ON A CLASS OF n-SEMIGROUPS

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1. **Introduction.** In this article we generalize the notion of λ-semigroups from binary to $(n+1)$ -ary case, establishing some properties for the generalized algebraic systems obtained in that way.

An *n-semigoup* $S([\ldots])$ is a non-empty set S with an $(n+1)$ -ary operation $[\ldots]$ such that for every $x_j \in S$, $j = 0,1,\ldots, 2n$ and every $i = 1,2,\ldots, n$ the following holds:

$$[[x_o\, x_1 \ldots x_n]\, x_{n+1} \ldots x_{2n}] = [x_o \ldots x_{i-1}\, [x_i \ldots x_{i+n}]\, x_{i+n+1} \ldots x_{2n}]$$

A non-empty subset Q of an n-semigroup S is said to be *n-subsemigroup* (*left ideal*) of S if $[Q \ldots Q] \subseteq Q$ ($[S \ldots SQ] \subseteq Q$). An n-semigroup S is said to be *λ-n-semigroup* if and only if each n-subsemigroup of S is a left ideal in S.

2. **Cyclic n-semigroups.** We shall start with some remarks about the cyclic n-semigroups. Let S be an n-semigroup and, for $a \in S$, let us put:

$$< a > = \{a, a^{n+1}, \ldots, a^{kn+1}, \ldots\},$$

where: $a^{n+1} = [a \ldots a]$, $a^{(k+1)n+1} = [a \ldots a\, (a^{kn+1})]$, $k > 0$. Obviously, $< a >$ is an n-subsemigroup of S; $< a >$ is called a *cyclic n-subsemigroup* of S and, if $S = < a >$ for some $a \in S$, then S is called a *cyclic n-semigroup*.

Examples. Let N be the set of all positive integers and $N^o = N \cup \{0\}$.

1. If we put

$$[a_o\, a_1 \ldots a_n] = 1 + \sum_{j=0}^{n} a_j,\ a_j \in N^o,$$

them N^o becames a cyclic n-semigroup such that $N^o = < 0 >$. It is easily seen that every cyclic n-semigroup of infinite order is isomorphic to N^o.

2. Let $I_{rm} = \{0,1,\ldots, r+m-1\}$, $r \in N^o$, $m \in N$. If we put

$$b = [a_o\, a_1 \ldots a_n],\ a_j,\ b \in I_{rm},$$

where $b = 1 + \sum_{j=0}^{n} a_j$ if $\sum_{j=0}^{n} a_j < r+m-1$ and $b = r+i$, $i \equiv 1 - r + \sum_{j=0}^{n} a_j \pmod m$, otherwise, then I_{rm} will turn out to be an n-semigroup such that $I_{rm} = < 0 >$. If $S = < a >$ is a cyclic n-semigroup of finite order k, then there exist $r \in N^o$, $m \in N$, $r+m=k$, such that S is isomorphic to I_{rm}. Nalemy, if s is the least positive integer with the property $a^{tn+1} = a^{su+1}$, $r \in N^o$, $r < s$, then we easily sce that $a^{(i+r)n+1} = a^{(j+r)n+1}$ if and only if $i \equiv j \pmod m$ where $m = s - r$, and all the elements in S are: a, $a^{n+1}, \ldots, a^{(r+m-1)n+1}$; then $f(i) = a^{tn+1}$ is an isomorphism from I_{rm} to S. We call r and m *index* and *period*, respectively, for every n-semigroup which is isomorphic with I_{rm}.

