\underline{G}$.

(c) $\underline{G}$ is a union of cyclic subsemigroups of $\underline{G}$. □

Proposition 2.4. *Mass* is a variety of groupoids and:

$$\{f(x) = x^{|f|} \mid f \in E\} \qquad (2.3)$$

is an axiom system for this variety. □

3. Free monoassociative groupoids

Assuming that B is a nonempty set, and $\underline{F}$ an absolutely free groupoid with the basis B, we are looking for a groupod $\underline{R} = (R, *)$ with the following properties:

(i) $B \subset R \subset F$;

(ii) $t \in R \Rightarrow P(t) \subseteq R$;

(iii) $t, u, tu \in R \Rightarrow t * u = tu$;

(iv) $\underline{R}$ is a free groupoid in *Mass* with the basis B.

Proposition 2.1 suggests the following set R as a candidate for the carrier of the desired groupoid $\underline{R}$:

$$R = \{t \in F \mid (\forall f \in E \setminus D, x \in F)\ f(x) \notin P(t)\}. \tag{3.1}$$

The following properties of R are obvious corollaries of (3.1).

Proposition 3.1. (a) R satisfies (i) and (ii).

(b) $t \in F$ & $m, n \in \mathbf{N}$, $n \geq 2 \Rightarrow t^m t^n \notin R$.

(c) $t \in F$ & $m, n \in \mathbf{N}$, $m \geq 2, n \geq 2 \Rightarrow (t^m)^n \notin R$.

(d) $\{t, u\} \subseteq R$ & $tu \notin R \Rightarrow (\exists \alpha \in R, m \geq 1, n \geq 2) tu = \alpha^m \alpha^n$. □

Now we will describe conditions under which $t^n \in R$.

Proposition 3.2. If $t \in F$ and $n \geq 2$, then:
$t^n \in R \iff t \in R$ & t is a base in $\underline{F}$.

Proof. Assume that $t \in R$ and t is a base in $\underline{F}$. By Proposition 3.1 (d), $t^2 \in R$. Assuming that $t^k \in R$, also by Proposition 3.1 (d), we obtain $t^{k+1} = t^k t \in R$.

Conversely, $t^n \in R$, by Proposition 3.1 (a), (d), implies: $t \in R$ and t i a base element in $\underline{F}$. □

Now we define an operation $*$ on R, as follows. If $t, u \in R$, then:

$$t * u = \begin{cases} tu, & tu \in R, \\ \alpha^{m+n} & tu = \alpha^m \alpha^n,\ m, n \in \mathbf{N},\ n \geq 2. \end{cases} \tag{3.2}$$

Proposition 3.3. $\underline{R} = (R, *)$ is a groupod which satisfies the conditions (iii) and (iv).

Proof. 1) By (3.2) and Proposition 3.2, $\underline{R}$ is a groupoid that B is the set of primes in $\underline{R}$, and the least generating subset of $\underline{R}$, as well. Moreover, we have:

$$|t * u| = |t| + |u| = |tu|, \tag{3.3}$$

$$\operatorname{cn}(t * u) = \operatorname{cn}(t) \cup \operatorname{cn}(u), \tag{3.4}$$

for any $t, u \in R$.

2) If $t \in R$, $n \in \mathbf{N}$, $f \in E$, then t_*^n, $f_*(t)$ are defined as follows:

$$t_*^1 = t, \quad t_*^{n+1} = t_*^n * t, \tag{3.5}$$

$$e_*(t) = t, \quad (f_1 f_2)_*(t) = (f_{1*}(t)) * (f_{2*}(t)). \tag{3.6}$$

By (3.2), (3.5), (3.6) and Proposition 3.2, we obtain that for any $t \in R$ is a base in $\underline{F}$, and any $m, n \in \mathbf{N}$, $f \in E$, the following equations hold:

$$t_*^n = t^n, \quad f_*(t) = t^{|f|}, \tag{3.7}$$

$$(t^m)_*^n = t^{mn}, \quad f_*(t^m) = t^{m|f|}. \tag{3.8}$$

Finally, from (3.7) and (3.8), by Proposition 2.1, we obtain that $\underline{R} \in Mass$.

