GLASNIK MAT. - FIZ. I ASTR.
Tom 18. — No. 3 — 1963.

# ON SEMIGROUPS $S$ IN WHICH EACH PROPER SUBSET $Sx$ IS A GROUP

*G'orgi Čupona, Skopje*

A semigroup $S$ is called *$F_{ep}$-semigroup* if there is a proper subset $Sx$ of $S$ (i. e. $Sx \neq S$) and if every such subset is a subgroup. Clearly, every $F_e$-semigroup [3] (and, therefore, every $F$-semigroup [2]) is an $F_{ep}$-semigroup too. The purpose of this paper is to describe the structure of $F_{ep}$-semigroups.

**1.** Let $S$ be an $F_{ep}$-semigroup and let us put:

$$P = \{x ; Sx = S\},\ Q = \{x ; x \in Sx \neq S\},\ B = \{x ; x \notin Sx \neq S\}.$$

Then we have: $S = Q \cup B \cup P$ and $Q \cap B = Q \cap P = B \cap P = 0$. The set $Q$ is non-empty; if $Q$ is not a group then it is an $F_{ep}$-semigroup; at least one of the sets $B, P$ is non-empty, when $Q$ is a group.

**1.1.** There is a group $G$ and a set $A$ such that $Q$ is isomorphic with the semigroup $G \times A$ defined by:

$$(x, \alpha)(y, \beta) = (xy, \beta) \quad (x, y \in G ; \alpha, \beta \in A). \tag{1}$$

Therefore, we may suppose that $Q = G \times A$. If $G_\alpha = \{(x, \alpha) ; x \in G\}$, then $\{G_\alpha ; \alpha \in A\}$ is a collection of all the left ideals of $S$ which are subgroups; and these groups are isomorphic with $G$.

**1.2.** Let us suppose that the set $B$ is non-empty. Then, there is a mapping $(\varphi(b), \xi_b)$ of $B$ in $Q\ (= G \times A)$ such that the following statements are satisfied:

$$(x, \alpha)\, b = (x\, \varphi(b), \xi_b)\ ; \tag{2}$$

$$b\,(x, \alpha) = (\varphi(b)\, x, \alpha)\ ; \qquad (x \in G ; \alpha \in A ; b, c \in B) \tag{3}$$

$$b\, c = (\varphi(b)\, \varphi(c), \xi_c). \tag{4}$$

**1.3.** If the set $P$ is non-empty then it is a left simple subsemigroup (i. e. $Pp = P$, for each $p \in P$) and moreover, $xp \in P \Leftrightarrow x \in P$. Then there is a homomorphism $\psi$ of $P$ in $G$ and a mapping $\tau : (\alpha, p) \to \alpha p$ of $A \times P$ in $A$ such that

$$\alpha(p\, q) = (\alpha\, p)\, q\ ; \tag{5}$$

$$A\, p = A\ ; \tag{6}$$

$$(x, \alpha)\, p = (x\, \psi(p), \alpha\, p)\ ; \quad (x \in G ; \alpha \in A ; b \in B ; p, q \in P) \tag{7}$$

$$p\,(x, \alpha) = (\psi(p)\, x, \alpha)\ ; \tag{8}$$

$$p\, b = (\psi(p)\, \varphi(b), \xi_b). \tag{9}$$

**1.4.** Let $B \neq 0, P \neq 0$ and let us put

$$a \in p[b] \Leftrightarrow a p = b. \tag{10}$$

Then we have

For each $p \in P$ and $b \in B$, $p[b]$ is a non-empty subset of $B$; (11)

$$p[q[b]] = pq[b] \text{ (where } p[q[b]] = \bigcup_{c \in p[b]} p[c]); \tag{12}$$

$$b \neq c \Rightarrow p[b] \cap p[c] = 0; \tag{13}$$

$$a \in p[b] \Rightarrow \varphi(b) = \varphi(a)\psi(p) \ \& \ \xi_b = \xi_a p. \tag{14}$$

If $ap \notin B$, then we have

$$a p = (\varphi(a)\psi(p), \xi_a p). \tag{15}$$

If there is an idempotent in $P$ (or if the set $B$ is finite) then in each subset $p[b]$ there is only one element, and $\{p[b]; p \in P\}$ is a collection of permutations of the set $B$. Therefore (in that case) the equation $xp = b$ is uniquely solvable in $B$, for every $p \in P$, $b \in B$.

**2.** Let $G$ be a group, $P$ a left simple semigroup and $A (\neq 0)$, $B$ two sets such that $G \times A, B$ and $P$ are mutually disjoint. Some of the sets $P, B$ may be empty, but at least one of them is non-empty if $A$ contains only one element. Let $(\varphi(b), \xi_b)$ be a mapping of $B$ into $G \times A$, $\psi$ a homomorphism of $P$ in $G$ and $\tau: (a, p) \to ap$ a mapping of $A \times P$ into $A$, such that the statements (5) and (6) are satisfied. If the sets $B$ and $P$ are non-empty, then let $\Gamma = \{p[b]; p \in P, b \in B\}$ be a collection of subsets of $B$ such that the statements (11)—(14) are satisfied.