Lemma 1. *If $S = < a >$ is a cyclic n-semigyoup with index r and period m, then $K = \{a^{rn+1}, \ldots, a^{(r+m-1)n+1}\}$ is an n-subgroup of $S \cdot K$ sontains an idempotent if and only if m and n are relatively prime and in that case the idempotent element is unique — it is in fact a neutral element in K.*

Proof. We call K an *n-subgroup* of S if for every $a_j \in K$, $j = 0,1,\ldots, n$ and every $k = 0,1,\ldots, n$ there exists $x_k \in K$ such that $[a_o \ldots a_{k-1}\, x_k\, a_{k+1} \ldots \ldots a_n] = a_k$. Let $a_j \in a^{(r+k_j)n+1}$, $b = a^{(r+k)n+1}$. If $k < 1 + (n-1)\,r + \sum_{j=1}^{n} k_j$ we can choose $p \in N^o$ such that $pm - s = 1 + (n-1)\ r + \sum_{j=1}^{n} k_j - k$, $0 \leqslant s < m$, and if $k \geqslant 1 + (n-1)\,r + \sum_{j=1}^{n} k_j$, we shall put $s = k - 1 - (n-1)\,r - \sum_{j=1}^{n} k_j$. If $x = a^{(r+s)n+1}$, then $[a_1 \ldots a_{j-1}\ xa_j \ldots a_n] = b$ for all $j = 1,2,\ldots, n$ and so K is an n-subgroup of S. If $b^{n+1} = b$, $b \in S$, we call b an *idempotent* in S. Let $b = a^{(r+q)n+1}$ be an idempotent in K; $b^{n+1} = a^{(r+(r+q)n+q+1)n+1} = a^{(r+q)n+1} = b$ implies that $(r+q)\,n+1 \equiv 0 \pmod m$ which means that m and n are relatively prime. Conversely, if m and n are relatively prime, from $um + vn = 1$, u and v integers, it follows that $(r+q)\,n+1 \equiv 0 \pmod m$ if we take q from $\{0,1,\ldots, m-1\}$ such that $r + q + v \equiv 0 \pmod m$. Then $b = a^{(r+q)n+1}$ will be an idempotent in K. If $b' = a^{(r+q')n+1}$ is an idempotent, then $(r+q')\,n + 1 \equiv 0 \pmod m$, and so, $(q-q')\,n \equiv 0 \pmod m$, i. e. $q - q' \equiv 0 \pmod m$ because m and

n are relatively prime. Finally, since $0 \leqslant q, q' < m$, we get $q = q'$, i. e. $b = b'$. If $x \in K$, from $(r+q)\, n + 1 \equiv 0 \pmod{m}$ it follows that $[\underbrace{b \dots b}_{i} xb \dots b] = x$ for every $i = 0, 1, \dots, n$, and therefore the idempotent b is in fact a neutral element in K.

3. **Some general properties of λ-n-semigroups.** From now on we suppose S to be a λ-n-semigroup. It is obvious that.

Lemma 2. *Every n-subsemigroup of a λ-n-semigroup is λ-n-semigroup.*

Lemma 3. *An n-semigroup S is a λ-n-semigroup if and only if* $[S \dots Sa] \subseteq \langle a \rangle$ *for all* $a \in S$.

Proof. If S is a λ-n-semigroup and $a \in S$, then

$[S \dots Sa] \subseteq S \dots S \langle \langle a \rangle] \subseteq \langle a \rangle]$.

Conversely, if Q is an n-subsemigroup of S and $a \in Q$, ihen $[S \dots Sa] \subseteq \langle a \rangle \subseteq Q$ which implies that $[S \dots SQ] \subseteq Q$, i. e. Q ts a left ideal in S and so S is a λ-n-semigroup.

Lemma 4. *Let* $x, y \in S$. *Then* $[x \dots xy] = y$ *if and only if* y *is an idempotent.*

Proof. Let y be an idempotent. Then $\langle y \rangle = \{y\}$ and for every $x_j \in S$, $j = 1, 2, \dots, n$, by Lemma 3, $[x_1 \dots x_n y] \in [S \dots S \langle y \rangle] \subseteq \langle y \rangle = \{y\} \subseteq \{[y \dots y]\} \subseteq [S \dots S \langle y \rangle]$, so,

$$(1) \qquad [x_1 \dots x_n y] = y,$$

and the assertion follows from (1). Conversely, let $[x \dots xy] = y$. Then $[y \dots yx] \in \langle x \rangle$ implies $[y \dots yx] = x^{kn+1}$, $k \in N^o$. For $k = 0$, i. e. if $[y \dots yx] = x$, we have $y = [x \dots xy] = [[y \dots yx]\, x \dots xy] = [y \dots y\, [x \dots xy]] = y^{n+1}$, so, y is an idempotent. Let $k > 0$ and let us write $x^{rn} y$ instead of $[(x^{(r-1)n+1})\, x \dots xy]$. Now,

