ON A CLASS OF GENERALIZED λ-SEMIGROUPS

B. Trpenovski

1. Introduction. Let S be a semigroup and n a fixed positive integer. We call S a *generalized* λ*-semigroup*, or λ^n*-semigroup*, if and only if all n-subsemigroups of S are left n-ideals in S (i. e. $Q^{n+1} \subseteq Q$ implies $S^n Q \subseteq Q$). *A* λ^1-semigroup is simply a λ-semigroup as defined in [2]; λ-semigroups are also studied in [3]. Starting with λ-semigroups and taking the $(n+1)$-ary generalization instead of the two statements in the definition of λ-semigroups (to be subsemigroup, or, to be left ideal), one could define two new classes of semigroups, one more special than λ-semigroups and the other more general. The more special class is defined as follows: a semigroup S is said to be λ^**-semigroup* if and only if all n-subsemigroups of S, for some fixed $n > 1$, are left ideals in S. The structure of this kind of semigroups is very simple as the following theorem shows:

Theorem 1. *A semigroup S is a* λ^**-semigroup if and only if S is periodic and* $xy = e_y$, *for all* $x, y \in S$, *where* e_y *is the idempotent in* $< y >$. *(Here* $n > 1$*).*

Proof. If S is a λ^*-semigroup, then T is a λ-semigroup, too, which implies that S is periodic ([2], Lemma 3); furthermore, $xy = e_y$ or y^2 for all $x, y \in S$ ([2], lemma 3). If $y^2 \neq e_y$ then $Q = \{y, e_y\}$ will be an n-subsemigroup of S, $n > 1$, which does not contain the element y^2; since Q is a left ideal, then $y^2 = yy \in SQ \subseteq Q$ and this contradicts the previous statement. So, $xy = e_y$. Conversely, for every $x \in S$, $< x >$ contains unique idempotent e_x. From $xy = e_y$ it follows that every n-subsemigroup Q of S, with every $x \in Q$ contains the corresponding idempotent e_x and then it is clear that Q is a left ideal.

The more general class than λ-semigroups consists of all semigroups in which all subsemigroups are left n-ideals. The class of λ^n- semigroups is more special than the last one and it seems natural and easier to start the study by λ^n-semigroups. On the other hand, λ^n-semigroups represent more special algebraic systems than λ-n-semigroups studied in [4] and [5] which makes possible to use some of the properties of λ-n-semigroups in studying λ-n-semigroups.

In this article we shall establish some general properties of λ^n-semigroups and then we shall give a charaeterization of a λ^n-semigroup.

2. Some general properties of λ^n-semigroups. Throughout this part S will be a λ^n-semigroup with $n \geqslant 2$ fixed. If we put $[x_0 x_1 \ldots x_n] = x_0 x_1 \ldots x_n$, then S becomes a λ-n-semigroup with respect to $[\ldots]$. In an n-semigroup may exist a cyclic n-subsemigroup which does not contain idempotents. As we shall see below, if S is a λ-n-semigroup where the $(n+1)$-ary operation is defined as above, then in every cyclic n-subsemigroup of S there exists an idempotent and S will be a $\overline{\lambda\text{-}n}$-semigroup ([4]). So, all the properties of $\overline{\lambda\text{-}n}$-semigroups are true in S when we consider S as an n-semigroup, or they may be vestated appropriately when we consider S as a semigroup. For example, Lemmas 2 and 3 of [4] can be stated as follows:

Lemma 1. *Every subsemigroup of a λ^n-semigroup is λ^n-semigroup.*

Lemma 2. *A semigroup S is a λ^n-semigroup if and only if $S^n a \subseteq \langle\!\langle a \rangle\!\rangle$ for every $a \in S$, where $\langle\!\langle a \rangle\!\rangle$ is the cyclic n-subsemigroup of S generated by a. If S is a λ^n-semigroup then $S^n a \subseteq \langle a \rangle$ for every $a \in S$ where $\langle a \rangle$ is the cyclic subsemigroup of S generatedbya.*

Let us prove, now, the following

Lemma 3. *Let R be a λ^n-semigroup. Then S is periodic and for every $a \in S$ the order of $\langle a \rangle$ is not greater than $n+2$. The set E of all idempotents of S is right zero subsemigroup of S; E is in fact a left ideal in S.*

