ON REPRESENTATION OF n-ASSOCIATIVES INTO SEMIGROUPS

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Introduction. The subject of this paper are n-associatives, i.e. algebras with one n-ary associative operation. First we give the necessary definitions and simple proofs of known results, that any n-associative A can be covered by a semigroup $A^\frown$, called the free covering of A, and hat if A is an n-group, then $A^\frown$ is a group. (The second result is in fact the Post's Coset Theorem ([2], p. 218)). Then we show that if A and B are n-associatives, then any homomorphism $\varphi : A \to B$ may be extended to a homomorphism $\varphi^\frown : A^\frown \to B^\frown$. (Thus $^\frown$ is a covariant functor from the category of n-associatives into the category of semigroups.) The extension $\varphi^\frown$ is not in general a monomorphism although φ is a monomorphism. An n-subassociative B of an n-associative A is called compatible in A if the extension $\varepsilon^\frown : B^\frown \to A^\frown$ of the embedding monomorphism $\varepsilon : B \to A$ is a monomorphism. It is shown by examples that neither isomorphisms nor intersections of subassociatives preserve the property of compatibility. We show that any n-subgroup of an n-associative and any subassociative which is a complement of an ideal are compatible. In the fourth part of the paper the compatible subassociatives of cyclic associatives are described. Finally, associatives in which all subassociatives are compatible are considered. Namely, it is shown that such an associative A must be periodic and if r_a is the index of an element $a \in A$, then $r_a \leqslant 2$. As a corollary, we get the theorem that each subassociative of an n-group A is compatible in A iff A is periodic.

1. Associatives and covering semigroups. An algebra $A(\omega)$ with an $(n+1)$-ary operation ω is said to be an *n-associative* if the following identity equations are satisfied:

$$\omega\omega x_0 x_1 \dots x_{2n} = \omega x_0 \omega x_1 \dots x_{2n} = \dots = \omega x_0 \dots x_{n-1} \omega x_n \dots x_{2n}. \quad (1)$$

Then the following "general associative law" is also satisfied:

$$\omega^k x_0 x_1 \dots x_{kn} = \omega x_0 \dots x_{i_0-1}\, \omega x_{i_0} \dots \dots x_{i_k-1}\, \omega x_{i_k} \dots x_{kn}, \tag{2}$$

where $i_0, i_1, \dots, i_k$ is a sequence of non-negative integers such that $\nu \leqslant i_{\nu+1} \leqslant \nu n$.

Further on we will omit the operator symbol ω and write $[x_0 x_1 \dots {}^i x_{kn}]$ instead of $\omega^k x_0 x_1 \dots x_{kn}$; we put $[x] = x$ if $k = 0$.

An n-associative A is said to be an *n-subassociative* of a semigroup $S(\cdot)$ if $A \subseteq S$ and

$$(\forall a_0, \dots a_n \in A)\, [a_0 \dots a_n] = a_0 \cdot a_1 \cdot \dots \cdot a_n. \tag{3}$$

Let $F = \bigcup_{k \in \mathbf{N}} A^k = \{(a_1, \dots, a_k) \mid k \in \mathbf{N},\ a_\nu \in A\}$ be the semigroup which is freely generated by the carrier A of an n-associative. Two elements $\mathbf{a} = (a_1, \dots, a_p)$, $\mathbf{b} = (b_1, \dots, b_q)$ of F $(a_\nu, b_\lambda \in A)$ are said to be *strongly linked* in A iff there is an element $\mathbf{e} = (e_0, \dots, e_t) \in F$ $(e_\nu \in A)$ and two sequences of non-negative integers $k_1, k_2, \dots, k_p$, and $m_1, \dots, m_q$ such that:

$$a_1 = [e_0 \dots e_{k_1}],\ a_2 = [e_{k_1+1} \dots e_{k_2}], \dots,\ a_p = [\dots e_t],$$

$$b_1 = [e_0 \dots e_{m_1}], b_2 = [e_{m_1+1} \dots e_{m_2}], \dots, b_q = [\dots e_t]; \tag{4}$$

then we write $\mathbf{a}\ sl\ \mathbf{b}$. The transitive extension of sl will be denoted by l_A, i. e.

$$\mathbf{a}\ l_A\ \mathbf{b} \Leftrightarrow (\exists \mathbf{c}_1, \dots, \mathbf{c}_r \in F)\ \mathbf{a}\ sl\ \mathbf{c}_1\ sl \dots sl\ \mathbf{c}_r\ sl\ \mathbf{b}. \tag{5}$$

If $\mathbf{a}\ l_A\ \mathbf{b}$, then we say that $\mathbf{a}$ and $\mathbf{b}$ are *linked* in A.