$$x^{kn}\, y = x^{kn}\, (x^n\, y) = [(x^{kn+1})\, x \dots xy] =$$
$$= [(y^n\, x)\, x \dots xy] = y^n\, (x^n\, y) = y^n\, y = y^{n+1},$$

and,

$$y^{n+1} = x^{kn}\, y = x^{(k-1)n}\, (x^n\, y) = x^{(k-1)n}\, y = \dots = x^n\, y = y,$$

and again y is an idempotent.

Lemma 5. *For every* $a \in S$, $\langle a \rangle$ *has finite order. If* $\langle a \rangle$ *contains an idempotent, then* $|\langle a \rangle| \leqslant 3$. *if* $\langle a \rangle$ *does not contain any idempotent, then the index of* $\langle a \rangle$ *is not greatei than* 2 *and its n-subgroup* K_a *is generated by every of its elements. The set E of all the idempotents of S is a right zero n-subsemigroup of S.*

Proof. If the order of $<a>$ is infinite, then $T=<a^{n+1}>$ will be an n-subsemigroup of $<a>$ which does not contain the element a^{2n+1}; by Lemma 2 $<a>$ is a λ-n-semigroup and $a^{2n+1}=[a\dots a\,(a^{n+1})]\in T$ which is a contradiction. Let $<a>=\{a,\ a^{n+1},\dots,\ a^{kn+1}\}$ has an idempotent e_a, which by Lemma 1 is unique. Let K_a be the corresponding n-subgroup of $<a>$ defined as in Lemma 1. If $x\in K_a$, taking into account (1) and the fact that e_a is neutral element in K_a, we get $x=[xe_a\dots e_a]=e_a$, and so $K_a=\{e_a\}$. Now, if the order of $<a>$ is greater than 3, i. e. if $k>2$, then $Q=\{a^{n+1},\ a^{3n+1},\ \dots,\ a^{kn+1}\}$, where $a^{kn+1}=e_a$, is an n-snbsemigroup of $<a>$ and $a^{2n+1}\in[<a>\dots<a>\ Q]\subseteq Q$ which is, again, a contradiction. So, $k\leqslant 2$, i. e. $|<a>|\leqslant 3$. Let $<a>$ does not contain any idempotent and let the index of $<a>$ be greater than 2. Then the above applies with the only difference that this time a^{kn+1} in Q is not an idempotent. Now, let $a^{sn+1}\in K_a$. By Lemma 2 we have that

$$a^{s+1\,(n+1)\,n+1}=[a\dots a\ (a^{sn+1})]\in[<a>\dots<a>\ <a^{sn+1}>]\subseteq<a^{sn+1}>.$$

and then $$<a^{(s+1)n+1}>\subseteq<a^{sn+1}>.$$

Continuing in the same way as above, we have that $a^{(s+2)\,n+1}$ belongs to $<a^{(s+1)\,n+1}>\subseteq<a^{sn+1}>$ and so on. So, $<a^{sn+1}>$ contains the all elements of K_a i. e. K_a is generated by every of its elements. The last statement of the lemma follows from (1).

Let J be an index set for the family of all n-subgroups K_a of S where K_a is defined as above, i. e. it is the periodic part of the cyclic n-subsemigroup $<a>$. Let $K=\cup\{K_j\,|\,j\in J\}$. By Lemma 2 all K_j are jeft ideals in S which implies that K is an left ideal in S, too and therefore K is an n-semigroup which is a union of n-groups. Since K_j is a left ideal in S, then $[SK_j\dots K_j]\subseteq K_j$ and so, all K_j are feft ideals in S in the sense ot the definition in [3]. By the dual of Theorem 1 of [3], all K_j are isomorphic with $K_j\times J$ for a fixed K_j where in $K_j\times J$ the $(n+1)$ — ary operation is defined as follows:

