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ON (F, G)-RINGS

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The subject of this paper are algebras with finitary operations which are called here (F, G)-rings. The concept of (F, G)-algebra is introduced in the first section and it is shown there that for any such algebra A there exists a ring $A^{\wedge}$, unique up to isomorphism, called the universal ring for A. In the section 2, the concept of (F, G)-ring, i.e. (J, K)-ring is introduced and some other definitions are given. It is proved there that any K-associative is a K-subassociative of some (J, K)-ring. The main part of the paper is the section 3, where it is proved that any (J, K)-ring, which multiplicative associative is a semigroup associative, can be embedded in a ring. (This is a generalization of the main result of [1].)

1. (F, G)-Algebras. Universal ring homomorphisms

Let F and G be two nonempty disjoint sets of finitary operators and let F_n and G_n be the set of all n-ary operators in F and G respectively. We assume that $F_0 \cup F_1 = \varnothing$, $G_0 \cup G_1 = \varnothing$, and $F_n \neq \varnothing$, $G_m \neq \varnothing$ for some $n, m \geqslant 2$. If A is a non-empty set and if every n-ary operator of $F \cup G$ is interpreted as an n-ary operation on A, then the ordered triple $\mathbf{A} = A(F, G)$ is said to be an *(F, G)-algebra*.

Let $\mathbf{R} = R(+, .)$ be a ring[1]). If $\xi: A \to R$ is a mapping such that for any operators $f \in F_n$, $g \in G_m$ and for any sequences $x_1, \ldots, x_n \in A$, $y_1, \ldots, y_m \in A$ the following conditions are satisfied:

$$\xi(f x_1 \ldots x_n) = \xi(x_1) + \ldots + \xi(x_n), \tag{1.1}$$

$$\xi(g y_1 \ldots y_m) = \xi(y_1) \ldots \xi(y_m), \tag{1.2}$$

then $\xi: \mathbf{A} \to \mathbf{R}$ is said to be a *ring homomorphism*. A ring homomorphism $\lambda: \mathbf{A} \to \mathbf{A}^{\wedge}$ is said to be a *universal* one iff[2]) for any ring homomorphism $\xi: \mathbf{A} \to \mathbf{R}$ there exists a unique homomorphism $\varphi: \mathbf{A}^{\wedge} \to \mathbf{R}$, such that $\xi = \varphi\lambda$. It is clear that:

1.1. Lemma. *If $\lambda: \mathbf{A} \to \mathbf{A}^{\wedge}$ and $\mu: \mathbf{A} \to \mathbf{V}$ are universal ring homomorphisms, then there exists a unique isomorphism $\eta: \mathbf{A}^{\wedge} \to \mathbf{V}$ such that $\mu = \eta\lambda$.*

[1]) Here and further on, "ring" means an "associative ring".
[2]) "iff" stands for "if and only if".

If $\lambda : \mathbf{A} \to \mathbf{A}^{\wedge}$ is a universal ring homomorphism, then $\mathbf{A}^{\wedge}$ is called a *universal ring* for the (F, G)-algebra $\mathbf{A}$. We shall consider now the existence of universal ring homomorphisms.

1.2. Theorem. *For any (F, G)-algebra $\mathbf{A}$ there exists a universal ring $\mathbf{A}^{\wedge}$.*

Proof. Denote by $\mathbf{U}_A$ the ring wich is freely generated by the carrier of the algebra $\mathbf{A}$. Then form the ideal H, generated by the set

$$\{c \mid (\exists n \in \mathbf{N}, f \in F_n, a_1, \dots, a_n, a \in A)\ c = $$
$$= a_1 + \dots + a_n - a \text{ in } \mathbf{U}_A,\ a = f a_1 \dots a_n \text{ in } \mathbf{A}\} \cup$$
$$\cup \{d \mid (\exists m \in \mathbf{N},\ b \in G_m,\ b_1, \dots, b_m,\ b \in A)\ d =$$
$$= b_1 \dots b_m - b \text{ in } \mathbf{U}_A,\ b = g\, b_1 \dots b_m \text{ in } \mathbf{A}\}.$$

It is clear that H consists of all elements of the foım

$$\sum \varepsilon_i\, u_i (a_{i1} + \dots + a_{in_i} - a)\, v_i + \sum \varepsilon_j{}'\, u_j{}' \,(b_{j1} \dots b_{jm_j} - \mathrm{b}_j)\, v_j{}', \qquad (1.3)$$

where $\varepsilon_i,\ \varepsilon_j{}' = \pm 1,\ u_i,\ v_i,\ u_j{}',\ v_j{}' \in U_A,\ a_{i\nu},\ b_{j\mu} \in A$ and $a = f\, a_1 \dots \dots a_{in_i},\ b_j = g_j\, b_{j1} \dots b_{jm_j}$ for some $f_i \in F_{n_i},\ g_j \in G_{m_j}$ (some of the symbols $u_i, v_i, u_j{}', v_j{}'$ may be the empty symbol).