1. 1. *The relation $l\,(= l_A)$ is a congruence on the free semigroup F, $A^l = \{a^l \mid a \in A\}$ is an n-subassociative of F/l and $\xi : a \to a^l$ is an isomorphism from A onto A^l.*

Proof. Obviously, l is a congruence on F, A^l is an n-subassociative of F/l and ξ is a homomorphism. If $a \in A$ and (a) is strongly linked with $\mathbf{b} = (b_1, \dots, b_q)$, then there is a sequence $\mathbf{e} = (e_0, \dots, e_{kn})$ of elements of A and non-negative integers $m_1, \dots, m_q$ such that

$$a = [e_0 \dots e_{kn}],$$

$$b_1 = [e_0 \dots e_{m_1}],\ \ b_2 = [e_{m_1+1} \dots e_{m_2}], \dots\ \ b_q = [\dots e_{kn}],$$

and therefore $n \mid q - 1$ and

$$[b_1 \dots b_q] = \big[\, [e_0 \dots e_{m_1}] \dots [\dots e_{kn}]\, \big] = [e_0 \dots e_{kn}] = a.$$

This implies that (a) is linked with $\mathbf{c}=(c_1, \ldots, c_s)$ if and only if $n \mid s-1$ and $a=[c_1 \ldots c_s]$ in A. Therefore, if $a, b \in A$ and $a^l=b^l$, then $a=b$, i. e. ξ is a monomorphism. Obviously ξ is an epimorphism and thus it is an isomorphism. ∎

The factor-semigroup F/l will be said to be the *free covering semigroup* of the given n-associative A and it will be denoted by $A^\frown$. It is convenient to identify the element $a \in A$ with the l-equivalent class a^l and thus $A^\frown$ obtains the following form:

$$A^\frown = A \cup A^2 \cup \ldots \cup A^n, \tag{6}$$

where $A^i=\{a_1 \ldots a_i \mid a_\nu \in A\}$ and $A^i \cap A^j=\varnothing$ if $i \neq j$. The equation $a_1 \ldots a_i=b_1 \ldots b_i$ holds in $A^\frown$ iff the sequences $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_i)$ are linked in A.

An (non-empty) n-associative A is said to be an *n-group* iff:

$$(\forall a_1, \ldots, a_n \in A)\,[A a_1 \ldots a_n]=A=[a_1 \ldots a_n A]. \tag{7}$$

1. 2. *The free covering semigroup $A^\frown$ of an n-group A is a group.*

(Now $A^\frown$ is said to be the *free covering group* of the given n-group A.)

Proof. Assume that $a=a_1 \ldots a_i$, $b=b_1 \ldots b_j \in A^\frown$, a_ν, $b_\lambda \in A$, $i \leqslant n$, $j \leqslant n$, and put $k=2n+j-i$. If $c_{j+1}, \ldots, c_k$ are arbitrary elements of A and if we choose $x_j \in A$ in such a way that $[x_j c_{j+1} \ldots c_k a_1 \ldots a_i]=b_j$, then we get $xa=b$, where $x=b_1 \ldots b_{j-1} x_j c_{j+1} \ldots c_k$. We find in a similar way a $y=y_1 \ldots y_k$ such that $ay=b$. ∎

1. 3. If $A^\frown$ is the free covering group of the n-group A, then: $a_1, \ldots, a_i, b_1, \ldots, b_i \in A \Rightarrow$

$$a_1 \ldots a_i = b_1 \ldots b_i \Leftrightarrow (\exists c_0, \ldots, c_{n-i} \in A)\,[c_0 \ldots c_{n-i}\, a_1 \ldots a_i] = [c_0 \ldots c_{n-i}\, b_1 \ldots b_i]$$

$$\Leftrightarrow (\forall c_0, \ldots c_{n-i} \in A)\,[c_0 \ldots c_{n-i}\, a_1 \ldots a_i] = [c_0 \ldots c_{n-i}\, b_1 \ldots b_i].$$

This theorem is a corollary of the preceding one. ∎

2. Extensions of homomorphisms.

2.1. If $\varphi: A \to A'$ *is a homomorphism of an n-associative A into an n-associative A', then* φ *can be extended in a unique way to a homomorhism* $\varphi^{\frown}: A^{\frown} \to A'^{\frown}$. *Also, if* $\mathbf{a} = (a_1, \ldots, a_i)$ *and* $\mathbf{b} = (b_1, \ldots, b_i)$ *are linked in A, then* $\varphi(a)$ *and* $\varphi(b)$ *are linked in A'.*