$$[(x_0,\ j_0)\ (x_1,\ j_1)\dots(x_n,\ j_n)]=([x_0\,x_1\ \dots x_n],\ j_n),$$

$x_k\in K_j$, $j_k\in J$, i. . K is isomorphic with the direct product of an n-group K_j and a right zero n-semigropup J. This implies that any two cyclic n-subsemigroups $<a>$ and $<b>$ of S with the same index are isomorphic, since their covresponding n-subgroups K_a and K_b are isomorphic. If G is an n-subgroup of S and if $c\in G$, then there exist some $d_j\in G$, $j=$ $=1,\ 2,\dots n$ such that $c=[d_1\dots d_n\ (c^{2n+1})]$ and then, by Lemma 5, $c\in K_c\subseteq K$. So, every n-subgroup of S is contained in K.

If S contains an idempotent e, then thare exists a cyclic n-subsemigroup $<a>$ of S such that $e\in<a>$ (for example, $e\in<e>$). Then by

Lemma 5 we have that $K_a = \{e\}$. Now, since all K_j are isomorphic to each other, we get that every cyclic n-subsemigroup of S contains an idempotent.

In summary we obtain the following

Theorem 1. *Let S be a λ-n-semigroup. Then:*

(i) all cyclic n-subsemigroups of S with the same index are isomorphic to each other,

(ii) the union $K = U\{K_j \mid j \in J\}$ of the all cyclic n-subgroups of S is an n-subsemigroup of S which is isomorphic to the direct product of an n-subgroup of S and a right zero n-semigroup, and every n-subgroup of S is contained in K,

(iii) if S contains an idempotent, then every cyclic n-subsemigroup of S contains unique idempotent.

If S is a λ-n-semigroup, then by Theorem 1 we have trat, either S does not contain any idempotent, either every cyclic n-subsemigroup of S contains an idempotent. In the last case we have a similar situation as in the λ-semigroups studied in [2]. In the next part we shall establish some more (general) properties of the λ-n-semigroups in which every cyclic n-subsemigroup contains an idempotent; if every cyclic n-subsemigroup of a λ-n-semigroup S contains an idempotent, then we call S a $\bar{\lambda}$-n-semigroup.

4. Decompozition of a $\bar{\lambda}$-n-semigroup into union of unipotent $\bar{\lambda}$-n-semigroups. Troughout this part S will be a $\bar{\lambda}$-n-semigroup.

Lemma 6. If $[x_1 \dots x_n y] = y$, *then every of the following assertions:*

(i) $x_j = z^{n+1}$ *for some* $j = 1, 2, \dots, n$, $z \in S$,

ii)) $[x_j \dots x_n x_1 \dots x_j] \neq x_j$ *for some* $j = 1, 2, \dots, n$,

implies that y is an idempotent.

Proof. If (i) holds with $z = x_j$, i. e. if some x_j is an idempotent, then f $j = 1$, $y = [x_1 \dots x_n y] = [(x_1^{n+1}) x_2 \dots x_n y] = x_1^n [x_1 \dots x_n y] = x_1^n \acute{y}$ and by Lemma 4 y is an idempotent. If $j > 1$, then

$$y = [x_1 \dots x_{j-1} (x_j^{n+1}) x_{j+1} \dots x_n y] =$$
$$= [[x_1 \dots x_{j-1} x_j \dots x_j] x_j \dots x_n y].$$

By (1), $[x_1 \dots x_{j-1} x_j \dots x_j] = x_j$, so $y = [x_j x_j \dots x_n y]$ with x_j an idempotent and the previous applies. If $x_j = z^{n+1}$, $z \in S$, then

$$y = [x_1 \dots x_{j-1} (z^{n+1}) x_{j+1} \dots x_n y]$$
$$= [x_1 \dots x_{j-1} z (z^n x_{j+1}) x_{j+2} \dots x_n y].$$