Consider the factor ring $\mathbf{U}_A/H$. It is easy to see that the mapping

$$\lambda : a \to a + H \qquad (1.4)$$

is a universal ring homomorphism from A into $\mathbf{U}_A/H$. Thus, $\mathbf{A}^{\wedge} = \mathbf{U}_A/H$ is a universal ring for the (F, G)-algebra $\mathbf{A}$.

(Note that the univerasl ring $\mathbf{A}^{\wedge}$ for an (F, G)-algebra $\mathbf{A}$, by **1.1**, is uniquely determined up to isomorphism.) ∎

Clearly, the restriction $\lambda_1 : \mathbf{A} \to \mathbf{A} + H$ of the homomorphism $\lambda : \mathbf{A} \to \mathbf{A}^{\wedge}$, defined by (1.4), is an epimorphism. If λ is a monomorphism, i.e. λ_1 is an isomorphism, then $\mathbf{A}$ is an (F, G)-subalgebra of the ring $\mathbf{A}^{\wedge}$ and so $\mathbf{A}$ can be called a *ring algebra.* We shall consider a class of ring algebras in the next secion.

2. (J, K)-rings

Let $A = A(F)$ be an F-algebra such that $F_0 \cup F_1 = \varnothing$ and $F_k \neq \varnothing$ for some $k \geqslant 2$, where F_n is the set of all n-ary operators in F. An F-algebra $A = A(F)$ is said to be an F-associative ([3], [6]) iff the following two conditions hold:

i) For any $f \in F_n$, $g \in F_m$ and $i : 1 \leqslant i \leqslant n$ the following identity is satisfied:

$$fg\, x_1 \ldots x_{m+n-1} = f\, x_1 \ldots x_{i-1}\; g\, x_i \ldots x_{m+n-1}.$$

ii) For any pair of sequences $f_1, \ldots, f_r;\ g_1, \ldots, g_s$ such that

$$f \in F_{n_i+1},\ g_j \in F_{m_j+1},\ n_1 + \ldots + n_r = n = m_1 + \ldots + m_s,$$

the following identity equality is satisfied:

$$f_1 \ldots f_r\, x_0 \ldots x_n = g_1 \ldots g_s\, x_0 \ldots x_n.$$

(In other words, $A(F)$ is an F-associative iff any two "continued products" $\Pi' x_1 \ldots x_n$ and $\Pi'' x_1 \ldots x_n$ (defined in an obvious way) are equal, i.e. iff the "general associative law" holds in $A(F)$.)

An F-associative $A = A(F)$ is also called a *J-associative*, denoted by $A = A(J)$, where

$$J = \{n - 1 \mid n \in \mathbf{N},\ F_n \neq \varnothing\}. \tag{2.1}$$

It follows by ii) that any two (n +1)-ary operations on A, for any fixd $n \in J$, are equal as mappings. No confusion will result if it is used the same symbol (ex. $*$) for all operations on $A(F)$.

If $\langle J \rangle$ is the additive subsemigroup of $\mathbf{N}(+)$ which is generated by the set J, then for any $s \in \langle J \rangle$ it can be defined an $(s+1)$-ary operation on A, and so it obtains a new algebra $A = A(\langle J \rangle)$ which is a $\langle J \rangle$ -associative. The distinction between the associatives $A\ J)$ and $A\ (\langle J \rangle)$ is not essenial for the questions we are going to discuss here and thus we may regard them as the same associative (i.e. we may consider the set J as $\langle J \rangle$, if needed).

We shall use later the following result ([5]):

2.1. Lemma. *If J is a subsemigroup of $\mathbf{N}(+)$ and $d = GCD(J)$*[1]*, then there exists a $r \in \mathbf{N}$ such that*

$$(x \in J \wedge x \geqslant r) \Leftrightarrow (x \geqslant r \wedge d|x).$$

(The subset $J_* = \{r,\ r + d,\ r + 2d, \ldots\}$ is called the *regular part* of J.) ∎

It is known ([3], [6]) that for any J-associative A there exists a semigroup $A^\wedge$, called a *universal semigroup* for A, which is unique up to isomorphism. (In fact, the semigroup $A^\wedge$ is the factor semigroup U_A/l, where U_A is the semigroup which is freely generated by the set A and l is a suitable congruence on $A(J)$.)