**2.1.** Let $S = S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma) = G \times A \cup B \cup P$, and let the product of two elements of $S$ be defined by (2)—(4), (7)—(10) and by (15) if $a \notin p[B]$; if $p, q \in P$ then $pq = s$ in $P \Leftrightarrow pq = s$ in $S$. Then $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ is an $F_{ep}$-semigroup.

**2.2.** For an arbitrary group $G$, left simple semigroup $P$ and sets $A (\neq 0)$, $B$ there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ -semigroup. Namely it is sufficient to put $\varphi(b) = u, \xi_b = \gamma, \psi(p) = e, ap = a$, for every $b \in B, p \in P, a \in A$, where $u \in G$ and $\gamma \in A$ are fixed elements, and $e$ is the identity of the group $G$.

For an arbitrary group $G$, a left simple semigroup without idempotents $P$, and a non-empty set $A$, there is an $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$-semigroup such that (for each $p \in P$ and $b \in B$) $p[b]$ contains only one element and $p[B] = \bigcup_{b \in B} p[b]$ is a proper subset of $B$.

I do not know whether there are semigroups $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$ such that some of the subsets $p[b]$ contain more than one element.

**3.** From **1.** and **2.** it follows:

T h e o r e m. *Every $F_{ep}$-semigroup is isomorphic with some $S(G, A, B, P; \varphi, \xi, \psi, \tau; \Gamma)$-semigroup.*

**4.** A semigroup $S$ is called an $F_{em}$-semigroup if there is a proper minimal left ideal and if all such ideals are subgroups. Clearly, every $F_{ep}$-semigroup is an $F_{em}$-semigroup too.

**4.1.** Let $S$ be an $F_{em}$-semigroup such that $SS = Q$, where $Q\,(= G \times A)$ is the class sum of all the minimal left ideals; and let the set $B = S \setminus Q$ be non-empty. Then there are mappings $\varphi(b)$ of $B$ in G and $\xi(a, b)$ of $A \times B$ in $A$ such that the following statements are valid:

$$b\,(x, a) = (\varphi(b)\,x, a)\,; \tag{16}$$

$$(x, a)\,b = (x\,\varphi(b), \xi(a, b))\,; \tag{17}$$

$$a\,b = (\varphi(a)\,\varphi(b), \xi(\xi(a, a), b))\,; \; (x \in G, b \in B, a, \beta \in A) \tag{18}$$

$$\xi(\xi(a, a), b) = \xi(\xi(\beta, a), b)\,. \tag{19}$$

Therefore $\xi(\xi(a, a), b)$ is a mapping of $B \times B$ into $A$.

**4.2.** Let $G$ be a group and $A\,(\neq 0), B$ two sets. Let $\varphi$ be a mapping of $B$ into G and $\xi(a, b)$ of $A \times B$ into $B$, such that (19) holds. If the product of two elements of $S = G \times A \cup B$ is defined by (1), (3), (17) and (18), then $S$ becomes an $F_{em}$-semigroup.

**4.3.** Let $G$ be a group and $u$ a fixed element of $G$, and let $A = \{a, \beta, \gamma\}, B = \{b\}$. If we put $\xi(a, b) = \xi(\beta, b) = a, \xi(\gamma, b) = \beta, \varphi(b) = u$, then (19) is satisfied, and therefore $S = G \times A \cup \{b\}$ is an $F_{em}$-semigroup. But $S$ is not an $F_{ep}$-semigroup because $Sb = G \times A \neq S$ is not a subgroup of $S$.

REFERENCES:

[1] *G. Pollak — L. Redei*, Die Halbgrupnen, deren alle echte Teilhalbgruppen Gruppen sind, Publicationes Math. Debrecen **6** (1959), 126—130,

[2] *Š. Schwarz*, Semigroups in which every proper subideal is a group, Acta Sci. Math. Szeged **21** (1960), 125—134,

[3] *R. Hrmová*, Pologrupy, v ktorých každý l' avý vlastný ideál je grupou, Mat.-Fyz. Časop. **11** (1961), 75—80 (RŽ Mat, 1962, 2A 238),

[4] *G. Čupona*, On completely simple semigroups, Glasnik Mat.-Fiz. Astr. **18** (1963), 159—164.

## O POLUGRUPAMA $S$ U KOJIMA JE SVAKI PRAVI PODSKUP OBLIKA $Sx$ PODGRUPA

G'orgi Čupona, Skopje

*Sadržaj*

Nazovimo $F_{ep}$-*polugrupom* svaku polugrupu $S$ sa osobinom da su svi pravi podskupovi oblika $S x$ grupe.

