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ON LEFT-IDEAL n-SEMIGROUPS

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A structure desscription of the class of n-semigroups in which every n-subsemigroup is a left ideal, is given in this note.

1. Introduction

Let $S=(S, [\ ])$ be an n-semigroup, $[\ ]:(x_0, x_1, \ldots, x_n) \to [x_0 x_1 \ldots x_n])$. We call S a *left-ideal n-semigroup* iff every n-subsemigroup Q of S is a left ideal of S, i. e.

$$Q \subseteq S, \quad [Q^{n+1}] \subseteq Q \Rightarrow [S^n Q] \subseteq Q.$$

If S contains an idempotent, then every cyclic n-subsemigroup of S has an idempotent ([1], Th. 1 (iii)). The structure of such n-semigroups is described in [2]. We shall give here a structure desrciption of left-ideal n-semigroups, from which will follow, as a corollary, another description of left-ideal n-semigroups wilh idempotents ([2]).

We shall use here the following results from [1]:

Lemma 1. *If S is a left-ideal n-semigroup, then for any $a \in S$, the cyclic n-subsemigroup*

$$<a> = \{a, a^{n+1}, a^{2n+1}, \ldots\}$$

is finite, such that:

(*i*) *the index $r=r_a$ of a is not greater than* 2;

(*ii*) *the n-subgroup K_a (the periodic part) of $<a>$,*

$$K_a = \{a^{rn+1}, a^{(r+1)n+1}, \ldots, a^{(r+m-1)n+1}\}$$

is generated by any of its elements. ∎

Lemma 2. *An n-semigroup S is left-ideal n-semigroup iff for any* $a \in S$

$$[S^n a] \subseteq < a >. \blacksquare$$

So, if S is a left-ideal n-semigroup, then S contains an n-group which is a left ideal of S and therefore the class $\mathcal{L}$ of left-ideal n-semigroups is a subclass of the class $\mathcal{M}$ of n-semigroups described in [3]. Note that $\mathcal{L}$ is a proper subclass of $\mathcal{M}$. Namely, according to Theorem of [3], $S \in \mathcal{M}$ if and only if $S \cong < G, A, P, \varphi, \xi >$, where G is an n-group; $S \in \mathcal{L}$ only if G in the above description is an n-group which is generated by any of its elements and clearly $\mathcal{L} \neq \mathcal{M}$. This implies that the srtucture of the n-semigroups in $\mathcal{L}$ should be simpler than the structure of those in $\mathcal{M}$.

Let us state one definition and two lemmas before we consider the main question of this paper.

An n-group G is said to be *elementary* iff it is generated by any of its elements. As a consequence of Lemma 1 and 2 we have:

Lemma 3. *An n-group G is left-ideal iff it is elementary.* $\blacksquare$

We have the following description ([4] pp. 283—285) of elementary n-groups:

Lemma 4. *A nontrivial cyclic n-group of order g is elementary iff g is a prime.*

2. A structure description of left-ideal n-semigroups

Now we are interested in finding necessary and sufficient conditions for $< G, A, P, \varphi, \xi >$ to be a left-ideal n-semigroup, making an apropriate modification in the Theorem of [3].

First, let $S = < G, A, P, \varphi, \xi >$ be a left-ideal n-semigroup, where:

(1) $S = (G \times A) \cup P$ and $(G \times A) \cap P = \varnothing$;

(2) G is an elementary n-group;

(3) A is a right-zero n-semigroup;

(4) P is a partial n-semigroup;

(5) $\varphi : P \to G$ is a homomorphism, which can be extended to a mapping from S into G by

$$(\forall (x, \alpha) \in G \times A) \quad \varphi(x, \alpha) = x;$$

(6) $\xi : P \to T_A$ is a homomorphism from P into the semigroup T_A of all transformations on A such that

$p_0, p_1, \ldots, p_n \in P, \quad p_0 p_1 \ldots p_n \notin P \to \xi p_0 \, \xi p_1 \ldots \xi p_n$ is a constant;

(7) ξ can be extended to a mapping from S into T_A by

$$(\forall (x, \alpha) \in G \times A) \;\; \xi(x, \alpha) = \xi_{(x, \alpha)},$$

where $(\forall \gamma \in A) \; \gamma \, \xi_{(x, \alpha)} = \alpha$;

(8) the operation [ ] in S is defined by

$$[x_0 x_1 \ldots x_n] = \begin{cases} x_0 x_1 \ldots x_n \text{ if } x_i \in P \text{ and } x_0 x_1 \ldots x_n \in P, \\ (\varphi(x_0) \varphi(x_1) \ldots \varphi(x_n), \;\; \gamma \xi_{x_0} \xi_{x_1} \ldots \xi_{x_n}) \text{ otherwise,} \end{cases}$$

where γ is an arbitrary element of A.