[1]) GCD(J) means the greatest common divisor of the elements of J.

Any semigroup can be consodered as a J-associative for any non-empty subset J of $\mathbf{N}$. If there exists a semigroup S such that a J-associative A may be regarded as a J-subassociative of S, then A is called a *semigroup associative*; in that case the universal semigroup $A^\wedge$ for A is called the *free covering semigroup* for A. If A is a semigroup J-associative and $d=GCD(J)$, then the free covering semigroup $A^\wedge$ has the following form:

$$A^\wedge = A_1 \cup A_2 \cup \ldots \cup A_d, \tag{2.2}$$

where $A \;= \{(a_1, \ldots, a\;)^l \mid a_\nu \in A\}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

A J-associative G is said to be a *J-group* iff

$$(\forall n \in J)\ (\forall a_1, \ldots, a_n,\ b \in G)\ (\exists\, x, y \in G)$$
$$* x a_1 \ldots a_n = b,\ * a_1 \ldots a_n\, y = b. \tag{2.3}$$

One defines a *commutative* J-group in an obvious way.

It is known ([3], [6]) that every J-group can be reduced to a $(d+1)$-group, where $d=GCD(J)$, i.e. it can be defined a $(d+1)$-operation [ ] on G in a such way that G becomes a $(d+1)$-group and every J-operation is an extension of the $(d+1)$-operation. (This can be effected by putting

$$[x_0 x_1 \ldots x_d] \overset{\mathrm{Df}}{=} * x_0 x_1 \ldots x_{d-1} y_0 y_1 \ldots y_r, \tag{2.4}$$

where $x_d = * y_0 y_1 \ldots y_r$ and r is as in **2.1**.) If the J-group G is commutative, then the $(d+1)$-group G is commutative too.

J-groups and $(n+1)$-semigroups are semigroup associatives ([3], [7]).

Let P be a commutative J-group in which the J-operations are denoted by $\boxed{+}$, and a K-associative in which the operations are denoted by $*$. The algebra $P = P(\boxed{+}, *)$ is said to be a *(J, K)-ring* iff

$$* x_1 \ldots x_i (\boxed{+}\, y_0 y_1 \ldots y_m)\, x_{i+1} \ldots x_n = \tag{2.5}$$
$$= \boxed{+}\, (* x_1 \ldots x\;\; y_0\, x_{\;+1} \ldots x_n) \ldots (* x_1 \ldots x\;\; y_m\, x_{i+1} \ldots x_n),$$

for any $x_\nu,\ y_\mu \in P$, $m \in J$, $n \in K$ and $i \in \{0, 1, \ldots, n\}$.

The equalities (2.5) are called *distributive laws* of the K-operations $*$ over the J-operations $\boxed{+}$. Thus, a (J, K)-ring P si an (F, G)-algebra such that $P(F)$ is a commutative J-group, $P(G)$ is a K-associative and the G-operations on P are distributive over the F-operations on P. We call $P(G)$ the *multiplicative associative* of the (J, K)-ring P.

The concept of (J, K)-ring is a generalization of $(m+1,\ n+1)$-ring (a concept which is introduced in [1] and, in a special case, in [2]). For, if $J = \langle m \rangle$ and $K = \langle n \rangle$, then P is a commutative $(m+1)$-group and an $(n+1)$-semigroup for which (2.5) hold. If $m = 1 = n$, then a (2,2)-ring is a binary ring or simply a ring. The sence of the terms $(d+1,\ K)$-ring and $(2, K)$-ring is clear.

A generalization of a theorem concerning semigroups and rings is the following theorem.