Proof. If $a = a_1 \ldots a_i \in A^{\frown}$, $1 \leqslant i \leqslant n$, then $\varphi^{\frown}(a)$ is defined by

$$\varphi^{\frown}(a) = \varphi(a_1). \ldots \varphi(a_i).$$

It is necessary to show that

$$a_1 \ldots a_i = b_1 \ldots b_i \text{ in } A \Rightarrow \varphi(a_1) \ldots \varphi(a_i) = \varphi(b_1) \ldots \varphi(b_i) \text{ in } A'^{\frown}.$$

Assume that the sequences $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_i)$ are strongly linked in A. Then there is a sequence $\mathbf{e} = (e_0, \ldots, e_t)$ of elements of A such that the equations (4) of § 1 hold for $p = q = i$. Therefore

$$\begin{aligned} \varphi(a_1) \ldots \varphi(a_i) &= [\varphi(e_0) \ldots \varphi(e_{k_1})] \ldots [\ldots \varphi(e_t)] \\ &= \varphi(e_0) \ldots \varphi(e_{k_1}) \ldots \varphi(e_t) \\ &= [\varphi(e_0) \ldots \varphi(e_{m_1})] \ldots [\ldots \varphi(e_t)] \\ &= \varphi(b_1) \ldots \varphi(b_i). \ \blacksquare \end{aligned}$$

The following statement is clear:

2. 2. *If* $\varphi: A \to A'$ *is an epimorphism, then* $\varphi^{\frown}: A^{\frown} \to A'^{\frown}$ *is an epimorphism too.* ■

The analogous proposition is not true in general for monomorphisms, as the following example shows.

Example. Let $A = \{a, b, c\}$. If a ternary operation is defined on A by:

$$[ccc] = b, \quad [xyz] = a \quad \text{if} \quad \{a, b\} \cap \{x, y, z\} \neq \varnothing,$$

then A becomes a 2-associative. The subset $B = \{a, b\}$ is a 2-subassociative of A. Find the free covering semigroups $A^{\frown}$ and $B^{\frown}$:

$$A^{\frown} = A \cup A^2, \quad A^2 = \{aa = ab = ba = bb = ac = ca, \ bc = cb, \ cc\},$$

$$B^{\frown} = B \cup B^2, \quad B^2 = \{aa = ab = ba \neq bb.\}.$$

If $\varepsilon: a \to a,\ b \to b$ is the embedding monomorphism of B in A, then the extension $\varepsilon^\frown$ is defined by:

$$\varepsilon^\frown: a \to a,\ b \to b,\ aa \to aa,\ bb \to bb = aa$$

and thus $\varepsilon^\frown$ is not a monomorphism.

2.3. *If* φ *is a monomorphism of an n-group* G *into an n-group* G', *then* $\varphi^\frown: G^\frown \to G'^\frown$ *is a monomorphism too.*

Proof. Let $a = a_1 \ldots a_i$, $b = b_1 \ldots b_i$ $(a_\nu, b_\nu \in G)$ be elements of $G^\frown$ such that $\varphi^\frown(a) = \varphi^\frown(b)$. Then

$$\varphi(a_1) \ldots \varphi(a_i) = \varphi(b_1) \ldots \varphi(b_i) \text{ in } G'^\frown.$$

If $c_0, \ldots, c_{n-i}$ are arbitrary elements of G then

$$\begin{aligned}\varphi(c_0 \ldots c_{n-i} a_1 \ldots a_i) &= \varphi(c_0) \ldots \varphi(c_{n-i}) \varphi(a_1) \ldots \varphi(a_i)\\ &= \varphi(c_0) \ldots \varphi(c_{n-i}) \varphi(b_1) \ldots \varphi(b_i)\\ &= \varphi(c_0 \ldots c_{n-i} b_1 \ldots b_i), \text{ i. e.}\end{aligned}$$

$$[c_0 \ldots c_{n-i} a_1 \ldots a_i] = [c_0 \ldots c_{n-i} b_1 \ldots b_i] \text{ in } G$$

and this (by 1.3) implies

$$a_1 \ldots a_i = b_1 \ldots b_i \text{ in } G^\frown. \ \blacksquare$$

3. Compatible subassociatives. Let A be an n-associative. $B \subseteq A$ is said to be a *subassociative* of A if $[B^{n+1}] \subseteq B$, i. e. if B is a subalgebra of the algebra A. A subassociative B is said to be *compatible* in A if the extension $\varepsilon^\frown: B^\frown \to A^\frown$ of the embedding monomorphism $\varepsilon: B \to A$ is a monomorphism too. The set of all subassociatives of A will be denoted by $\mathcal{L}(A)$, and the set of all compatible subassociatives of A by $\mathcal{K}(A)$. (Clearly $A \in \mathcal{L}(A) \cap \mathcal{K}(A)$.) Some statements about compatibility will be established below.