Lemma 5. *If $x_0, x_1, \ldots, x_n \in P$ and $x_0 x_1 \ldots x_n \in P$, then*

$$x_0 x_1 \ldots x_n = x_n^{sn+1} \textit{ for some } s \in \mathbf{N}.$$

Proof. Since S is a left-ideal n-semigroup, by Lemma 2 we have $[x_0 x_1 \ldots x_n] \in \langle x_n \rangle$ for any $x_0, x_1, \ldots, x_n \in S$. Since, by hypothesis, $[x_0 x_1 \ldots x_n] = x_0 x_1 \ldots x_n \in P$, according to (8) it follows that $[x_0 x_1 \ldots x_n] = x_n^{sn+1} \in P$ for some $s \in \mathbf{N}$. ∎

A partial n-semigroup P is said to be *partial left-ideal n-semigroup* iff

$$x_0, x_1, \ldots, x_n \in P, \; [x_0 x_1 \ldots x_n] \in P \Rightarrow (\exists s \in \mathbf{N}) \; [x_0 x_1 \ldots x_n] = x_n^{sn+1}.$$

Lemma 6. $(\forall a \in P) \langle a \rangle \not\subseteq P$, *$P$ being as in* (4).

Proof. If some $x_i \notin P$, then by (8), $[x_1 \ldots x_n a] \notin P$. On the other hand, if $\langle a \rangle \subseteq P$, since $[x_1 \ldots x_n a] \in \langle a \rangle$ for any $x_1, \ldots, x_n \in S$, it would be obtained $[x_1 \ldots x_n a] \in P$, which is a contradiction. ∎

Let us describe the cyclic n-subsemigroups of S.

Let $a \in S$. By Lemma 6, $a^{sn+1} \notin P$ for some $s \in \mathbf{N}$. If k is the least integer with this property and $a^{kn+1} = (g, \theta) \in G \times A$, then we write $\theta = \theta_a$. Then

$$\langle a^{kn+1} \rangle = \langle (g, \theta_a) \rangle = (\langle g \rangle, \theta_a),$$

for A is a right-zero n-semigroup. Since G is generated by any of its elements we have that $\langle a^{kn+1} \rangle = G \times \{\theta_a\}$. So

$$\langle a \rangle = \{a, a^{n+1}, \ldots, a^{(k-1)n+1}\} \cup G \times \{\theta_a\}.$$

By Lemma 1 (i), $r_a \leqslant 2$ and so $a^{2n+1} \in K_a$, which by, Lemma 1 (ii), implies that $a^{2n+1} = (a^{kn+1})^{sn+1} \in G \times \{\theta_a\}$, for some $s \in \mathbf{N}$.

From the above discussion it follows that:

Lemma 7. *For any* $a \in S$

$$< a > = L_a \cup G \times \{\theta_a\},$$

where $L_a = \{a, a^{n+1}, \dots \; a^{(k-1)n+1}\}$.

If S is left-ideal, then: $L_a = \emptyset$ *if* $a \notin P$, $L_a = \{a\}$ *if* $a \in P$ *and* $a^{n+1} \notin P$, $L_a = \{a, a^{n+1}\}$ *if* $a, a^{n+1} \in P$. ∎

If $S = < G, A, P, \varphi, \xi >$ is a left-ideal n-semigroup, we can replace ξ by a homomorphism $\psi : P \to A$ defined by

$$(9) \qquad (\forall p \in P) \;\; \psi(p) = \theta_p,$$

θ_p being as above.

