

n-SUBSEMIGROUPS OF PERIODIC SEMIGROUPS

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An algebra $Q []$ with an $n + 1$ -ary operation

$$[]: (x_0, x_1, \dots, x_n) \rightarrow [x_0 x_1 \dots x_n]$$

is said to be an n -subsemigroup of a semigroup S if $Q \subseteq S$ and

$$[a_0 a_1 \dots a_n] = a_0 a_1 \dots a_n$$

for all $a_0, \dots, a_n \in Q$.

A description of the class of n -subsemigroups of periodic semigroups is given in this note.

0. First we state some preliminaries.

0.1. $Q []$ is an n -subsemigroup of a semigroup iff¹⁾ $Q []$ is an n -semigroup, i.e. an algebra with an associative $n + 1$ -ary operation. Every commutative n -semigroup is an n -subsemigroup of a commutative semigroup. (These results, and convenient descriptions of some other classes of n -semigroups, can be found in [2]). Further on we shall usually say „ n -semigroup Q “ instead of „ n -semigroup $Q []$ “. A continued product on a sequence $(a_0, a_1, \dots, a_{sn}) \in Q^{sn+1}$ will be denoted by $[a_0 a_1 \dots a_{sn}]$; if $s = 0$, then $[a] = a$.

0.2. A semigroup S is said to be periodic iff every cyclic subsemigroup is finite. Then, to every element $a \in S$ are associated two positive integers $r = r_a$ (the index of a), $m = m_a$ (the period of a) such that:

$$a^i = a^j \Leftrightarrow i = j \text{ or } (i \geq r, j \geq r, i \equiv j \pmod{m})$$

(For example, [1], p. 19.) A periodic semigroup is said to have a finite index (period) iff the set of indices (periods) is finite. An n -semigroup Q is said to be periodic iff every cyclic subalgebra of Q is finite.

¹⁾ „iff“ means „if and only if“.

0.3. Usually, by x, y, \dots, x_1, \dots will be denoted individual variables, and by $\mathbf{x}, \mathbf{y}, \mathbf{z}$ sequences of variables. If $\mathbf{x} = x_1 \dots x_k$, then $|\mathbf{x}| = k$.

1. The main result of this note is the following:

Theorem. An n -semigroup Q is an n -subsemigroup of a periodic semigroup iff the following statement is satisfied:

$$(\forall \mathbf{x}) (\exists r \geq 0, m > 0) (\forall \mathbf{y}, \mathbf{z}) [\mathbf{y} \mathbf{x}^r \mathbf{z}] = [\mathbf{y} \mathbf{x}^{r+mn} \mathbf{z}]^2. \quad (1.1)$$

Proof. Let Q be an n -subsemigroup of a periodic semigroup S . If $a_1, \dots, a_k \in Q$ and if r is the index and m the period of $b = a_1 \dots a_k$, then $b^r = b^{r+mn}$, and this implies that

$$[\mathbf{y} b^r \mathbf{z}] = [\mathbf{y} b^{r+mn} \mathbf{z}]$$

for every pair \mathbf{y}, \mathbf{z} of sequences of elements of Q such that $rk + |\mathbf{y}| + |\mathbf{z}| \equiv 1 \pmod{n}$. Thus, Q satisfies the condition (1.1).

Assume now that the n -semigroup Q satisfies the statement (1.1). Let Q^\wedge be the universal covering semigroup of Q ([3] p. 25). We note that Q is a generating n -subsemigroup of Q^\wedge and that the following statement is satisfied:

$$a_0, \dots, a_s, b_0, \dots, b_t \in Q, a_0 \dots a_s = b_0 \dots b_t \Rightarrow s \equiv t \pmod{n}. \quad (1.2)$$

By (1.1), for every sequence $\mathbf{a} = (a_1, \dots, a_k) \in Q^k$, there exist $r = r_{\mathbf{a}} \geq 0, m = m_{\mathbf{a}} > 0$ such that

$$[\mathbf{y} \mathbf{a}^r \mathbf{z}] = [\mathbf{y} \mathbf{a}^{r+mn} \mathbf{z}]$$

for every pair \mathbf{y}, \mathbf{z} of sequences of elements of Q such that $rk + |\mathbf{y}| + |\mathbf{z}| \equiv 1 \pmod{n}$. (We can assume that $r_{\mathbf{a}}, m_{\mathbf{a}}$ are the least numbers with the mentioned property).

Let τ_0 be a relation defined in Q^\wedge in the following way. If $a_1, \dots, a_k, c_1, \dots, c_q \in Q$ and $1 \leq j \leq q+1$, then

$$c_1 \dots c_{j-1} (a_1 \dots a_k)^r c_j \dots c_q \tau_0 c_1 \dots c_{j-1} (a_1 \dots a_k)^{r+mn} c_j \dots c_q,$$

where r and m are the numbers which are associated to the sequence $\mathbf{a} = (a_1, \dots, a_k)$. Let τ_1 be defined by:

$$u \tau_1 v \Leftrightarrow u \tau_0 v \text{ or } u = v \text{ or } v \tau_0 u, (u, v \in Q^\wedge)$$

and τ be the transitive extension of τ_1 , i. e. τ is defined by:

$$u \tau v \Leftrightarrow (\exists u_1, \dots, u_p) u \tau_1 u_1 \tau_1 u_2 \tau_1 \dots \tau_1 u_p \tau_1 v.$$

Then τ is a congruence on Q^\wedge , and $S = Q^\wedge / \tau$ is a periodic semigroup.