(Throughout the paper A will denote an n-associative if it is not stated otherwise.)

3.1. If $B \in \mathcal{L}(A)$, *then* $B \in \mathcal{K}(A) \Leftrightarrow l_A \mid B = l_B$.

Proof. Assume that l_B is the restriction of l_A. If $\varepsilon^\frown(a_1 \ldots a_i) = \varepsilon^\frown(b_1 \ldots b_i)$, then $(a_1, \ldots, a_i)\ l_A\ (b_1, \ldots, b_i)$, $a_\nu,\ b_\nu \in B$, hence $(a_1, \ldots, a_i)\ l_B\ (b_1, \ldots, b_i)$, i. e. $a_1 \ldots a_1 = b_1 \ldots b_i$ in $B^\frown$. Thus $\varepsilon^\frown$ is a monomorphism, i. e. $B \in \mathcal{K}(A)$.

Conversely, if $\varepsilon^\frown$ is a monomorphism and if $(a_1, \ldots, a_i)\ l_A$ $(b_1, \ldots, b_i)$, $a_\nu, a_\nu \in B$, then

$$\varepsilon^\frown (a_1 \ldots a_i) = a_1 \ldots a_i = b_1 \ldots b_i = \varepsilon^\frown (b_1 \ldots b_i),$$

i. e. $a_1 \ldots a_i = b_1 \ldots b_i$ in $B^\frown$, which means that $(a_1, \ldots, a_i)\ l_B$ $(b_1, \ldots, b_i)$. ∎

By **3.1.** we obtain:

3.2. *A subassociative B of A is compatible in A if and only if $B^\frown$ is a subsemigroup of $A^\frown$.* ∎

3.3. Examples.

2) If we define in the set $A = \{a, b, c, d, e\}$ a ternary operation by:

$$[eee] = d, \quad x\ y, z \in \{a, b\} \Rightarrow [xyz] = a,$$

$$\{x, y, z\} \cap \{c, d, e\} \neq \varnothing, \quad (x, y, z) \neq (e, e, e) \Rightarrow [xyz] = c,$$

hen we get a 2-associative. The free covering $A^\frown$ is defined by:

$$A^\frown = \{a, b, c, d, e, \ \ aa, bb, cc, ee, be, eb, de\},$$

where: $aa = ab = ba,\ cc = ac = ca = ad = da = ae = ea = bc = cb = cd =$ $= dc = dd = ec = ce, \quad de = ed.$

$B = \{a, b\}$, $C = \{c, d\}$ are isomorphic subassociatives of A and $B^\frown =$ $= \{a, b,\ aa = ab = ba,\ bb\}, C^\frown = \{c, d,\ cc = cd = bc,\ dd\}$. Thus $B \in \mathcal{K}(A)$, but $C \notin \mathcal{K}(A)$ since $cc = dd$ in $A^\frown$ and $cc \neq dd$ in $C^\frown$.

3) The set $A = \{1', 1'', 3, 5, 7, \ldots\}$ with the ternary operation $[xyz] = \varphi(x) + \varphi(y) + \varphi(z)$, where the mapping $\varphi : A \to \mathbf{N}$ is defined by $\varphi(1') = 1 = \varphi(1'')$ and $\varphi(a) = a$ for $a \neq 1', 1''$, is a 2-associative and the subsets $B = \{1', 3, 5, \ldots\}$ and $C = \{1'', 3, 5, \ldots\}$ are 2-subassociatives of A. The free coverings of A, B, C are:

$$A^\frown = \{1', 1'', 3, 4, 5, 6, \ldots\} \cup \{(1', 1'), (1', 1''), (1'', 1'), (1'', 1'')\},$$
$$B^\frown = \{1', (1', 1'), 3, 4, 5, 6, \ldots\},$$
$$C^\frown = \{1'', (1'', 1''), 3, 4, 5, 6, \ldots\}.$$

Thus $B, C \in \mathcal{K}(A)$. The intersection $D = \{3, 5, 7, \ldots\}$ of B and C is also a subassociative of A, but D is not compatible in A, for $(3, 5)$ and $(5, 3)$ of D^2 are linked in A but they are not linked in D. This example shows that (in general) the intersection does not preserve the compatibility of subassociatives.

Some corollaries of **3.1.** will be stated here.