If $z^n x_{j+1} = e_{j+1}$, where e_{j+1} is the idempotent in $< x_{j+1} >$ then we have the case already considered; the same is if $z^n x_{j+1} = x_{j+1}$, since in this case, by Lemma 4, x_{j+1} will be an idempotent. If $z^n x_{j+1} = x_{j+1}^{n+1}$, then

$$y = [x_1 \dots x_{j-1} z x_{j+1} (x_j{}^n{}_{+1} x_{j+2}) x_{j+3} \dots x_n y].$$

Here we can repeat the previous considerations with x_{j+1} instead of z and x_{j+2} instead of x_{j+1}. Continuing in that way, or, at some part of that chain we shall conclude that y is an idempotent, or, at the end we shall come to

$$y = [x_1 \dots z x_{j+1} \dots x_{n-1} (x_n{}^n y)].$$

Here, again each of $x_n{}^n y = e_y$ and $x_n{}^n y = y$ imples that y is an idepotent. Finally, if $x_n{}^n y = y^{n+1}$, then $y = [x_1 \dots z \dots x_{n-1} (y^{n+1})] \in < y^{n+1} > = \{y^{n+1}, e_y\}$ because $|< y >| \leqslant 3$, and in both cases y is an idempotent.

If (ii) holds, then any of the following two cases: a) $[x_j \dots x_n x_1 \dots x_j] = x_j{}^{n+1}$ and b) $[x_j \dots x_n x_1 \dots x_j] = x_j{}^{2n+1} (= e_j)$ implies that (i) is satisfied. Namely, when a) is true,

$$\begin{aligned} y = [x_1 \dots x_n y] &= [x_1 \dots x_n [x_1 \dots x_n y]] = \\ &= [x_1 \dots x_{j-1} [x_j \dots x_n x_1 \dots x_j] x_{j+1} \dots x_n y] = \\ &= [x_1 \dots x_{j-1} (x_j{}^{n+1}) x_{j+1} \dots x_n y, \end{aligned}$$

and when b) is true, $y = [x_1 \dots x_{j-1} e_j x_{j+1} \dots x_n y]$ with e_j an idempotent.

Lemma 7. Let $[x_1 \dots x_n y] = y$. *If y is not an idempotent, then: (i)* $x_j \neq e_j$, $x_j{}^{n+1} = e_j$, $j = 1\ 2, \dots, n$; (ii) $x_j = z_j{}^{n+1}$ *for no* $z_j \in S$, $j = 1, 2, \dots, n$; (iii) *for every* $j = 1, 2, \dots, n$ $[x_j \dots x_n x_1 \dots x_j] = x_j$; (iv) $y^{n+1} = e_y$, $y = z^{n+1}$ *for no* $z \in S$.

Proof. The statements (ii), (iii) and the first part of (i are consequences of Lemma 6. Let us prove that $y^{n+1} = e_y$; *by* Lemma 5 $[y x_1 \dots x_n] \in \{x_n, x_n{}^{n+1}, x_n{}^{2n+1} = e_n\}$. If $[y x_1 \dots x_n] = x_n$ then $[x_1 \dots x_{n-1} y x_1] = x_1$ as in the contray, by lemma 6, x_1 will be an idempotent which the first part of (i). Now,

$$\begin{aligned} y = [x_1 \dots x_n y] &= [[x_1 \dots x_{n-1} y x_1] x_2 \dots x_n y] = \\ &= [x_1 \dots x_{n-1} y [x_1 \dots x_n y]] = [x_1 \dots x_{n-1} y y] = \\ &= [x_1 \dots x_{n-1} [x_1 \dots x_n y] y] = [[x_1 \dots x_{n-1} x_1 x_2] x_3 \dots x_n y y]. \end{aligned}$$

Let $[x_1 \dots x_{n-1} x_1 x_2] = t$; by Lemma 6, from $[t x_3 \dots x_n y y] = y$, if $t = x_2{}^{n+1}$ or $t = e_2$, it would follow that y is an idempotent. So, it must be $[x_1 \dots x_{n-1} x_1 x_2] = x_2$ in which case $[x_2 \dots x_n y y] \dots] = y$. Continuing in in the same way we shall obtain $y = [x_n y \dots y]$ and then,

$$y = [x_n y \dots y] = [[y x_1 \dots x_n] y \dots y] = [y [x_1 \dots x_n y] y \dots y] = y^{n+1}.$$