3) It remains to show that $\underline{R}$ is free in *Mass* with the basis B.

Let $\underline{G} \in Mass$, $\lambda : B \to G$, and φ be the homomorphism from $\underline{F}$ into $\underline{G}$, which extends λ. Then, for any $t, u \in R$, we have:

$$\varphi(t*u) = \begin{cases} \varphi(tu) = \varphi(t)\varphi(u), & tu \in R, \\ \varphi(\alpha^{m+n}) = \varphi(\alpha)^m \varphi(\alpha)^n = \varphi(t)\varphi(u), & tu = \alpha^m \alpha^n, \ m, n \in \mathbf{N}, \ n \geq 2, \end{cases}$$

and this implies that the restriction $\psi = \varphi|R$ of φ on R is a homomorphism from $\underline{R}$ into $\underline{G}$, which extends λ. □

The following properties of $\underline{R}$ can be also easily shown.

Proposition 3.4. If $t \in R$, then t is a base element in $\underline{R}$ iff t is a base element in $\underline{F}$. □

Propositon 3.5. If $u \in R$, then there exists a unique pair $(t, k) \in R \times \mathbf{N}$ such that t is a base in $\underline{R}$ and $u = t_*^k (= t^k)$. □

We say that t is the *base*, and k is the *exponent* of u in $\underline{R}$. In the case $k \geq 2$, the equation $u = v * w$ holds in $\underline{R}$ iff $v = t^r$, $w = t^s$, and $r + s = k$.

Proposition 3.6. If $u \in R$ is a base element and $u \in R \setminus B$, then there is a unique pair $(v, w) \in R^2$ such that $u = v * w (= vw)$; moreover, v and w have different bases. □

Proposition 3.7. If $t, u, v \in R$, then:

(a) $t * u = u * t$ iff t and u have the same base.

(b) $(t * u) * v = t * (u * v)$ iff t, u, and v have the same base. □

Proposition 3.8. If $B = \{b\}$ is a one element set, then $R = \{b^n \mid n \geq 1\}$, and $b^m * b^n = b^{m+n}$. (Therefore, $\underline{R}$ is isomorphic with the additive semigroup of positive integers.) □

4. Injective objects in the variety of monoassociative groupoids

Looking for a convenient class of "injective groupoids" in a variety $\mathcal{V}$ of groupoids we choose as axioms of such a class corresponding properties of free objects in $\mathcal{V}$ that are "near" the statement (1.1). In the case of *Mass*, such statements are Proposition 3.5 and Proposition 3.6, and that is why we give the following definition.

We say that a groupoid $H \in$ *Mass* is *injective* in *Mass*, i.e. it is in *Massin*, iff it satisfies the following conditions:

(i) For any $a \in H$ there is a unique pair $(b, k) \in H \times \mathbf{N}$ such that $a = b^k$ and b is a base in $\underline{H}$. (We say that b is the *base* and k is the *exponent* of a in $\underline{H}$, and write $b = \beta(a)$, $k = \varepsilon(a)$.)

(ii) Let $a \in H$ be not prime in $\underline{H}$.

(ii.1) If $b = \beta(a)$ and $\varepsilon(a) \geq 2$, then

$$a = cd \Rightarrow \beta(c) = \beta(d) = b \ \ \& \ \ \varepsilon(c) + \varepsilon(d) = \varepsilon(a).)$$

(ii.2) If $c, d \in H$ are such that $\beta(c) \neq \beta(d)$, then $\beta(cd) = cd$, and: $cd = c'd' \Rightarrow (c, d) = (c', d')$.

As corollaries of the given definition and Propositions 3.5–3.7, we obtain the following poperties of *Massin*.

Proposition 4.1. The class of free groupoids in *Mass* (shortly: *Massfr*) is a subclass of *Massin*. □

Proposition 4.2. A groupoid $\underline{H} \in$ *Massin* contains only one base element iff $\underline{H}$ is isomorphic to the additive semigroup of positive integers. □