ON CYCLIC ASSOCIATIVES

Naum CELAKOSKI

1. *Introduction*

Let J be a subsemigroup of $\mathbf{N}^{\circ}(+)$ (the additive semigroup of the nonnegative integers), such that $0 \in J$ and $|J| > 1$. An algebra $\mathbf{A} = (A; (f_n)_{n \in J})$, where f_n, for $n \geqslant 1$, is an $(n+1)$-ary operation on the set A and f_0 is the unary identity operation, i.e. $f_0\, x = x$ for any $x \in A$, is called a *J-associative* iff[1] for any $n, n_0, \ldots, n_k \in J$ $(n = n_0 + n_1 + \ldots + n_k)$ and for any sequence $i_1, \ldots, i_k \in \mathbf{N}^{\circ}$, such that $0 \leqslant i_\nu \leqslant i_{\nu+1}$, $i_\nu \leqslant n_0 + n_1 + \ldots + n_{\nu-1}$ $(\nu = 1, 2, \ldots, k)$, the following identity equality holds:

$$f_n\, x_0\, x_1 \ldots x_n = f_{n_0}\, x_0 \ldots x_{i_1 - 1}\, f_{n_1}\, x_{i_1} \ldots x_{i_{k-1}}\, f_{n_k}\, x_{i_k} \ldots x_n. \quad (1.1)$$

Further on we shall write $[x_0\, x_1 \ldots x_n]$ instead of $f_n\, x_0\, x_1 \ldots x_n$, for any $n \in J$.

(A more general definition of the notion J-associative is given in [1] — [3], but this generalization is not essential.)

Here we give a characterization of the *cyclic J-associatives*, i.e. J-associatives generated by one of its elements. We show that the infinite cyclic J-associative is the free cyclic J-associative and its universal semigroup (see ex. [3], p. 8, or here in Sec. 3) is the additive semigroup of the positive integers. Every cyclic J-associative is entirely determined by a corresponding congruence α on $J(+)$ and any congruence α on $J(+)$ determines a cyclic J-associative, which is denoted by J_α. Finally we give a characterization of the congruences α for which J_α is a *semirgoup J-associative* (i.e. a J-associative which can be embedded in a semigroup). J_α usually is not a semigroup associative.

2. *The free J-associative* $J^*(J)$

Let $J^* = \{n' \mid n' = n + 1,\ n \in J\}$ and define an $(m+1)$-ary operation on J^*, for any $m \in J$, by:

$$[n_0'\ n_1' \ldots n'_m] = n_0' + n_1' + \ldots + n'_m. \quad (2.1)$$

1) "iff" stands for "if and only if".

It is clear that J^* becomes a J-associative with the operations defined by (2.1) and that J^* is cyclic with 1 as its generator. We shall denote it by $J^*(J)$. If A is any J-associative and $a \in A$, then the mapping $1 \mapsto a$ can be uniquely extended to a homomorphism $\varphi: n' \mapsto a^{n'}$ from $J^*(J)$ into $A\,(J)$. Therefore:

2.1. *$J^*(J)$ is a free J-associative with one generator.*

Hence any cyclic J-associative is isomorphic with a factor associative of $J^*(J)$. □

It can be easily checked that:

2.2. *A relation α is a congruence on the semigroup $J\,(+)$ iff the relation α^* on J^* defined by*

$$m' \,\alpha^*\, n' \Leftrightarrow m \,\alpha\, n \tag{2.2}$$

is a congruence on the J-associative $J^*(J)$. □

Hence, $J^*(J)\,/\alpha^*$ is a J-associative, which we denote by J_α.

Clearly, if $\langle a \rangle_J = \{a^{n+1} \mid n \in J\}$ is a cyclic J-associative generated by a, then $\langle a \rangle_J \cong J_\alpha$ iff

$$a^{m+1} = a^{n+1} \Leftrightarrow m \,\alpha\, n. \tag{2.3}$$

Further on we shall usually identify J_α with $\langle a \rangle_J$.

Note that $J_\alpha \cong J_\beta$ iff $\alpha = \beta$.

3. *Universal semigroups for cyclic J-associatives*

If **A** is a J-associative and $\mathbf{S}=(S,.)$ is a semigroup, then a mapping $\xi: A \to S$ is said to be a *semigroup homomorphism* if

$$\xi([x_0\ x_1 \ldots x_n]) = \xi(x_0)\,\xi(x_1)\ldots\xi(x_n)$$

for any $n \in J$ and $x_0, x_1, \ldots, x_n \in A$. A semigroup homomorphism $\lambda: \mathbf{A} \to \mathbf{A}^\wedge$ is said to be *universal* one if for any semigroup homomorphism $\xi: \mathbf{A} \to \mathbf{S}$ there exists a unique homomorphism $\varphi: \mathbf{A}^\wedge \to \mathbf{S}$ such that $\varphi\lambda = \xi$. The semigroup $\mathbf{A}^\wedge$, called the universal semigroup for **A**, is uniquelly determined up to an isomorphism. If λ is an injection, then we say that **A** is a *semigroup associative*. We may assume in this case that $A \subseteq A^\wedge$.

We shall determine here the universal semigroups for the cyclic J-associatives.