2.2. Theorem. *Let J and K be nonempty subsets of* $\mathbf{N}$. *If A is a K-associative, then A is a K-subassociative*[1]) *of the multiplicative associative of some* (J, K)*-ring.*

Proof. It suffices to show that the theorem holds for $(2, K)$-rings. Let A be a K-associative and let P be the abelian group whch is freely generated by the set A. Denote any element $x = n_1 a_1 + \ldots + n_k a_k$ of P shortly by $\sum_{\nu} n_\nu a_\nu$ and put

$$\left[\sum_{\nu_0} k_{0\nu_0} a_{0\nu_0}, \sum_{\nu_1} k_{1\nu_1} a_{1\nu_1}, \ldots, \sum_{\nu_n} k_{n\nu_n} a_{n\nu_n}\right] \overset{Df}{=}$$

$$\overset{Df}{=} \sum_{\nu_0, \ldots, \nu_n} k_{0\nu_0} \ldots k_{n\nu_n} [a_{0\nu_0}, \ldots, a_{n\nu_n}], \tag{2.6}$$

where $n \in K$ and $[x_0, x_1, \ldots, x_n]$ stands for $* x_0 x_1 \ldots x_n$.

Since A is an associative and the set A generates P, it is clear that any two continued products $\Pi' x_0 x_1 \ldots x_n$ and $\Pi'' x_0 x_1 \ldots x_n$ in P are equal, i.e. P with (2.6) becomes a K-associative.

It is a matter of simple computation to see that, for any $x_0, x_1, \ldots, x_n$, $y, z \in P$ $(n \in K)$, where $x = \sum_{\nu_i} k_{\nu_i} a_{\nu_i}$ $(i = 0, 1, \ldots, n)$, $y = \sum_{\mu} r_\mu a_\mu$, $z = \sum_{\lambda}{}' s_\lambda a_\lambda$, using (2.6), one obtains

$$* x_0 \ldots x_{i-1} (y + z) x_{i+1} \ldots x_n =$$

$$= (* x_0 \ldots x_{i-1} y x_{i+1} \ldots x_n) + (* x_0 \ldots x_{i-1} z x_{i+1} \ldots x_n),$$

i.e. that the K-operations $*$ are distributive over $+$ in P. Thus P is a $(2, K)$ -ring. ∎

3. The embedding of (J, K)-rings into binary rings

We shall investigate now the question of "embedding" of a (J, K)-ring into a binary ring by the following plan: first, we shall reduce the (J, K)-ring P to a $(d+1, K)$-ring, then we shall „embed" the $(d+1, K)$-ring into a $(2, K)$-ring and, finaly, we shall "embed" the $(2, K)$-ring, which multiplicative associative is a semigroup associative, into a ring. According to the transitivity of "embeddings", we may consider the stated problem solved.

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1) A nonempty subset B of a K-associative A is called a K-subassociative of A iff $(\forall n \in K)\ (\forall\, b_0, \ldots, b_n \in B)\ * b_0 \ldots b_n \in B$

In order to shorten some expressions in formulas, we shall assume the following notation:

1) x_1^k or $(x_j)_{j=1}^k$ for $x_1 x_2 \ldots x_k$,

2) $\underset{j=1}{\overset{m}{\boxplus}} y_j$ or $\boxplus\, y_0^m$ for $\boxplus\, y_0 y_1 \ldots y_m$ (in J-groups),

3) $\underset{j=1}{\overset{n}{*}} x_j$ for $* x_0 x_1 \ldots x_n$ (in K-associatives),

4) $\sum_{j=1}^{k} x_j$ for $x_1 + x_2 + \ldots + x_k$ (in additive groups).

Using 1) and 2), the distributive laws (2.5) can be written in the following form:

$$* (x_1^i \boxplus y_0^m\, x_{i+1}^{\ n}) = \underset{j=0}{\overset{m}{\boxplus}} (* x_1^i\, y_j\, x_{i+1}^{\ n}). \tag{3.1}$$

3.1. Lemma. *If P is a (J, K)-ring, then P is a $(\langle J \rangle, \langle K \rangle)$-ring too.*

Proof. Since P is a commutative J-group and a K-associative, it follows ([3]) that P is a commutative $\langle J \rangle$ l-group and a $\langle K \rangle$ -associative. It remains to verify only the distributive laws (3.1), and it can be done by induction. ∎

3.2. Lemma. *Let P be a (J, K)-ring. If the J-group P is reduced to a $(d+1)$-group, $d = GCD(J)$, then it obtains an algebra $P(d+1, K)$, which is a $(d+1, K)$-ring.*

Proof. As we noted in the previous section (see (2.4)), the J-group P can be reduced to a $(d+1)$-group. Denote by $[\ \ ]$ the $(d+1)$-operation on P. It is necessary to check only the distributivity of the K-operations over the $(d+1)$-operation. Let $x_1, \ldots, x_n$, $y_0, \ldots, y_d$ $(n \in K)$ be any elements of P, and let $y_0 = \boxplus\, c_0^r$ in the J-group P (where $r \in J_*$, as in **2.1.**) Then