3.4. $B \in \mathcal{L}(A) \Rightarrow \mathcal{L}(B) \cap \mathcal{K}(A) \subseteq \mathcal{K}(B)$. $\blacksquare$

3.5. $B \in \mathcal{K}(A) \Rightarrow \mathcal{K}(B) \subseteq \mathcal{K}(A)$. $\blacksquare$

3.6. *For any n-associative A the set* $\mathcal{K}(A)$ *is inductive, i. e. if* $\{B_i \mid i \in I\}$ *is a chain in* $\mathcal{K}(A)$, *then* $B = \bigcup_{i \in I} B_i \in \mathcal{K}(A)$.

Proof. Assume that $\mathbf{u}, \mathbf{v} \in B^k$ and $\mathbf{u}\ l_A\, \mathbf{v}$. Then there is an $i \in I$ such that $\mathbf{u}, \mathbf{v} \in B_i^k$, and then $\mathbf{u}\ l_{B_i}\, \mathbf{v}$ since $l_A \mid B_i = l_{B_i}$. Thus $\mathbf{u}\, l_B\, \mathbf{v}$, for $l_{B_i} \subseteq l_B$. $\blacksquare$

3.7. *If* $B \in \mathcal{L}(A)$ *and* $A \setminus B$ *is an ideal*[1] *in A, then* $B \in \mathcal{K}(A)$.

Proof. Assume that $\mathbf{u} = (u_1, \ldots, u_k) \in B^k$ and $u_1 = [e_1 \ldots e_{p_1}]$, $u_2 = [e_{p_1+1} \ldots e_{p_2}], \ldots, u_k = [\ldots e_t]$ in A. This implies that $(e_1, \ldots, e_t) \in B^t$, for if $e_\alpha \in A \setminus B$ and $u_\beta = [\ldots e_\alpha \ldots]$, then $u_\beta \in A \setminus B$. Thus $l_B = l_A \mid B$, and by 3. 1 we have $B \in \mathcal{K}(A)$. $\blacksquare$

3.8. Let φ *be an automorphism of A,* $B \in \mathcal{L}(A)$ *and* $C = \varphi(B)$. *Then*: $B \in \mathcal{K}(A) \Leftrightarrow C \in \mathcal{K}(A)$.

Proof. Assume that $C \in K(A)$. If $\mathbf{u}, \mathbf{v} \in B^k$ and $\mathbf{u}\, l_A\, \mathbf{v}$, then $\varphi(\mathbf{u})$ $\varphi(\mathbf{v}) \in C$ and $\varphi(\mathbf{u})\ l_A\, \varphi(\mathbf{v})$. This implies $\varphi(\mathbf{u})\ l_C\ \varphi(\mathbf{v})$ since $l_C = l_A \mid C$. By **2.1.** we have $\mathbf{u}\ \ l_{\varphi^{-1}(C)}\ \ \mathbf{v}$, i. e. $l_B = l_A \mid B$, i. e. $B \in \mathcal{K}(A)$. $\blacksquare$

It may happen that B is isomorphic with C and $B \in \mathcal{K}(A)$, but $C \notin \mathcal{K}(A)$ as the example 2) shows.

3.9. *Any n-subgroup of an n-associative A is a compatible subassociative of A.*

Proof. Let B be an n-subgroup of A and $\mathbf{b}\ l_A\ \mathbf{b}'$, where $\mathbf{b} = (b_1, \ldots, b_i)$, $\mathbf{b}' = (b_1', \ldots, b_i') \in B^i$. Then $b_1 \ldots b_i = b_1' \ldots b_i'$ in $A^\frown$. Denote by C the subsemigroup of $A^\frown$ generated by B. Then C is a subgroup of A and $b_1 \ldots b_i = b_1' \ldots b_i'$ in C. If $d_o, \ldots, d_{n-i}$ are arbitrary elements of B, then $d_0 \ldots d_{n-i}\, b_1 \ldots b_i = d_0 \ldots d_{n-i}\, b_1' \ldots b_i'$ in C, and also

$$[d_o \ldots d_{n-i}\, b_1 \ldots b_i] = d_o \ldots d_{n-i}\, b_1' \ldots b_i']$$

in B. Hence (cf. **1. 3**) $\mathbf{b}\ l_B\ \mathbf{b}'$. $\blacksquare$

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[1] i. e. $[a_o \ldots a_n] \notin A \setminus B \Rightarrow a_o, \ldots, a_n \notin A \setminus B$.

4. Compatible subassociatives of cyclic associatives. The set of compatible subassociatives of a cyclic n-associative will be discussed here.