Proof. The statement that S is periodic can be proved as in Lemma 5 of [4]. So, in every cyclic subsemigroup of S there exists an idempotent. If e_a is the idempotent in $\langle a \rangle$ and if $n \in S$, then

$$xe_a = xe_a \ldots e_a \in S^n \langle e_a \rangle \subseteq \langle e_a \rangle \subseteq \{e_a\},$$

so that,

$$xe_a = e_a. \tag{1}$$

If K_a is the corresponding subgroup of $\langle a \rangle$ and if $y \in K_a$, then $y = ye_a = e_a$ and so $K_a = \{e_a\}$. This particularly implies that e_a belongs to $\langle\!\langle a \rangle\!\rangle$, i. e. that every cyclic n-subsemigroup of S contains an idempotent. Let

$< a > = \{a, a^2, \ldots a^s\}$. If $s > n + 2$ then $T = \{a, a^{n+1}. a^{n+3} \ldots, a^s = e_a\}$ will be an n-subsemigroup of S since the periodic part of $< a >$ is $K_a = \{e_a\}$. T does not contain the element a^{n+2}; by Lemma 1, $a^{n+2} = a^2 a \ldots aa \in < a > \ldots < a > T \subseteq T$ and this is a contradiction. So, $|< a >| \leqslant n+2$. The assertion that E is a left ideal in S follows from (1).

If $x^{n+1} = x$, $x \in S$, we call x an *n- idempotent* in S. It is obvious that every idempotent is n-idempotent, too. When S is a λ^n-semigroup the converse is also true.

Lemma 4. *An element $x \in S$ is idempotent if and only if it is n-idempotent.*

Proof. It is a part of the proof of the next lemma.

Lemma 5. *Let $x, y \in S$. Then $xy = y$ if and only if y is an idempotent.*

Proof. If y is an idempotent, the assertion follows from (1). Conversely, if $xy = y$ then $x^n y = y$ and this implies that y is an n-idempotent ([4], Lemma 4). From $y^{n+1} = y$ it follows that $y^2 = y \cdot y^{n+1} = y^2 \cdot yy \ldots y = y$ ([4], (1)) and so y is an idempotent.

Lemma 6. *Let $x_j \in S$, $j = 0,1,\ldots,n+1$ and let e_{n+1} be the idempotent in $< x_{n+1} >$. Then $x_0 x_1 \ldots x_n x_{n+1} = e_{n+1}$.*

Prof. According to Lemmas 2,3 and 5, $x_1 \ldots x_n x_{n+1} = e_{n+1}$ or x_{n+1}^{n+1}. In the first case the assertion of the lemma follows from (1); in the second case, $x_0 x_1 \ldots x_n x_{n+1} = x_0 x_{n+1}^{n+1} = (x_0 x_{n+1}) x_{n+1} \ldots x_{n+1}$, and again the last product can take one of the values: e_{n+1}, x_{n+1}^{n+1}. If $(x_0 x_{n+1}) x_{n+1} \ldots x_{n+1} = x_{n+1}^{n+1}$, then by Lemma 5 it follows that $x_{n+1}^{n+1} = e_{n+1}$ and so, $x_0 x_1 \ldots x_n x_{n+1} = e_{n+1}$.

Let $S(e) = \{x \in S \mid x^p = e \text{ for some } p \in N\}$. The following holds:

Theorem 2. *If S is a λ^n_semigroup then $S = \cup \{S(e) \mid e \in E\}$ where the union is disjoint and: E is the right zero subsemigroup of all the idempotents in S, $S(e) . S(f) \subseteq S(f)$, $S(e)$ is a left ideal in $S - S(e)$ is the maximal unipotent subsemigroup of S with e as its idempotent which is zero in $S(e)$. In other words: S is the union of a band, which is right zero semigroup, of unipotent λ^n_semigroups, and this is the greatest decomposition of S such that the factor semigroup is a band* ([1]).