$* (x_1^i\, [y_0\, y_1 \ldots y_d]\, x_{i+1}^{\ n}) = * (x_1^i\, [\boxplus\, c_0^r\, y_1^d]\, x_{i+1}^{\ n}) =$

(here $r + d = s \in J$ and therefore $[\boxplus\, c_0^r\, y_1^d] = \boxplus\, c_0^r\, y_1^d$

is a „sum" in the J-group P; thus (3.1) can be applied here and after some simple computation it will be obtained:)

$= [(* x_1^i\, y_0\, x_{i+1}^{\ n})\, (* x_1^i\, y_1\, x_{i+1}^{\ n}) \ldots (* x_1^i\, y_d\, x_{i+1}^{\ n})]$,

which means that the K-operations are distributive over the $(d+1)$-operation on P. Hence the obtained algebra $P(d+1, K)$ has the $(d+1, K)$-ring structure. ∎

Let P be a (J, K)-ring and let $J' \subseteq J$, $K' \subseteq K$. A subset $P' \subseteq P$ is said to be a (J', K')-*subring* of the (J, K)-ring P iff:

a) P' is a K'-subassociative of the K-associative P (i. e. for any $n' \in K'$ and $a_0, \ldots, a_{n'} \in P'$ it follows that $* a_0 \ldots a_{n'} \in P'$),

b) P' is a J'-subgroup of the J-group P (i.e. P' is a J'-subassoiative of P and all equations in P' of the form (2.3) are solvable in P').

(Clearly, any (J', K')-subring P' of a (J, K)-ring can be considered as a (J, K)-ring.)

Since every ring R can be regarded as a (J, K)-ring for any nonempty subsets J, K of $\mathbf{N}$, the concept of (J, K)-subring of a ring is clear.

If P is a (J, K)-subring of a ring R and if the set P generates R, we call the ring R a *covering* of the (J, K)-ring P. If a ring R is a covering of a (J, K)-ring P and all coverings of P are homomorphic images of R, then R is called the *free covering ring for P*.

It is convenient here to use the "language" of the previous sections. Namely, if $\xi : P \to R$ is a ring homomorphism of a (J, K)-ring P into a ring R which is a monomorphism, then P can be considered as a (J, K)-subring of R. We say in that case that P is isomorphically *embeddeble* into the ring R and we call the monomorphism ξ an *embedding*. It is clear the sense of the expression "embedding of a $(d+1, K)$-ring into a $(2, K)$-ring".

3.3. Lemma. *Any $(d+1, K)$-ring can be embedded into a $(2, K)$-ring.*

Proof. Let P be a $(d+1, K)$-ring and let $M(+)$ be the free covering group of the commutative $(d+1)$-group $P = P([\ \ ])$.[1]) The K-operations $*$ of $P(K)$ are defined on the set M only partially, namely just for the elements of P. We shall extend these partially defined K-operations on the whole set M, step by step, requiring the distributivity over the operation $+$ (in the group M).

Let $a_1, \ldots, a_n \in P$ and $u = b_0 + \ldots + b_k \in M$, where $n \in K$ and $b_\nu \in P$, and put

$$* a_1^i \, u \, a_{i+1}^n \overset{Df}{=} \sum_{j=1}^{k} (* a_1^i \, b_j \, a_{i+1}^n). \tag{3.2}$$

The right-hand side of (3.2) is defined since $*$ is a K-operation on P and it does not depend on the representation of $u \in M$ as a sum of elements of P. For, let $u = c_0 + \ldots + c_k$ $(c_j \in P)$. Since $c_0 + \ldots + c_k = b_0 + \ldots + b_k$ is an equality in the free covering group of the $(d+1)$-group P, it follows that in $M(+)$,

$$c_0 + \ldots + c_k + x_1 + \ldots + x_{d-k} = b_0 + \ldots + b_k + x_1 + \ldots + x_{d-k},$$

for some $x_1, \ldots, x_{d-k} \in P$, and thus $[c_0^k \, x_1^{d-k}] = [b_1^k \, x_1^{d-k}]$ in the $(d+1)$-group P and

$$* a_1^i \, [c_0^k \, x_1^{d-k}] \, a_{i+1}^n = * a_1^i \, [b_1^k \, x_1^{d-k}] \, a_{i+1}^n.$$

[1]) i.e. ([7]): $M = P \cup 2P \cup \ldots \cup dP$, where $iP \cap jP \neq \emptyset \Rightarrow i = j$ and if $a_\nu, b_\nu \in P$, the $(a_0 + \ldots + a_i = b_0 + \ldots + b_i \Leftrightarrow (\exists\, c_1, \ldots, c_{n-i} \in P)\ [a_0 \ldots a_i \, c_1 \ldots c_{n-i}] = [b_0 \ldots b_i c_1 \ldots \ldots c_{n-i}])$.