Ako je $S$ $F_{ep}$-polugrupa i ako stavimo $Q = \{x\,;\, x \in Sx \neq S\}$, $B = \{x\,;\, x \notin Sx \neq S\}, P = \{x\,;\, Sx = S\}$, dobijamo $S = Q \cup B \cup P$, pri čemu su skupovi $Q$, $B$ i $P$ međusobno disjunktni. Daćemo sada opis strukture $F_{ep}$-polugrupa.

1°. $Q$ je levi ideal i on je (kao polugrupa) izomorfan sa nekom polugrupom oblika $G \times A$, gde je $G$ grupa, $A$ neki skup i operacija određena sa (1). Zato možemo smatrati da je $Q = G \times A$.

2°. Postoji preslikavanje $b \to (\varphi(b), \xi_b)$ skupa $B$ u $Q$ koje zadovoljava jednačine (2)—(4).

3°. $P$ je potpolugrupa polugrupe $S$. Pri tome imamo $Pp = P$, $Qp = Q$ i $B \subseteq Bp \subseteq B \cup Q$ za svako $p \in P$. Postoji $F_{ep}$-polugrupa gde je $B \subset Bp$ i takva gde je $B = Bp \subset B \cup Q$.

4°. Postoji homomorfizam $\psi$ polugrupe $P$ u grupi $G$ i preslikavanje $(a, p) \to ap$ skupa $A \times P$ u $A$, koji zadovoljavaju jednačine (5)—(9). Ako za $a \in B, p \in P$ imamo $ap \in Q$ onda je tačna i jednačina (15).

5°. Prema 4°, za svaki par $p \in P, b \in B$ postoji bar jedno rješenje jednačine $xp = b$ koje se nalazi u skupu $B$. Ako je $p[b]$ skup svih takvih rešenja onda familija $\Gamma = \{p[b]; p \in P, b \in B\}$ tih podskupova skupa $B$ zadovoljava uslove (12)—(14).

Ako postoji idempotentan elemenat u polugrupi $P$, ili ako je konačan skup $B$, onda $p[b]$ sadrži samo jedan elemenat, t. j. jednačina $xp = b$ je jednoznačno rešiva. Pri tome imamo $Bp = B$ i $p[B] = \bigcup_{b \in B} p[b] = B$. Postoji $F_{ep}$-polugrupa u kojoj je (za svako $p \in P$) $p[B]$ pravi podskup skupa $B$, t. j. bar jedan elemenat skupa $B$ nije rešenje jednačine oblika $xp = b$. Nije poznat primer $F_{ep}$-polugrupe u kojoj neka takva jednačina ima više od jednog rešenja.

Jasno je da neki od skupova $Q, B$ i $P$ može biti prazan. Pri tome imamo: $(i)$ $G \times A = 0 \Rightarrow B = 0, P = S \neq 0$, a u ovom slučaju je $Sx = S$, za svako $x \in S$; $(ii)$ $B = 0$ ili $P = 0 \Rightarrow \Gamma = 0$.

6°. Neka je $G$ grupa, $P$ polugrupa sa osobinom $Pp = P$ za svako $p \in P$ i neka su $B$ i $A$ dva skupa; pri tome pretpostavljamo da su međusobno disjunktni skupovi $G \times A, P, B$ i dopuštamo da je neki od njih prazan, ali treba da je zadovoljen gore pomenuti uslov $(i)$. Neka je $b \to (\varphi(b), \xi_b)$ preslikavanje skupa $B$ u $G \times A, \psi$ homomorfizam polugrupe $P$ u grupi $G$ i $(a, p) \to ap$ preslikavanje skupa $A \times P$ u $A$ i neka su pri tome zadovoljene jednačine (5)—(6). Neka je $\Gamma = \{p[b]; p \in P. b \in B\}$ familija podskupova skupa $B$ koja zadovoljava uslove (12)—(14).

Skup $S = G \times A \cup B \cup P$ postaje $F_{ep}$-polugrupa ako se operacija odredi sa: $(i)$ $p_1 p_2 = p$ u $P \Leftrightarrow p_1 p_2 = p$ u $S$; $(ii)$ jednačinama (1)—(4), (7)—(9) i (10) ili (15) u zavisnosti od toga da li je $a \in p[B]$ ili $a \notin p[B]$.

Prema tome, sa 1°—5° je potpuno opisana struktura klase $F_{ep}$-polugrupa.

$F_{ep}$-polugrupe čine potklasu klase polugrupa u kojima su svi minimalni levi ideali i podgrupe. U radu je dat primer polugrupe koja nije $F_{ep}$-polugrupa iako su svi minimalni levi ideali grupe.

(Primljeno 13. IX 1963.)