Algebraic conference
Skopje 1980

BI-IDEAL SEMIGROUPS

B. Trpenovski

We call a semigroup S a bi-ideal semigroup iff all subsemigroups of S are bi-ideals in S, i.e. $B \subseteq S$, $B^2 \subseteq B \Rightarrow BSB \subseteq B$. Bi-ideal semigroups were introduced in [6] in an analogous way as the left-ideal semigroups were introduced and studied in [3] and [7]. It seems, however, that the way the structure of left-ideal semigroups is described in [3] and [7] is not appropriate in the case of bi-ideal semigroups. So, we explore here the idea from [2] to give a structural description for bi-ideal semigroups. First, let us quote some of the results from [6]:

<u>Theorem 1.</u> Let S be a bi-ideal semigroup. Then the following hold:

(i) $(\forall a \in S)$ $aSa \subseteq \langle a \rangle$, where $\langle a \rangle$ is the cyclic subsemigroup of S generated by a;

(ii) S is periodic and $|\langle a \rangle| \leq 5$ for all $a \in S$;

(iii) The set E of all the idempotents of S is a rectangular band;

(iv) $(\forall e \in E)(\forall x \in S)$ $xe, ex \in E$. ∎

In what follows we suppose S to be a bi-ideal semigroup.

Let us put $P = S \setminus E$, where E is as in Theorem 1. We shall establish some properties about P and S.

a) It is easily seen that P is a partial semigroup, i.e. $(\forall x,y,z \in P)$ if one of the elements $(xy)z$ and $x(yz)$ belongs to P, then $(xy)z, x(yz) \in P$ and $(xy)z = x(yz)$. ■

From Theorem 1 it follows that

b) $(\forall x \in P)(\exists m \in N)$, where N is the set of positive integers, such that $x^m \notin P$. (In fact, $(\forall x \in P)$ $x^5 \notin P$). Because of this property we may call P a periodic partial semigroup. ■

A subset R of a partial semigroup Q is said to be a partial subsemigroup of Q iff $[x,y \in R$ and $xy \in Q$, then $xy \in R]$. A partial subsemigroup R of a partial semigroup Q is said to be a bi-ideal in Q iff $x,y \in R$, $xqy \in Q$, $q \in Q$, implies $xqy \in R$. If all partial subsemigroups of a partial semigroup Q are bi-ideals in Q, then we call Q a partial bi-ideal semigroup. We can show, now,

c) P is a partial bi-ideal semigroup.

Really, if B is a partial subsemigroup of P, $x,y \in B$ and $xpy \in P$, $p \in P$, then $B^* = \langle B \rangle$ in S is a bi-ideal in S and therefore $xpy \in B^*$. But, from $B^* \setminus B \subseteq E$ and $P \cap E = \emptyset$ it follows that $xpy \in B$. ■

Let e_x be the idempotent in $\langle x \rangle$ and let us put $\phi(x) = e_x$. Then,

d) $\phi : P \to E$ is a homomorphism.

If $xy=z$, $x,y,z \in P$, then $zx=xyx \in \langle x \rangle$ in S, i.e. $zx= =x^k$, $k \in \{1,2,3,4,5\}$. Let $x^m=e_x$. From $zx=x^k$ it follows that $zx^m=x^{m+k-1}$, i.e. $ze_x=e_x$, since $x^{m+k-1} = e_x x^{k-1}$ is a idempotent (th. 1 (iv)) which belongs to $\langle x\rangle$. Now, $z^2e_x=ze_x=e_x$, $z^3e_x=e_x$ and so on, so that $e_ze_x=e_x$. Similarly we have $e_ye_x=e_x$ and then $e_xe_y=e_ze_xe_ye_z \in \langle e_z\rangle$ so that $e_xe_y=e_z$ which means that $\phi(x)\phi(y) = \phi(xy)$. ■

Let $x,y \in S$ and $xy \in E$. Then in a similar way as above we can prove that $ee_x=e_x$ and $e_ye=e_y$ where $e=xy$. Now, $xy=e=ee_xe_ye=e_xe_y=\phi(x)\phi(y)$:

e) $x,y \in S,\ xy \in E \Longrightarrow xy = \phi(x)\phi(y)$. ■

If we put $\phi(e)=e$ for all $e \in E$, then from the definition of ϕ and e) it follows that we can extend $\phi: S \to E$ to be an epimorphism.

Conversely, assume that P is a periodic partial bi-ideal semigroup, E a rectangular band, $P \cap E = \emptyset$ and $\phi: P \to E$ a homomorphism. By putting $\phi(e)=e$ for all $e \in E$, we can consider ϕ as a mapping from $S = P \cup E$ onto E such that $\phi_{|P}$ is a homomorphism. We define an operation in S by

$$xy = \begin{cases} xy \text{ as in P if } x,y \in P \text{ and } xy \text{ is defined in P,} \\ \phi(x)\phi(y) \text{ otherwise.} \end{cases}$$

Let us show that S is a semigroup. Let $x,y,z \in S$. We consider the following three cases:

(i) If one of $(xy)z$ and $x(yz)$ belongs to P, then as P is a partial semigroup, we have that $(xy)z$, $x(yz) \in P$ and $(xy)z = x(yz)$.