Using the distributive laws of the K-operations $*$ over the $(d+1)$-operation [ ] one obtains an equality in $P(d+1)$, wich can be written in the group $M(+)$ in the form

$$\sum_{j=0}^{k} (* a_1^i c_j a_{i+1}^{\ n}) + \sum_{j=1}^{d-k} (* a_1^i x_j a_{i+1}^{\ n}) =$$
$$= \sum_{j=0}^{k} (* a_1^i b_j a_{i+1}^{\ n}) + \sum_{j=1}^{n} (* a_1^i x_j a_{i+1}^{\ n}).$$

This equality implies (after the cancellation) the desired result. Thus the K-operations $*$ (still partial in M) are well-defined by (3.2).

Let $a_0, a_1, \ldots, a_n \in P$ and $u, v \in M$, where $n \in K$, $u = b_0 + \ldots + b_k$ and $v = c_0 + \ldots + c_r$ $(b_\mu, c_\nu \in P)$ and put

$$* a_0^{i-1} u a_{i+1}^{j-1} v a_{j+1}^{\ n} = \sum_{\nu=0}^{r} (* a_0^{i-1} u a_{i+1}^{j-1} c_\nu a_{j+1}^{\ n}) =$$
$$= \sum_{\nu=0}^{r} \sum_{\mu=0}^{k} (* a_0^{i-1} b_\mu a_{i+1}^{j-1} c_\nu a_{j+1}^{\ n}). \tag{3.3}$$

It is easy to show that the K-product given by the left-hand side of (3.3) is uniquely determined. Assume that the products $* u_1^i a u_{i+1}^{\ n}$, for any $u_1, \ldots, u_n \in M$ $(n \in K)$, $a \in P$ and $i \in \{0, 1, \ldots, n\}$ are uniquey determined and that $*$ (as a partial K-operation on M) is distributive over the $(d+1)$-operation [ ]. Let $u \in M$ and $u = b_0 + \ldots + b_k$ $(b_j \in P)$ and define

$$* u_1^i u u_{i+1}^{\ n} \stackrel{Df}{=} \sum_{j=0}^{k} (* u_1^i b_j u_{i+1}^{\ n}). \tag{3.4}$$

By the same argument as for (3.2) one obtains that $*$ are well-defined by (3.4). Thus, the K-operations of the K-associative P are extended to K-operations on the set M. Since $P(K)$ is a K-associative and P is a generating set for the group M, it follows that $M(K)$ is a K-associative.

By straightforward verification one can see that the K-operations $*$ on M are distributive over the operation $+$.

Thus, M is a $(2, K)$-ring and the given $(d+1, K)$-ring P is a $(d+1, K)$-subring of M, i. e. P is embedded into a $(2, K)$-ring. ∎

Now we turn to the problem of embedding of a $(2, K)$-ring into a (binary) ring. First we shall prove three lemmas.

Let $M(\oplus, *)$ be a $(2, K)$-ring, where the multiplicative associative $M(*)$ is a semigroup K-associative. Denote by $S = S(\cdot)$ the free covering semigroup of the K-associative $M(*)$ and by $U = U_S(+, .)$ the free S-ring (i.e. the free ring over the semigroup S). Let T[1]) be the subgroup of the group $U(+)$ which is generated by the set M.

[2]) From now on, the meaning of M, P, S, T and also H (which will be introduced later by (3.5)) remains unchanged to the end of the paper.

3.4. Lemma. *The subgroup T is freely generated by the set M, and T, as a subset of U, is a $(2, K)$-subring of the ring U.*

Proof. The set M is a subset of the set S, which freely generates the group $U(+)$. Therefore M is a set of free generators for the subgroup T.