This shows that it can not happen to be $[y x_1 \dots x_n] = x_n$. If $[y x_1 \dots x_n] = e_n$ then

$$\begin{aligned} y^{n+1} &= [y [x_1 \dots x_n y]\ [x_1 \dots x_n y] \dots [x_1 \dots x_n y]] = \\ &= [[y x_1 \dots x_n]\ [y x_1 \dots x_n] \dots [y x_1 \dots x_n] y] = e_n{}^n y, \end{aligned}$$

and

$$e_n^{\ n}(y^{n+1}) = e_n^{\ n}(e_n^{\ n} y) = [(e_n^{\ n+1})\, e_n \dots e_n\, y] = e_n^{\ n}\, y = y^{n+1},$$

and by Lemma 4, y^{n+1} is an idempotent; the idempotent in $< y >$ is unique and $y^{n+1} \in < y >$, so $y^{n+1} = e_y$. Finally, if $[yx_1 \dots x_n] = x_n^{\ n+1}$, then

$$x_n^{\ n+1} = [yx_1 \dots x_n] = [[x_1 \dots x_n\, y]\, x_1 \dots x_n] =$$

$$= [x_1 \dots x_n\, [yx_1 \dots x_n]] = [x_1 \dots x_n\, (x_n^{\ n+1})] = [[x_1 \dots x_n\, x_n]\, x_n \dots x_n].$$

Now, $[x_1 \dots x_n\, x_n] = e_n$ or $x_n^{\ n+1}$ implies $x_n^{\ n+1} = e_n$ and we have the previous situation. If $[x_1 \dots x_n\, x_n] = x_n$, then

$$y = [x_1 \dots x_{n-1}\, [x_1 \dots x_n\, x_n] y] = [[x_1 \dots x_{n-1}\, x_1\, x_2]\, x_3 \dots x_n\, x_n\, y].$$

$[x_1 \dots x_{n-1}\, x_1\, x_2] \neq x_2$ implies $y = [ux_3 \dots x_n\, x_n\, y]$ with $u = y^{n+1}$ which, by Lemma 6, implies that y is an idempotent; so, we must take $[x_1 \dots x_{n-1}$ $x_1\, x_2] = x_2$ and then,

$$y = [x_2 \dots x_n\, x_n\, y] = [x_2 \dots x_{n-1}\, [x_n\, x_2 \dots x_n\, x_n]\, x_n\, y].$$

From $y = [x_2 \dots x_n\, x_n\, y]$, by (iii) of this Lemma, it follows that $x_n =$ $= [x_n\, x_2 \dots x_n\, x_n]$, and by similar reasons as before, $[x_2 \dots x_{n-1}\, x_n\, x_2\, x_3] =$ $= x_3$. So,

$$y = [[x_2 \dots x_{n-1}\, x_n\, x_2\, x_3]\, x_4 \dots x_n\, x_n\, x_n\, y] = [x_3 \dots x_n\, x_n\, y_n\, y].$$

Continuing in the same way we shall come to $y = [x_n \dots x_n\, y]$ which implies that y is an idempotent. Summarizing all we have done till now we can conclude that $y^{n+1} = e_y$. If we apply this to $[x_j \dots x_n\, x_1 \dots x_j] = x_j$, taking into account that x_j is not an idempotent, we shall get that $x_j^{\ n+1} = e_j$ which completes the statement (i). To complete the proof of the Lemma we have, finally, to prove that $y = z^{n+1}$ for no $z \in S$. If $y = z^{n+1}$ for some $z \in S$, then

$$y = [x_1 \dots x_n\, y] = [x_1 \dots x_n\, (x_n\, z^{n+1})] = [[x_1 \dots x_n\, z]\, z \dots z]$$

implies that $y = e_z$. Namely, if $[x_1 \dots x_n\, z] = z$, by the first part of (iv); we have $z = e_z$ or $z^{n+1} = e_z$ and in both cases $y = e_z$; if $[x_1 \dots x_n\, z] =$ $= z^{n+1}$, then $y = z^{2n+1} = e_z$, and if $[x_1 \dots x_n\, z] = z^{2n+1} = e_z$, it is obvious that $y = e_z$. From $y = z^{n+1} \in < z >$ it follows thatt $e_y = e_z$, since $< z >$ contains unique idempotent.