4.1. *Let* $A=\{a^{kn+1} \mid k\in \mathbf{N}\}=\langle a\rangle$ *be an infinite cyclic* n*-associative. An* n*-subassociative* B *is compatible in* A *iff* B *is cyclic.*

Proof. Note first that $a^{in+1}=a^{jn+1}\Rightarrow i=j$. It is easy to see that

$$(a^{\nu_1 n+1},\ldots,a^{\nu_i n+1})\, l_A\, (a^{\lambda_1 n+1},\ldots,a^{\lambda_i n+1})\Leftrightarrow$$

$$\nu_1+\ldots+\nu_i=\lambda_1+\ldots+\lambda_i. \qquad (1)$$

Let $B=\langle b\rangle$ be a cyclic n-subassociative of A, where $b=a^{pn+1}$. If

$$(b^{\nu_1 n+1},\ldots,b^{\nu_i n+1})\, l_A\, (b^{\lambda_1 n+1},\ldots,b^{\lambda_i n+1}),$$

then by (1) we have

$$((pn+1)\nu_1+p)+\ldots+((pn+1)\,\nu_i+p)=((pn+1)\,\lambda_1+p)+\ldots+$$
$$+((pn+1)\,\lambda_i+p),$$

i.e. $\nu_1+\ldots+\nu_i=\lambda_1+\ldots+\lambda_i$, and this again by (1) implies

$$(b^{\nu_1 n+1},\ldots,b^{\nu_i n+1})\quad l_B\, (b^{\lambda_1 n+1},\ldots,b^{\lambda_i n+1}).$$

Thus $B\in \mathcal{K}(A)$.

Let C be a non-cyclic n-subassociative of A and let p be the least positive integer such that $a^{pn+1}=b\in C$. Denote by q the least positive integer such that $c=a^{qn+1}\in C$ and $c\notin\langle b\rangle$, i. e. $pn+1$ is not a divisor of $qn+1$. Then $(b,\, c)\ l_A\ (c,\, b)$. We will show that $(b,\, c)$ and $(c,\, b)$ are not linked in C and this would imply that $C\notin \mathcal{K}(A)$.

Assume that $b=[b_o\ldots b_{kn}]$, $c=[c_o\ldots c_{sn}]$ in C. Then we have $k=0$ and $b=b_o$ since p is the least non-negative integer such that $a^{pn+1}=b\in C$. It is not possible to be $s>0$, for then $c_o,\ldots,c_{sn}\in\langle b\rangle$, and c would be an element of $\langle b\rangle$. Therefore $s=0$ and $c=c_o$. Thus we have proved that

$$(b,\, c)\ l_C\ (b',\, c')\Leftrightarrow b=b',\ c=c',$$

and hence $(b,\, c)$ and $(c,\, b)$ are not linked in C. ∎

As in the class of semigroups (cf. [1], p. 19) it is easy to prove the following "*structural theorem*" for finite cyclic n-associatives.

4.2. If $A=\langle a\rangle$ is a finite cyclic n-associative and if $r+m$ $(r\geqslant 0,$ $m>0)$ is the least positive integer such that $a^{(r+m)n+1}=a^{rn+1}$, then

(i) $A = \{a, \dots, a^{rn+1}, \dots, a^{(r+m-1)n+1}\}$ contains $r+m$ elements and

$$a^{(r+i)n+1} = a^{(r+j)n+1} \Leftrightarrow i \equiv j \pmod{m}; \tag{2}$$

$r = r_a$ is said to be the *index* and $m = m_a$ the *period* of a and A.

(ii) The set $P = \{a^{rn+1}, \dots, a^{(r+m-1)n+1}\}$ is an n-subgroup of A.

(iii) If $b = a^{pn+1} \in A$, then the index of b is the least non-negative integer r_b such that

$$(pn+1)\, r_b + p \geqslant r, \tag{3}$$

and the period of b is the least positive integer m_b which satisfies the following condition:

$$(pn+1)\, m_b \equiv 0 \pmod{m}. \ \blacksquare \tag{4}$$

4.3. *Let A be a cyclic n-associative with the index r and period m, and let B be an n-subassociative of A. Then:*

(i) $B \subseteq P \Rightarrow B \in \mathcal{K}(A)$.

(ii) *If* $B = \langle b \rangle$, where $b = a^{pn+1}$ and $0 < p < r$, then:

(ii.1) $pn < r \leqslant pn + p + 1 \Rightarrow B \in \mathcal{K}(A)$;

(ii.2) $r \leqslant pn \Rightarrow B \notin \mathcal{K}(A)$.