If $t_0, \ldots, t_n \in T$ $(n \in K)$ and if $t_\nu = \sum \varepsilon_{i\nu} a_{i\nu}$ $(a_{i\nu} \in M)$, then

$$t_0 t_1 \ldots t_n = \sum_{i, j_0, \ldots, j_n} (\varepsilon_{ij_0} \ldots \varepsilon_{ij_n}) (* a_{ij_0} \ldots a_{ij_n}) = \sum_\nu \varepsilon_\nu a_\nu,$$

where $a_\nu = (* a_{ij_0} \ldots a_{ij_n}) \in M$. Therefore the set T is closed under the K-operations. Since the K-operations on T are "extensions" of the multiplication in the ring U which is an associative operation, it follows that T is a K-associative. The distributivity of $\cdot$ over $+$ implies the distributivity of $*$ over $+$ in T. Hence $T(+, *)$ is a $(2, K)$-subring of the ring U. ∎

As a corollary of **3.4.** we obtain the following lemma.

3.5. Lemma. *The identity mapping on the set M, $1_M(a) = a$ $(a \in M)$, can be extended to a homomorphism ψ from the group $T(+)$ onto the group $M(\oplus)$. Here ψ can be treated as a homomorphism from the $(2, K)$-ring $T(+, *)$ into the $(2, K)$-ring $M(\oplus, *)$.*

Proof. The first part follows by the fact that $T(+)$ is freely generated by the set M, and the second part — since the K-operations on T are extensions of the corresponding K-operations on M, i.e. they are "the same" operations on the set M. ∎

Denote by H the set of all elements of the ring U which can be represented as finite sums of elements of the form $\varepsilon u (a + b - c) v$, where $\varepsilon = \pm 1$, $u, v \in U$ and $c = a \oplus b$ in the group $M(\oplus)$ (it is allowed u or v to be the empty symbol), i.e.

$$x \in H \Leftrightarrow x = \sum_{i=1}^{k} \varepsilon_i u_i (a_i + b_i - c_i) v_i, \tag{3.5}$$

where $\varepsilon_i = \pm 1$, $u_i, v_i \in U$, $c_i = a_i \oplus b_i$ in $M(\oplus)$ (see also (1.3)). Clearly H is an ideal of the ring U. We shall show that:

3.6. Lemma. *The ideal H, defined by (3. 5), separates the elements of M, i.e. $(a, b \in M \wedge b \in a + H \Rightarrow a = b)$.*

Proof. Let $a, b \in M$ and $b \in a + H$. Then an equality of the form

$$b = a + \sum \varepsilon_\nu a_{1_\nu} \ldots a_{i_\nu} (c_\nu + d_\nu - e_\nu) a_{(i+1)_\nu} \ldots a_{n_\nu} \tag{3.6}$$

holds in the ring $U(+, .)$, where $c_\nu \oplus d_\nu = e_\nu$ in the group $M(\oplus)$, $a_{j_\nu} \in M$ and $\varepsilon_\nu = \pm 1$. Since $U(+)$ is the free commutative group which is freely generated by the set S (where $S(\cdot)$ is the free covering semigroup of the associative $M(K)$), it follows that (3.6) is an identity equality. The right-hand side of (3.6) is in the $(2, K)$-ring M (since $b \in M$), and thus we may regard it as a sum of "admissible products" (i. e. K-products), since all n_v-products, where $n_\nu \notin K$, mutually cancel. Therefore we may assume that $n_\nu \in K$ in (3.6).

Let $\psi : T \to M$ be the homomorphism from the $(2, K)$-ring T into the $(2, K)$-ring M defined in **3.5.** If c, d, e are any elements of M such that $c \oplus d = e$ in $M(\oplus, *)$, then

$$\psi(c + d - e) = \psi(c + d) \ominus \psi(e) = c \oplus d \ominus e = o^{\wedge}, \tag{3.7}$$

where $\ominus$ is the subtraction and $o^{\wedge}$ is the zero in the group $M(\oplus)$. Considering the fact that $o^{\wedge} \oplus o^{\wedge} = o^{\wedge}$ and the distributive laws of the K-operations $*$ over $\oplus$, we obtain

$$o^{\wedge} \oplus (* \, a_1^i \, o^{\wedge} \, a_{i+1}^{\,n}) = * \, a_1^i \, o^{\wedge} \, a_{i+1}^{\,n} = * \, a_1^i \, (o^{\wedge} \oplus o^{\wedge}) \, a_{i+1}^{\,n} =$$

$$= (* \, a_1^i \, o^{\wedge} \, a_{i+1}^{\,n}) \oplus (* \, a_1^i o^{\wedge} \, a_{i+1}^{\,n}),$$