Let us put $S(e) = \{x \in S \mid$ the idempotent of $< x >$ is e$\}$, $T =$ $= \{x \in S \mid x \neq e_x,\ x^{n+1} = e_x,$ there is no $z \in S$ such that $x = z^{n+1}\}$, and $R = T \cup E$, where E is the set of all the idempotents in S. Let Q be an n-subsemigroup of S with the following properties: Q contains unique idempotent and, every n-subsemigroup of S which contains the idempotent Q as its unique idempotent is contained in Q. Then we call Q the *maximal unipotent n-subsemigroup* of S.

Theorem 2. *Let S be a* $\overline{\lambda}$*-n-semigroup. Then:*

(i) R is a left ideal in S,

(ii) $S = \cup \{S(e) \mid e \in E\}$, *disjoint union, and E right zero n-subsemigroup of all the idempotents of S,*

(iii) Every S (e) is a left ideal in S; S (e) is the maximal unipotent n-subsemigroup of S with e as its idempotent which is a zero in S (e).

Proof. If $x_j \in R$ and $y = [x_0 x_1 \ldots x_n]$, then there are two possibilities for y: $y = e_n$; in the first case $y \in E$ and in the second one, by Lemma 7, $y \in T$. This proves (i). The part (ii) od the Theorem follows from Lemma 5 and the difinitions of S(e) and of λ-n-semigroup. If $y_j \in S(e)$, then from $y = [y_0 y_1 \ldots y_n] \in < y_n >$ it follows that the idempotent e corresponds to y, i. e. $y \in S(e)$, so $S(e)$ is an n-subsemigroup of S an therefore a left ideal in S. By (1), $[z_1 \ldots z_n e] = e$, for every $z_j \in S(e)$, and then,

$$\begin{aligned} e &= [z_1 \ldots z_n (z_n{}^{k^n+1})] = [z_1 \ldots z_{n-1} (z_n{}^{k^n+1}) z_n] = \\ &= [z_1 \ldots z_{n-1} e z_n] = [z_1 \ldots z_{n-1} (z_{n-1}^{sn+1}) z_n] = \\ &= [z_1 \ldots z_{n-2} (z_{n-1}^{sn+1}) z_{n-1} z_n] = [z_1 \ldots z_{n-2} e z_{n-1} z_n] = \ldots \end{aligned}$$

where k, s, ... are some of the integers 0, 1, 2. This poves that e is a zero in S (e). Finally, let Q be a unipotent n-subsemigroup of S with e as its idempotent. If $x \in Q$, then $< x > \subset Q$; Q is unipotent and so, the corresponding idempotent of x must be e, i. e. $x \in S(e)$ which proves that $S(e)$ is the maximal unipotent n-subsemigroup of S.

The left ideal R of Therem 2 is itself a $\overline{\lambda}$-n-semigroup. We call R the *reduced* $\overline{\lambda}$*-n-semigroup.* If S is a λ-semigroup and R its corresponding reduced λ-subsemigroup, then $xy = e_y$ for every $x, y \in R$. This is not the case when S is a $\overline{\lambda}$-n-semirroup if $n > 1$, as the following example shows:

Example. Let $R = \{a_1, a_2, \ldots, a_n\, e\}$ and let us define an $(n+1)$-ary operation by; $[a_j a_{j+1} \ldots a_n a_1 \ldots a_j] = a_j$, $[x_0 x_1 \ldots x_n] = e$ otherwise. It is easy to see that R is an n-semigroup and that $[R \ldots R a_j] = \{a_j, e\} = < a_j >$ which, by Lemma 3 implies that R is a λ-n-semigroup; R is the reduced $\overline{\lambda}$-n-semigroup such that $E = \{e\}$ and $T = \{a_1, a_2 \ \ldots, a_n\}$.

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