Proof. Since S is periodic, it is union of $\{S(e) \mid e \in E\}$. If $x \in S(e) \cap \cup S(f)$, $x^p = e$, $x^m = f$ and, if $p < m$, by (1), $f = x^{m-p} x^p = x^{m-p} e = e$, and so, the union is disjoint. We have already proved that E is right zero subsemigroup of S. Let $x \in S(e)$, $y \in S(f)$, and let e_{xy} it the correspoding

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idempotent of xy, i.e. $(xy)^s = e_{xy}$. If $q \geqslant \frac{n+2}{s}$ then by Lemma 6, $e_{xy} = e^q{}_{xy} = u_0 u_1 \ldots u_n y = f$, where u_j is equal to x or y or to some produot of x's and y's. This proves trat $S(e)S(f) \subseteq S(f)$, and, at the same time, because S is union of $\{S(e) \mid e \in E\}$, that all $S(e)$ are left ideals in S. If Q is a unipotent subsemigroup of S with e as its idempotent, and if $x \in Q$ then $< x > \subseteq Q$; by Lemma 3 $< x >$ contains an idempotent and since Q is unipotent, the idempotent in $< x >$ is e, so, $x \in S(e)$ and $Q \subseteq S(e)$ which proves that $S(e)$ is the maximal unipotent subsemigroup of S with e as its idempotent. Finally, if $x \in S(e)$, $x^r = e$, then $ex = x^r x = xx^r = xe = e$ and e is zero in $S(e)$.

3. Unipotent λ^2-semigroups. Here, for the sake of simplicity, we shall give a characterization of a unipotent λ^2-semigroup; so, in what follows, S will be a unipotent λ^2-semigroup and e its idempotent.

If we consider S as a $\overline{\lambda—2}$-semigroup, its corresponding reduced $\overline{\lambda—2}$-semigroup R ([5], Theorem 1) will be a zero 2-semigroup (Lemma 5). Taking this into account, by Theorem 1 of [5] we have that,

Theorem 3. *Let S be a unipotent λ^2-semigroup with e as its idempotent. Then there exist subsets A, B and C of S and, if $A \neq \varnothing$, there exist mappings $f\colon Q \times Q \times A \to \{0,1\}$ and $g\colon A \to C$, where $Q = A \cup B$ such that:*

(i) $S = A \cup B \cup C \cup \{e\}$ — *disjoint union;*

(ii) $f(a,a,a) = 1$ *if* $a \in A$;

(iii) g *is a surjection and so* $|A| \geqslant |C|$;

(iv) $xyz = g(z)$ *if* $f(x,y,z) = 1$ *and* $xyz = e$ *in all other cases.*

Conversely, let A, B and C be pairwise disjoint sets with $|A| \geqslant |C|$ and $\{e\}$ an one-element set disjoint with $A \cup B \cup C$. If $A \neq \varnothing$, let $f\colon Q \times Q \times A \to \{0,1\}$ and $g\colon A \to C$ be two mappings such that $f(a,a,a) = 1$ if $a \in A$ and g a surjection; here $Q = A \cup B$. Let $S = A \cup B \cup C \cup \{e\}$. If we put

$$(xyz =)\begin{cases} g(z) & \text{if } f(x,y,z) = 1 \\ e & \text{otherwise,} \end{cases}$$

then $S((\ldots))$ will be $\overline{\lambda—2}$-semigroup with unique idempotent e which is zero in S and such that its reduced 2-subsemigroup is a zero 2-subsemigroup.

If S is a unipotent λ^2-semigroup then the subsets and the mappings in the above theorem are defined as follows: $A = \{x \in S \mid x^3 \neq e\}$, $B = \{x \in S \mid x \neq e,\ x^3 = e,\ x = y^3 \text{ for no } y \in S\}$, $B = \{x \in S \mid x \neq e,\ x = y^3 \text{ for some } y \in S\}$, $f(x,y,z) = 1$ if $xyz \neq e$, $f(x,y,z) = 0$ if $xyz = e$, $g(x) = x^3$.

Theorem 3 gives a description of the structure of S if it is considered as a $\overline{\lambda-2}$-semigroup. To describe the structure of S as a semigroup in a similar way, is more difficult than in the case of λ-semigroups. Here we shall give another characterization of a λ^2-semigroup.