and by cancellation in the group $M(\oplus)$:

$$* \, a_1^i \, o^{\wedge} \, a_{i+1}^{\,n} = o^{\wedge}. \tag{3.8}$$

By (3.6), using (3.7) and (3.8), we obtain:

$$b = \psi(b) = \psi(a + \textstyle\sum \ldots) = \psi(a) \oplus \psi(\textstyle\sum \ldots) =$$

$$= a \oplus \textstyle\sum \varepsilon_\nu \, a_{1_\nu} \ldots a_{i_\nu} \, o^{\wedge} \, a_{(i+1)_\nu} \ldots a_{n_\nu} = a \oplus o^{\wedge} = a.$$

Thus H separates the elements of M. ∎

Note that there exist (J, K)-rings which multiplicative associative is not a semigroup associative; such a (J, K)-ring can be obtained if A in **2.2** is not a semigroup associative. Now we can prove:

3.7. Lemma. *If the multiplicative associative of a $(2, K)$-ring M is a semigroup associative, then M can be embedded into a ring.*

Proof. Let R be the factor ring of the ring U over the ideal H, $R = U/H$ (where H and T are as before, in **3.4** — **3.6**). By **3.6**, the mapping $\xi : a \to a + H$ of the $(2, K)$-ring M into the ring R is a monomorphism. Thus M can be considered as a $(2, K)$-subring of the ring R $(M = \overline{M} = \{a + H \,|\, a \in M\})$. ∎

As a consequence of **3.2**. **3.3** and **3.7**, we obtain the following

3.8. Theorem. *If the multiplicative associative of a (J, K)-ring P is a semigroup associative, then P can be embedded into a (binary) ring.* ∎

Note that the factor ring $R = U/H$ is a covering of the (J, K)-ring P. Thus we may regard R as generated by the set P and, moreover, P is a (J, K)-subring of R.

By the construction of the covering U/H for the (J, K)-ring P follows that U/H is, in fact, the universal ring $P^{\wedge}$ for $P(J, K)$ which, by **1.1**, is uniquely determined. Also, any covering of $P(J, K)$ is a homomorphic image of the ring P. Therefore P may be called a *maximal covering* (ring) of $P(J, K)$.

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Наум Целакоски

ЗА (F, G) — ПРСТЕНИТЕ

Резиме

Нека F и G се непразни дисјунктни множества од финитарни операции во кои нема нуларни и унарни. Ако A е непразно множество и ако секој n-арен оператор од $F \cup G$ се интерпретира како n-арна операција на A, тогаш подредената тројка $\mathbf{A} = A(F, G)$ се вика (F, G)-*алгебра*.

Се покажува дека на секоја (F, G)-алгебра може, на соодветен начин, да ѝ се придружи бинарен прстен (еднозначно определен до изоморфизам).

Една (F, G)-алгебра P се вика (J, K)-прстен акко[1] $P(F)$ е комутативна J-група ([3]), $P(G)$ е K-асоцијатив ([3]) и G-операциите на P се дистрибутивни спрема F-операциите на P (т.е. важат идентитетите (2.5)). $P(G)$ се вика *мултипликативен асоцијатив* на (J, K)-прстенот P. Се покажува дека: ако J и K се непразни подмножества од N и A е K-асоцијатив, тогаш A е K-подасоцијатив од мултипликативниот асоцијатив на некој (J, K)-прстен. Ако P е (J, K)-прстен, тогаш P е и $(\langle J \rangle, \langle K \rangle)$-прстен (каде што $\langle X \rangle$ означува потполугрупата од $\mathbf{N}(+)$ што е генерирана од множеството X).

Главниот дел на работава е сместување на (J, K)-прстен, чијшто мултиплит кативен асоцијатив е полугрупен асоцијатив ([3], [6]), во бинарен прстен. Проблемо: е решаван како соодветениот проблем за $[m, n]$-прстените ([1]), во неколку етапи-прво, (J, K)-прстенот P се сведува на $(d + 1, K)$-прстен, потоа $(d + 1, K)$-прстенот се сместува во $(2, K)$-прстен и, на крајот, $(2, K)$-прстенот се сместува во бинарен пр. стен; со тоа може да се смета дека (J, K)-прстенот P е сместен во бинарен прстен (т.е- P може да се смета за (J, K)-потпрстен од бинарен прстен).

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[1]) „акко" стои наместо „ако и само ако".