

SEMIGROUPS IN WHICH SOME LEFT IDEAL IS A GROUP

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In this note we give a structural theorem for semigroups containing a left ideal which is a group. This theorem is a generalization of the main results of the papers [2], [3] and [4].

Theorem. Some left ideal of a semigroup S is a group if and only if S is isomorphic to a semigroup $G \times JUP = [G, J, P, \varphi, \xi]$, where

(i) G is a group, P is a partial semigroup¹⁾ (which may be empty), J is a non-empty set and $G \times J \cap P = \emptyset$;

(ii) $\varphi: p \rightarrow p\varphi$ is a homomorphism from P into G and $\xi: p \rightarrow \xi_p$ a homomorphism from P into T_J ²⁾ such that $\xi_p \xi_q$ is a constant if the product pq is not defined in P ;

(iii) the product of two elements of the set $G \times JUP$ is defined by:

$$(1) (x, i)(y, j) = (xy, j), \quad (2) (x, i)p = (xp\varphi, i\xi_p), \quad (3) p(x, i) = (p\varphi x, i),$$

(4) $pq = r$ in $P \Rightarrow pq = r$ in $G \times JUP$, and (5) $pq \in P \Rightarrow pq = (p\varphi q\varphi, i\xi_p \xi_q)$, where $x, y \in G, p, q, r \in P, i, j \in J$.

Proof. If G, J, P, φ and ξ satisfy the conditions (i) and (ii) and if an operation is defined in $G \times JUP$ by (iii), then it can be easily seen that $G \times JUP = [G, J, P; \varphi, \xi]$ is a semigroup; and if $G_i = \{(x, i); x \in G\}$ then $\{G_i; i \in J\}$ is a collection of left ideals which are groups isomorphic to G .

Suppose now that S is a semigroup in which some left ideal G_i is a group. Then G_i is a minimal left ideal and therefore (see, for example, [1]) $\{G_i s; s \in S\}$ is the collection of all minimal left ideals of S .

The element s may not belong to $G_i s$ but we can choose an $r \in G_i s$ such that $r \in G_i, r = G_i s$; thus we may assume $s \in G_i s$ and then $s = us$ for some $u \in G_i$. Let $x, y \in G_i s$ and $x = x_1 s, y = y_1 s$ where $x_1, y_1 \in G_i$; if e is the identity element of G_i then $se \in G_i$ and $se = x_1^{-1} y_1 u z_1^{-1}$ for some $z_1 \in G_i$. From $s = us$ it follows $xz = y$ where $z = z_1 s \in G_i s$. Thus $x G_i s = G_i s$ for every $x \in G_i s$, i. e. $G_i s$ is a right

¹⁾ P is a partial semigroup if a partial binary operation is defined in P such that $p \cdot q r \in P \Leftrightarrow pq \cdot r \in P$ and then $p \cdot qr = pq \cdot r$.

²⁾ T_J is the semigroup of all mappings from J into itself.

simple semigroup. As a minimal left ideal $G_i S$ is also a left simple semigroup, and therefore it is a group. Thus we have proved that every minimal left ideal of S is a group.

Let $\{G_i; i \in J\}$ be the collection of all (different) minimal left ideals of the semigroup S and let $K = \bigcup_{i \in J} G_i$. Then K is a two-sided ideal of S and $\{G_i; i \in J\}$

is a collection of left ideals of K which are groups. Therefore (see, for example, [5] Lemma 2) there is a group G such that K is isomorphic to a semigroup $G \times J$ where the product is defined by (1); and we assume that $K = G \times J$. The subset $P = S \setminus K$ is a partial semigroup and we assume it to be non-empty.