(ii) If both, xy and yz are not defined in P, then $(x,y)z$, $x(yz) \in E$ and by the definition of the opperation in S and the associativity in E we have that

$$(xy)z=[\phi(x)\phi(y)]\phi(z)=\phi(x)[\phi(y)\phi(z)]=x(yz).$$

(iii) Finally, if at least one of xy, yz (for instance xy) is defined in P but neither of $(xy)z$ and $x(yz)$ is defined in P, then

$$(xy)z=\phi(xy)\phi(z)=(\phi\text{-homomorphism})=$$
$$=[\phi(x)\phi(y)]\phi(z)=(\text{associativity in } E)=$$
$$=\phi(x)[\phi(y)\phi(z)]=(\text{definition of } \phi,$$
$$\text{or } \phi\text{-homomorphism})=\phi(x)\phi(yz)=x(yz).$$

Denote the semigroup just constructed by $S=(P,E,\phi)$. We shall prove, now, that $S=(P,E,\phi)$ is a bi-ideal semigroup.

Let B be a subsemigroup of S, $x,y\in B$ and $s\in S$. It is clear that $B^*=B\setminus E$ is a partial subsemigroup of P. So, if $xsy\in P$, then $xsy\in B^*\subseteq B$ since P is a partial bi-ideal semigroup. Let xsy is not defined in P. If $xy\in B^*$, then xy is not defined in P and

$$xsy=\phi(x)\phi(s)\phi(y)=(E \text{ is a rectangular band})=$$
$$=\phi(x)\phi(y)=xy\in B.$$

Finally, if $xy\in P$, then $xy\in B^*$ and, because of the periodicity of P, $(xy)^k\in E$ for some $k\in N$. Let $(xy)^k=e$ We have that $e\in B\setminus B^*$ and, since ϕ is a homomorphism, $\phi(x)\phi(y)=\phi(xy)\in E$. Now,

$$\phi(xy)=[\phi(xy)]^k=\phi[(xy)^k]=e.$$

So, again we have that

$$xsy=\phi(x)\phi(y)=\phi(xy)=e\in B,$$

which proves that B is a bi-ideal of S.

In summary, we have proved the following

Theorem 2. A semigroup S is a bi-ideal semigroup iff

$S=(P,E,\phi)$ where P is a periodic partial bi-ideal semigroup, E a rectangular band, $P\cap E=\emptyset$ and $\phi:P\to E$ a homomorphism.■

At the end, using Theorem 2, let us quote some examples of bi-ideal semigroups.

Examples

1) Every rectangular band is a bi-ideal semigroup.

2) Let A be a nonempty set, E-rectatgular band and $\phi:A\to E$ any mapping. Then $S=A\cup E$ is a bi-ideal semigroup with an operation defined as follows:

$$xy=\begin{cases}\phi(x)\phi(y) & \text{if } x,y\in A\\ xy & \text{if } x,y\in E\\ \phi(x)y & \text{if } x\in A, y\in E\\ x\phi(y) & \text{if } x\in E, y\in A.\end{cases}$$

3) Let E be a rectangular band and B_k a partial semigroup defined as follows: (i) $x,y\in B_k$, $x\neq y \Longrightarrow xy$ is not defined in B_k; (ii) $x\in B_k \Longrightarrow x^2\in B_k$ $(k=2)$, $x^2,x^3\in B_k$ $(k=3)$, $x^2,x^3,x^4\in B_k$ $(k=4)$. Further, let $\phi:B_k\to E$ be a mapping such that, if $x^k\in B_k$, then $\phi(x^k)=\phi(x)$. Let us extend ϕ to a mapping from $S=B_k\cup E$ onto E by $\phi(e)=e$ for all $e\in E$ If we define an operation in S by $xy=\phi(x)\phi(y)$, then S will be a bi-ideal semigroup.

REFERENCES

[1] Cliford A.H. and Preston G.B., The algebraic theory of semigroups, v.1, 1961, Providence

[2] Čupona G., Semigroups in which some left ideal is a group, God.zbor., PMF, A, T.14 (1963), 15-17

[3] Kimura Naoki, Tamura Takayki, Merkel Rudoplh, Semigroups in which all subsemigroups are left ideals, Can.J.Math., XVII (1965), 52-62

[4] Ляпин Е.С., Полугрупп , 1960, Москва

[5] Petrich Mario, Introduction to semigroups, Columbus, Ohio, 1973

[6] Трпеновски Б.Л., За една класа полугрупи, Год.Збор. ЕМФ, Скопје, кн. 2 (1968), 5-8

[7] Шутов Э.Г., Полугруппы с идеальными подполугруппами, Матем.Сбор. (новая серия), Т.57, 1962, 179-186