Clearly, by Lemma 6, ψ is a mapping. It remains to show that ψ is a homomorphism. Let $p_1, \dots, p_n, p \in P$. By Lemmas 2 and 6, $q = [p_1 \dots p_n \, p] \in < p > \not\subseteq P$ and so, for any $\gamma \in A$

$$\gamma \, \xi_{p_1} \dots \xi_{p_n} \xi_p = \gamma \, \xi_{[p_1 \dots p_n \, p]} = \gamma \, \xi_{p^{sn+1}}$$

for some $s \in \mathbf{N}$. If $p^{sn+1} \notin P$, then $\gamma \xi_{p^{sn+1}} = \theta_p$, since (by Lemma 7) $p^{sn+1} \in G \times \{\theta_p\}$. If $p^{sn+1} \in P$, then for $q = p_1 \dots p_n \, p$, we have that $q^{kn+1} \in G \times \{\theta_q\}$ for some k and $q^{kn+1} = (p^{sn+1})^{kn+1}$ is not in P. Therefore, $q^{kn+1} \in G \times \{\theta_p\}$ and $\theta_q = \theta_p$. So

$$\psi(p_1 \dots p_n \, p) = \psi([p_1 \dots p_n \, p]) = \theta_p = \theta_{p_1} \dots \theta_{p_n} \theta_p =$$

$$= \psi(p_1) \dots \psi(p_n) . \psi(p).$$

It is easily seen that ψ can be extended to a homomorphism from S into A by putting

$$(9') \qquad (\forall \, (x, \alpha) \in G \times A) \;\; \psi \, (x, \alpha) = \alpha.$$

Lemma 8. *If* $S = < G, A, P, \varphi, \xi >$ *is a left-ideal n-semigroup, then there is a homomorphism* $\psi : S \to A$ *such that the operation in S is*

$$(10) \;\; [x_0 \, x_1 \dots x_n] = \begin{cases} x_0 \, x_1 \dots x_n \text{ if } x_i \in P \text{ and } x_0 \, x_1 \dots x_n \in P, \\ (\varphi \, (x_0) \dots \varphi \, (x_n), \; \psi \, (x_0) \dots \psi \, (x_n)) \textit{ otherwise.} \end{cases}$$ ∎

The n-semigroup obtained from $S = < G, A, P, \varphi, \xi >$ by replacing ξ by ψ and the operation defined by (10), will be written by $S = < G, A, P, \varphi, \psi >$.

Now we shall consider the converse question.

Let $S = < G, A, P, \varphi, \psi >$, where G is an elementary n-group, A is a right-zero n-semigroup, P is a partial left-ideal n-semigroup such that $< a > \not\subseteq P$ for any $a \in S$, φ, ψ are homomorphisms defined by (5), (9) and (9') and the operation is defined by (10). Then S is a left-ideal n-semigroup which is seen from the following discussion.

By Lemma 2, it suffices to prove that for any $a \in S$

$$[S^n a] \subseteq < a >.$$

Therefore, consider the element $u = [x_1 \ldots x_n a] \in S$. If $u \in P$, then $u = a^{sn+1}$ (since P is a partial left-ideal n-semigroup), i.e. $u \in < a >$. If $u \in P$, then by the definition (10) of the operation in S, we have that

$$[x_1 \ldots x_n a] = (\varphi (x_1) \ldots \varphi (x_n) \varphi (a), \psi (x_1) \ldots \psi (x_n) \psi (a)) =$$

$$= (\varphi (x_1) \ldots \varphi (x_n) \varphi (a), \psi (a)) \in G \times \{\theta_a\},$$

which, by the first part of Lemma 7, implies that u belongs to $< a >$. Hence $[S^n a] \subseteq < a >$.

Thus we have proved the following

Theorem 1. *An n-semigroup S is a left-ideal n-semigroup iff*

$$S = < G, A, P, \varphi, \psi >,$$

where G is an elementary n-group, A is a right-zero n-semigroup, P is a partial left-ideal n-semigroup such that $< a > \not\subseteq P$ for any $a \in S$ and φ, ψ are homomorphisms such that (5), (9) *and* (9') *hold.* ∎

3. Examples and notes

1) Let G be an elementary n-group, A a right-zero n-semigroup. Then $S = G \times A$ is a left-ideal n-semigroup. ‖

2) Let G and A be as in 1) and P a nonempty set disjoint with $G \times A$. Let $\varphi : P \to G$ and $\psi : P \to A$ be any mappings. Then $S = (G \times A) \cup P$ will become a left-ideal n-semigroup if an $(n+1)$-ary operation in S is defined by

$$[x_0 x_1 \ldots x_n] = (\varphi (x_0) \varphi (x_1) \ldots \varphi (x_n), \psi (x_n)). \; \|$$