(iii) *Let $B \subseteq' P$ (where $\subseteq'$ means "is not a subset of") and p be the least positive integer such that $b = a^{pn+1} \in B$. If there is a $q < r$ such that $c = a^{qn+1} \in B$ and $pn+1$ is not a divisor of $qn+1$, then $B \notin \mathcal{K}(A)$.*

Proof. (i) If $B \subseteq P$, then B (as an n-subassociative of a finite n-group) is an n-subgroup of A and by **3.9**, $B \in \mathcal{K}(A)$.

(ii)Let $(b_1, \dots, b_i), (c_1, \dots, c_i) \in A^i$, $b_j = a^{\nu_j n+1}$, $c_j = a^{\lambda_j n+1}$. Then it is easy to see that

$$(b_1, \dots, b_i)\ l_A\ (c_1, \dots, c_i) \Leftrightarrow (*) \text{ or } (**), \tag{5}$$

where

$$(*)\ \nu_1 + \dots + \nu_i = \lambda_1 + \dots + \lambda_i < r,$$

$$(**) \begin{cases} \nu_1 + \dots + \nu_i,\ \lambda_1 + \dots + \lambda_i \geqslant r, \\ \nu_1 + \dots + \nu_i \equiv \lambda_1 + \dots + \lambda_i \pmod{m}. \end{cases}$$

(ii.1) Let $(c_1, \dots, c_i)\ l_A\ (d_1, \dots, d_i)$, where $c_j = b^{\nu_j n+1}$, $d_j = b^{\lambda_j n+1} \in B$. If

$$\nu_j' = (pn+1)\,\nu_j + p, \quad \lambda_j' = (pn+1)\,\lambda_j + p \tag{6}$$

then by (5) we have

$$\nu_1' + \dots + \nu_i' = \lambda_1' + \dots + \lambda_i' < r, \tag{*'}$$

or

$$\nu_1' + \ldots + \nu_i',\ \lambda_1' + \ldots + \lambda_i' \geqslant r \quad \text{and}$$
$$\nu_1' + \ldots + \nu_i' \equiv \lambda_1' + \ldots + \lambda_i' \pmod{m}, \qquad (**')$$

If (*′) holds, then (*) holds too, thus $(c_1, \ldots, c_i)\ l_B\ (d_1, \ldots, d_i)$. Assume that (**′) holds and

$$pn < r \leqslant pn + p + 1 \qquad (7)$$

By (7), $r_b = 1$. By (**′),

$$(pn+1)(\nu_1 + \ldots + \nu_i) + ip,\ (pn+1)(\lambda_1 + \ldots + \lambda_i) + ip \geqslant r$$

and

$$(pn+1)(\nu_1 + \ldots + \nu_i) \equiv (pn+1)(\lambda_1 + \ldots + \lambda_i) \pmod{m};$$

clearly

$$\nu_1 + \ldots + \nu_i,\ \lambda_1 + \ldots + \lambda_i \geqslant 1 = r_b \quad \text{and}$$
$$\nu_1 + \ldots + \nu_i \equiv \lambda_1 + \ldots + \lambda_i \pmod{m_b},$$

which implies $(c_1, \ldots, c_i)\ l_B\ (d_1, \ldots, d_i)$. Therefore $B \in \mathcal{K}(A)$.

(ii.2) Let $p < r \leqslant pn$. If $c = b^{mrn+1}$, $\mathbf{b} = (\underbrace{b, \ldots, b}_{n})$, $\mathbf{c} = (\underbrace{c, \ldots, c}_{n})$, by (5), **b** and **c** are linked in A, but they are not linked in B. Thus $B \in \mathcal{K}(A)$.

(iii) Let q be the least positive integer such that $a^{qn+1} = c \in B \setminus \langle b \rangle$ Then we have $(b, c)\ l_A\ (c, b)$. As in the proof of **4.1** it can be easily seen that $b = [b_o \ldots b_{kn}]$, $c = [c_o \ldots c_{kn}]$ in B imply $b = b_o$, $c = c_o$. Hence it would be obtained that (b, c) and (c, b) are not linked in B, i. e $B \notin \mathcal{K}(A)$. ∎

5. Associatives in which all subassociatives are compatible: As a corollary of the theorems **4.2** and **4.3** we get the followings

5.1. *If $\mathcal{L}(A) = \mathcal{K}(A)$, then the index r_a of any element $a \in A$ i. not greater than* 2.