Theorem 4. *A semigroup S is a unipotent λ^2-semigroup if and only if the following assertions are true:*

(i) *there exists an ideal (two-sided) J in S, with zero, which is a zero semigroup,*

(ii) *the Rees factor semigroup S/J is a zero semigroup (with J as its zero),*

(iii) *if $x \in J$, $y \in S$ then $xy = e$ or y^3 where e is the zero in J.*

Proof. Let S be a unipotent λ^2-semigroup and e its idempotent. If we put $J = SS$ then J will be an ideal in S which contains e and e, as a zero in S, will be a zero in J. If $x,y \in J$, i.e. if $x = u_1v_1$, $y = u_2v_2$, u_i, $v_i \in S$, then by Lemma 6, $xy = u_1v_1u_2v_2 = e$ and so (i) is proved. (ii) follows directly from the definition of J. If $x \in J$, $y \in S$ then $xy = uvy = e$ or y^3 ((iv), Theorem 3) since $x = uv$ for some $u,v \in S$, and this proves (iii). Conversely, let T be a 2-subsemigroup of S and let $x,y \in S$, $z \in T$. By (ii) $xy \in J$ and then by (iii) $xyz = e$ or z^3. On the other hand, $z \in T$ implies that $\ll z \gg \subset T$ where $\ll z \gg$ is the cyclic 2-subsemigroup of S generated by z; from $z^2, z^3 \in J$ it follows that $z^5 = e$ where e is the zero in J, so that $\ll z \gg = \{z, z^3, e\}$ (here it may happen $s^3 = e$). So, $xyz \in T$ and T is left 2-ideal. This proves that S is a 2-semigroup. The zero e of J is an idempotent in S; if $x \in S$ is an idempotent, then $x = x^4 = x^2 \cdot x^2 = e$ since $x^2 \in J$, and so S is unipotent.

Note. If S is a unipotent λ^n-semigroup where $n>2$ and if we put $J = S^n$ then, in order to have a similar characterization as for λ^2-semigroups, (ii) and (iii) in Theorem 4 should be replaced by (ii_1) the Rees factor semigroup S/J is a zero (n-1)-semigroup, and (iii_1) if $x \in J$, $y \in S$ then $xy = e$ or y^{n+1}.

4. General λ^2-semigroups. Let us suppose that S is a λ^2-semigroup with more than one idempotent. Analogous to Theorem 4 is the following

Theorem 5. *A semigroup S is a λ^2-semigroup if and only if the following assertions are true:*

(i) *there exists an ideal J in S which is a λ^*-semigroup,*

(ii) *the Rees factor semigroup S/J is a zero semigroup,*

(iii) *if* $x \in J$, $y \in S$ *then* $xy = e$ *or* y^3, *where* e *is the corresponding idempotent of* y (i.e. $y \in S(e)$).

Proof. If we put $J = SS$ then J will be an ideal in S. If $x, y \in J$ then $xy = u_1 v_1 u_2 v_2 = e$ (Lemma 6) since by Theorem 2, $v_2 \in S(e)$ for some $e \in E$ and then $y = u_2 v_2 \in S(e)$. This, and the fact that J is periodic (Lemma 3) imply rhat J is a λ^*-semigroup (Theorem 1). (ii) is obviously true and (iii) can be proved as in Theorem 4. The converse part also can be proved as the converse part of Theorem 4.

Notes. If $J(e)$ is the ideal in the unipotent λ^2-semigroup $S(e)$ which satisfies the conditions of Theorem 4, and if we put $J^\wedge = \cup \{J(e) \mid e \in E\}$, then it can be proved, in a similar way as in Theo.em 5, that $J^\wedge$ is an ideal in S which is a λ^*-semigroup. But, in general, $S/J^\wedge$ need not be a zero semigroup, since if $x \in S(e)$, $y \in S(f)$ and if $e \neq f$ then $xy \in S(7)$ but may happen that $xy \notin J(f)$ as the following example shows:

	x	y	e	z	f
x	e	e	e	f	f
y	e	e	e	f	f
e	e	e	e	f	f
z	y	x	e	f	f
f	e	e	e	f	f

Example. Let $S = \{x, y, z, e, f\}$ be the semigroup defined by the enclosed Cayley's table. It is easily seen that S is a λ^2-semigroup with e and f as idempotents. Here $J(e) = \{e\}$ and $J(f) = \{f\}$ and $zx = y \notin J(f)$.

If S is a λ^n-semigroup with $n > 2$, then, as for the unipotent case, by similar modifications in Theorem 5 we can get a characterization foi S.

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