3) Taking G and A to be as in 1), let $P = B \cup C$, $B \cap C = \emptyset$, $(G \times A) \cap P = \emptyset$ and $\varphi : P \to G$, $\psi : P \to A$, $f : B \to C$ be mappings such that for any $x \in B$

$$\varphi f(x) = [\varphi(x)]^{n+1},$$

$$\psi f(x) = \psi(x).$$

Define a partial $(n+1)$-ary operation in P by

$$(\forall\, x_0, x_1, \ldots, x_n \in B) \quad x_0 x_1 \ldots x_n = f(x_n).$$

(Note that from this we have $f(x) = x^{n+1}$ for any $x \in B$.)

By a sraightforeward verification it can be seen that $< G, A, P, \varphi, \psi >$ becomes a left-ideal n-semigroup. Note that

$$a \in B \Rightarrow < a > = \{a, a^{n+1}\} \cup < (\varphi(a), \psi(a)) >,$$

$$a \in C \Rightarrow < a > = \{a\} \cup < (\varphi(a), \psi(a)) >. \,\|$$

4) Let $G = \{e'\}$ and $A = \{\alpha\}$. Put $G \times A = \{e\}$. Let B, C, D be nonempty pairwise disjoint sets and $P = B \cup C \cup D$, where $e \notin P$. Let $g : B \to C$ be a surjection. Put $Q = B \cup D$ and define a function $f : Q^n \times B \to \{0, 1\}$ (here Q^n denotes the n-th Cartesian power of the set Q) such that

$$(\forall x \in B) \;\; f(x, \ldots, x) = 1.$$

Define a partial $(n+1)$-ary operation in P by

$$x_0 x_1 \ldots x_n = g(x_n) \text{ iff } f(x_0, \ldots, x_n) = 1.$$

Note that $g(x) = x^{n+1}$ for any $x \in B$. It is easily seen that P is a partial left-ideal n-semigroup. Define an $(n+1)$—ary operation in $S = P \cup \{e\}$ by

$$[x_0 x_1 \ldots x_n] = \begin{cases} x_0 x_1 \ldots x_n \text{ if it is defined in } P, \\ e \text{ otherwise.} \end{cases}$$

Then S becomes a left-ideal n-semigroup. ||

Note 1. If we take $n=1$ in the example 4), then we come to the structure description of unipotent left-ideal semigroups, given in [5] and [6].

Note 2. To obtain the structure descrption of the unipotent left-ideal n-semigroups of [2], it is necessary, in the examele 4), to define a partial $(n+1)$-ary operation in D in an appropriate way; namely, $D \cup \{e\}$ should become a reduced left-ideal n-semigroup if we put $x_0 x_1 \ldots x_n = e$ whenever the product $x_0 x_1 \ldots x_n$ is not defined in D.

Note 3. Taking T to be one-element set and A to contain more than one element, it could be constructed an example which will correspond to the structure description of left-ideal n-semigroups with more than one idempotent. However, we are not going to state this example here because it includes to much technicalities.

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ЗА ЛЕВОИДЕАЛНИТЕ n-ПОЛУГРУПИ

Резиме

За една n-полугрупа S велиме дека е *левоидеална* ако секоја n-потполугрупа е лев идеал на S. Еден структурен опис на левоидеалните n-полугрупи со идемпотенти е даден во [2]. Во оваа статија даваме структурен опис на левоидеалните n-полугрупи (било да имаат, било да немаат идемпотенти), т.е. главна цел ни е да ја докажеме следнава

Теорема. Една n-полугрупа S е левоидеална ако и само ако

$$S = \langle G, A, P, \varphi, \psi \rangle,$$

каде што G е елементарна n-група (т.е. n-група којашто е генерирана од секој свој елемент), A е деснонулта n-полугрупа, P е делумна левоидеална n-полугрупа (т.е. делумна n-полугрупа за која важи условот што стои непосредно пред лемата 6), таква што $\langle a \rangle \not\subseteq P$ за кој било $a \in S$ и φ, ψ се хомоморфизми такви што да важат (5), (9) и (9').

На крајот се даваат неколку примери на левоидеални n-полугрупи и неколку забелешки.