Proof. Assume that there is an element $a \in A$ such that $\langle a \rangle$ is infinite, or $\langle a \rangle$ is finite but $r_a \geqslant 3$, then $B = \{a^{kn+1} \mid k \geqslant 1\}$ is an n-subassociative of $\langle a \rangle$. By **4.2** and **4.3** (iii) $B \notin \mathcal{K}(\langle a \rangle)$ and $B \notin \mathcal{K}(A)$. ∎

5.2. *If A is an n-group, then*:

$$\mathcal{L}(A) = \mathcal{K}(A) \Leftrightarrow A \textit{ is periodic.}$$

Proof. If $\mathcal{L}(A) = \mathcal{K}(A)$, then by **5.1** A is periodic.

Assume now that A is a periodic n-group, i.e. that for each $a \in A$, $\langle a \rangle$ is finite. Then $r_a = 0$ and $\langle a \rangle$ is an n-subgroup of A. We will show that any n-subassociative B of A is an n-subgroup of A and then the conclusion will follow by **3.9.**

Let B be an n-subassociative of A and $a_1, a_2, \ldots, a_n \in B$ with periods $m_1, m_2, \ldots, m_n$ respectively. Then we have $a_i^{m_i n+1} = a_i$ and thus $a_i^{m_i n} = e$ is the identity of the group $A^\frown$ and so $a_i^{m_i n-1} = a_i^{-1}$ in $A^\frown$. Let b be any element of B. Then $[ba_n^{m_n n-1} \ldots a_1^{m_1 n-1}] = x \in B$, for $1 + (m_n n - 1) + \ldots + (m_1 n - 1) = 1 + (m_1 + \ldots + m_n - 1)\, n$, and

$$[xa_1 a_2 \ldots a_n] = $$
$$= ba_n^{-1} \ldots a_1^{-1} a_1 a_2 \ldots a_n \quad \text{in } A^\frown$$
$$= b.$$

Thus we have proved that

$$(\forall a_1, \ldots, a_n, b \in B)(\exists x \in B)\, [xa_1 \ldots a_n] = b,$$

and by symetry:

$$(\forall a_1, \ldots, a_n, b \in B)(\exists y \in B)\, [a_1 \ldots a_n y] = b,$$

. e. that B is an n-group. ■

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РЕЗИМЕ

ПРЕТСТАВУВАЊЕ НА АСОЦИЈАТИВИ ВО ПОЛУГРУПИ

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Предмет на оваа работа се n-асоцијативите, т.е. алгебри со една $(n+1)$-арна асоцијативна операција. Во првиот дел се даваат потребните дефиниции и едноставни докази на познатите резултати дека за секој n-асоцијатив A постои слободна покривка A^l (т.е. полугрупа, таква што: (1) множеството A ја генерира A^l, (2) $(n+1)$-арната операција во A е „реализирана" со операцијата на полугрупата A^l, (3) ако S е полугрупа со особи-

ните (1) и (2), тогаш постои епиморфизам од A^l на S кој го индуцира идентичното пресликување на A) и дека ако A е n-група, тогаш слободната покривка A^l е група. Потоа се покажува дека секој хомоморфизам $\varphi:A\rightarrow B$ од n-асоцијатив A во n-асоцијатив B може да се прошири до хомоморфизам $\varphi^l:A^l\rightarrow B^l$. Проширувањето φ^l не мора да е мономорфизам иако φ е мономорфизам. Ако B е n-подасоцијатив од n-асоцијативот A и ако проширувањето $\varepsilon^l:B^l\rightarrow A^l$ на инклузивниот мономорфизам $\varepsilon:B\rightarrow A$ е мономорфизам, тогаш B се вика *компатибилен подасоцијатив* од A. Покажано е со примери дека ни изоморфизмите ни пресеците на подасоцијативите не го задржуваат својството на компатибилност. Потоа се докажуваат некои резултати, како на пр.: секоја n-подгрупа од n-асоцијатив е компатибилен подасоцијатив; секој n-подасоцијатив што е комплемент на идеал е компатибилен; фамилијата компатибилни подасоцијативи на еден асоцијатив е индуктивна; ако φ е автоморфизам на асоцијативот A и ако C е автоморфна слика при φ на некој подасоцијатив B, тогаш C е компатибилен ако и само ако B е компатибилен. Во четвртиот дел на работава се опишани компатибилните подасоцијативи на циклични асоцијативи, а на крајот се разгледуваат асоцијативи во кои секој подасоцијатив е компатибилен. Притоа се покажува дека таков асоцијатив A мора да е периодичен и индексот на секој елемент од А не е поголем од 2. Користејќи го тоа и еден претходен резултат, се докажува дека во n-групата A секој подасоцијатив е компатибилен ако и само ако A е периодична.

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