Let $p \in P, i, j \in J$ be arbitrary elements and e the identity element of G . Then

$$p(e, i) = ((p, i)\varphi, i) \in G_i = \{(x, i); x \in G\}^3$$

(because G_i is a left ideal of S) and

$$((p, i)\varphi, i) = p(e, i) = p(e, j)(e, i) = ((p, j)\varphi, j)(e, i) = ((p, j)\varphi, i),$$

i. e. $(p, i)\varphi$ does not depend on i ; and so

$$(6) \quad p(e, i) = (p\varphi, i),$$

where φ is a mapping of P into G . Also

$$(e, i)p = ((i, p)\psi, i\xi_p) \in G \times J^3,$$

for $G \times J$ is a right ideal of S . Then

$$((i, p)\psi, i) = ((i, p)\psi, i\xi_p)(e, i) = (e, i)p(e, i) = (p\varphi, i),$$

i. e. $(i, p)\psi = p\varphi$ for every $i \in J, p \in P$; thus

$$(7) \quad (e, i)p = (p\varphi, i\xi_p)$$

where $\xi: p \rightarrow \xi_p$ is a mapping of P into T_J . From (6) and (7) there follow

$$p(x, i) = p(e, i)(x, i) = (p\varphi, i)(x, i) = (p\varphi x, i)$$

and

$$(x, i)p = (x, i)(e, i)p = (x, i)(p\varphi, i\xi_p) = (xp\varphi, i\xi_p),$$

i. e. the equations (2) and (3) are satisfied.

Let $p, q \in P$. If $pq = r \in P$ then

$$(r\varphi, i\xi_r) = (e, i)r = (e, i)pq = (p\varphi q\varphi, i\xi_p \xi_q),$$

whence follows that φ and ξ are homomorphisms. If $pq = (x, k) \in G \times J$, then, for every $i \in J$,

$$(x, k) = (e, i)(x, k) = (e, i)pq = (p\varphi q\varphi, i\xi_p \xi_q),$$

whence follows the equation (5) and that $\xi_p \xi_q$ is a constant. This completes the proof of Theorem.

³) $(p, i)\varphi, (i, p)\psi \in G, i\xi_p \in J$

Some notes.

1. For an arbitrary group G , a non-empty set J and a partial semigroup P , there exists a $[G, J, P; \varphi, \xi]$ -semigroup. For example, we can put $p\varphi = e$, $i\xi_p = k$ where k is a fixed element of J ; if $P = \emptyset$ then $[G, J, P; \varphi, \xi] = G \times J$.

2. Two semigroups $[G, J, P; \varphi, \xi]$ and $[G^*, J^*, P^*; \varphi^*, \xi^*]$ are isomorphic if and only if there exist an isomorphism α from G onto G^* , an isomorphism β from P onto P^* and a one-to-one mapping γ from J onto J^* , such that $\beta\varphi^* = \varphi\alpha$ and $\xi_p\gamma = \gamma\xi^*_{p\beta}$.

3. $G \times J$ is the minimal two-sided ideal and also the unique minimal right ideal of a $[G, J, P; \varphi, \xi]$ -semigroup; therefore it is the set of all right zeroid elements of the semigroup [1]. From this follows that if a semigroup contains a left ideal and a right ideal which are groups then the semigroup contains two-sided zeroid elements.

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ПОЛУГРУПИ ВО КОЈ НЕКОЈ ИДЕАЛ Е ГРУПА

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Резиме

Во работата е докажан следниот резултат. Некој лев идеал на полугрупата S е група ако и само ако S е изоморфна со некоја полугрупа од облик $G \times JUP = [G, J, P; \varphi, \xi]$ каде: (i) G е група, J е напразно множество, а P е делумична полугрупа дисјунктна со множеството $G \times J$ (P може да биде и празно множество); (ii) $\varphi: p \rightarrow p\varphi$ е хомоморфизам од P во G , а $\xi: p \rightarrow \xi_p$ е хомоморфизам од P во T_J таков да $\xi_p \xi_q$ е константна ако производот pq не е определен во P ; (iii) операцијата во множеството $G \times JUP$ е определена со (1)—(5).

При тоа под делумична полугрупа подразбираме алгебарска структура со една делумична бинарна операција која е асоцијативна, т.е. $pq \cdot r \in P \Leftrightarrow p \cdot qr \in P$ и при тоа $pqr = p \cdot qr$. Со T_J ја означуваме полугрупата од сите трансформации на множеството J , т.е. пресликувања од J во J ; операцијата во таа полугрупа е обичното множење на пресликувања.