²⁾ „ $[\mathbf{x}] = [\mathbf{y}]$ “ is an abbreviation for „ $|\mathbf{x}| \equiv |\mathbf{y}| \pmod{n} \Rightarrow [\mathbf{x}] = [\mathbf{y}]$ “.

If $a \in Q$, $u \in Q^\wedge$ and $a \tau_0 u$, then there exist $c_0, \dots, c_q, a_1, \dots, a_k \in Q$, and $j \in \{0, \dots, q+1\}$ such that:

$$a = c_0 \dots c_{j-1} (a_1 \dots a_k)^r c_j \dots c_q,$$

$$u = c_0 \dots c_{j-1} (a_1 \dots a_k)^{r+mn} c_j \dots c_q,$$

where r and m are the numbers associated to $\mathbf{a} = (a_1, \dots, a_k)$.

From the first of the last two equations (by (1.2)) it follows that $q + rk \equiv 0 \pmod{n}$, i.e. $q + rk + kmn \equiv 0 \pmod{n}$, and this, by (1.1), implies that:

$$\begin{aligned} a &= c_0 \dots c_{j-1} (a_1 \dots a_k)^r c_j \dots c_q = \\ &= [c_0 \dots c_{j-1} (a_1 \dots a_k)^r c_j \dots c_q] = [c_0 \dots c_{j-1} (a_1 \dots a_k)^{r+mn} c_j \dots c_q] \\ &= c_0 \dots c_{j-1} (a_1 \dots a_k)^{r+mn} c_j \dots c_q \\ &= u. \end{aligned}$$

It can be shown in the same way that: $a \in Q$, $u \in Q^\wedge$, $u \tau_0 a \Rightarrow u = a$.

Thus we have:

$$a \in Q, u \in Q^\wedge \Rightarrow (a \tau_1 u \Leftrightarrow a = u),$$

and this implies that:

$$a, b \in Q \Rightarrow (a \tau b \Leftrightarrow a = b). \quad (1.3)$$

From (1.3) it follows that the given n -semigroup Q can be embedded (as an n -subsemigroup) in the semigroup S , and this completes the proof.

We also note that the semigroup S satisfies the statement (1.2).

2. Most of the statements stated below are obvious corollaries from Theorem or its proof.

2.1. An n -semigroup Q is an n -subsemigroup of a periodic semigroup with a finite index iff:

$$(\exists r \geq 0) (\forall \mathbf{x}) (\exists m > 0) (\forall \mathbf{y}, \mathbf{z}) [\mathbf{y} \mathbf{x}^r \mathbf{z}] = [\mathbf{y} \mathbf{x}^{r+mn} \mathbf{z}]. \quad (2.1)$$

2.2. An n -semigroup Q is an n -subsemigroup of a periodic semigroup with a finite period iff:

$$(\exists m > 0) (\forall \mathbf{x}) (\exists r \geq 0) (\forall \mathbf{y}, \mathbf{z}) [\mathbf{y} \mathbf{x}^r \mathbf{z}] = [\mathbf{y} \mathbf{x}^{r+mn} \mathbf{z}]. \quad (2.2)$$

2.3. An semigroup Q is an n -subsemigroup of a semigroup with a finite index and a finite period iff:

$$(\exists r \geq 0, m > 0) (\forall x, y, z) [y x^r z] = [y x^{r+mn} z]. \quad (2.3)$$

2.4. An n -semigroup Q is an n -subsemigroup of a periodic group iff:

$$(\forall x) (\exists m > 0) (\forall y, z) [y z] = [y x^{mn} z]. \quad (2.4)$$

Then, Q is an n -group.

2.5. If $m > 0$ and $m \equiv 0 \pmod{n}$ then an n -semigroup is an n -subsemigroup of a group with an exponent a divisor of m iff

$$(\forall x, y, z) [y z] = [y x^m z] \quad (2.5)$$

2.6. If Q is an n -subsemigroup of a periodic semigroup, then Q is a periodic n -semigroup.

2.7. The class of n -subsemigroups of periodic commutative semigroups is equal to the class of periodic commutative n -semigroups.

Proof. If Q is a periodic commutative n -semigroup, then (by 0.1) Q is an n -subsemigroup of a commutative semigroup S , and it can be assumed that S is generated by Q . Then S is periodic.

We notice also that the propositions **2.1** — **2.5** have obvious analogies in the case of commutativity.

Example. Let $A = \{a_k \mid k \in \mathbf{Z}\}$, $B = \{b_k \mid k \in \mathbf{Z}\}$ be two disjoint sets and $k \rightarrow a_k, k \rightarrow b_k$ be two injections from \mathbf{Z} (the set of integers) into $G = A \cup B$. If we define a binary operation on G by:

$$a_i a_j = a_{i+j}, a_i b_j = b_{i+j}, b_i a_j = b_{i-j}, b_i b_j = a_{i-j},$$

then we obtain a group in which B is a ternary subsemigroup. B is periodic, but G is not periodic. It can be easily shown that if B is a ternary subsemigroup of a semigroup S , then the inclusion mapping of B into G can be extended to an injective homomorphism of G into S , and this implies that S is not periodic.

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n -ПОТПОЛУГРУПИ ОД ПЕРИОДИЧНИ ПОЛУГРУПИ

Резиме

Во трудов е даден опис на класата n -потполугрупи од периодични полугрупи.