First, it is clear that $\varepsilon: n' \to n'$ is a semigroup homomorphism from $J^*(J)$ into **N** (the additive semigroup of positive integers). Let $\xi: J^* \to \mathbf{S}$ be any semigroup homomorphism. If $\xi(0') = \xi(1) = a$, then $\xi(n') =$

$= \xi([1 \dots 1]) = a \dots a = a^{n'}$. Therefore there exists a unique homomorphism $\varphi : \mathbf{N} \to \mathbf{S}$, such that $\varphi\varepsilon = \varepsilon$ (namely, $\varphi(k') = a^{k'}$). Thus:

3.1. *$J^*(J)^\wedge = \mathbf{N}(+)$ and $J^*(J)$ is a semigroup associative.* □

In order to determine the universal semigroup $\mathbf{A}^\wedge$ for a cyclic J-associative $\mathbf{A}$ in general, i.e. $J_\alpha^\wedge$ for J_α, we need some suitable information about the congruence α and therefore we shall state the following result which is a slight modification of the main result of [4]:

3.2. *Let α be a congruence on $J(+)$ which is not the equality on J and let*

$$m_\alpha = \min \{ |a - b| : a\,\alpha\,b, \quad a \neq b \}. \tag{3.1}$$

Then

$$p\,\alpha\,q \Rightarrow p \equiv q \ (\mathrm{mod}\ m_\alpha).$$

If B_α consists of all the elements $b \in J$, such that $b^\alpha = \{x \in J |\ x \alpha b\}$ is an infinite set, then B_α is an ideal in $J(+)$ and

$$p,\ q \in B, \quad p \equiv q \ (\mathrm{mod}\ m_\alpha) \Rightarrow p\alpha q. \ \square \tag{3.2}$$

The number m_α is called the *modulus* of α and

$$s_\alpha = \min \{ x \in J |\ (\exists\, y \in J)\ x \neq y,\ x\,\alpha\,y \}, \tag{3.3}$$

is called the *index* of α. It can be easily seen that $s_\alpha = 0$ iff $B = J$.

We call the pair (B_α, m_α) the *characteristic* of α. We note that there can exist more than one congruence with the same characteristic (B_α, m_α); the set of all congruences on $J(+)$ with the characteristic (B_α, m_α) is finite and the number of possibilities to choose s_α is finite too.

Let α be a congruence on $J(+)$ with the modulus m and index s. (Here and further on, α is not the equality on $J(+)$.) As in Section 2, we shall identify here the J-associative J_α with $\langle a \rangle_J$ and we put $r = s + 1$.

If $\xi : J_\alpha \to S$ is a semigroup homomorphism, then the semigroup generated by $\xi(J_\alpha)$ is cyclic and thus we may assume that S is a cyclic semigroup. Since J_α is finite, it follows that S is finite, too.

Let $S = C_{\rho,\mu} = \langle b \rangle$ be a cyclic semigroup with the index ρ and period μ. If $\xi : J_\alpha \to C_{\rho,\mu}$ is a semigroup homomorphism, where $\xi(a) = b$, then $\xi(a^{n+1}) = b^{n+1}$. Since there exists an element $n \in J, n \neq s$, such that $n\,\alpha\,s$, it follows that $b^{n+1} = b^{s+1}$ (in $C_{\rho,\mu}$) and thus $\rho \leqslant s + 1 = r$.

If $q = p + m$ and $p\,\alpha\,q$, then $a^{p+1} = a^{p+m+1}$, which implies that $m \equiv 0 \pmod{\mu}$, i.e. $\mu | m$.

Hence, if the mapping $\xi : a \mapsto b$ can be extended to a homomorphism from J_α into $C_{\rho,\mu}$ then $\rho \leqslant s + 1 = r$ and $\mu | m$.

Conversely, let $\rho \leqslant s + 1 = r$ and $\mu | m$ and put $\xi(a^{n+1}) = b^{n+1}$. If $a^{p+1} = a^{q+1}$, $p \neq q$, then $p \equiv q \pmod{m}$ and $p, q \geqslant s$. Therefore $p + 1$,

$q+1 \geqslant s+1=\rho$, and (since $\mu \mid m$) $p+1 \equiv q+1 \pmod{\mu}$). Thus, $\xi : J_\alpha \to C_{\rho,\mu}$ is a mapping and, clearly, ξ is a homomorphism.

Hence, if $\rho \leqslant s+1=r$ and $\mu \mid m$, then by $\xi(a^{n+1}) = b^{n+1}$ is defined a homomorphism form J into $C_{\rho,\mu}$.

This and the fact that by $\varphi : b^k \mapsto c^k$ is defined a homomorphism from $C_{r,m} = \langle b \rangle$ into $C_{\rho,\mu} = \langle c \rangle$ iff $\rho \leqslant r$ and $\mu \mid m$, imply the following:

3.3. Theorem. *If α is a congruence on $J(+)$ with the modulus m and index $s = r-1$, then the universal semigroup for J_α* is $J_\alpha^\wedge = C_{r,m}$. □

Using 3.3, the following characterization for finite cyclic semigroup J-associatives can be derived:

3.4. Theorem. *If α is a congruence on $J(+)$ with the characteristic (B, m) and index s, then J_α is a semigroup J-associative iff*

$$B = \{x \in J \mid x \geqslant s\}. \tag{3.4}$$

Proof. Let J_α be a semigroup J-associative, i.e. $\lambda : J_\alpha \to C_{r,m} = \langle c \rangle$, $\lambda(a^{n+1}) = c^{n+1}$, is a monomorphism. If $x > s$, then there exist infinitely many elements c^{y+1} which are equal to $\lambda(a^{x+1}) = c^{x+1}$ (since c^{x+1} belongs to the periodic part of $\langle c \rangle$). If x were in a finite α-class, then we would have $a^{x+1} \neq a^{y+1}$ for some $y \in J$ and $c^{x+1} = c^{y+1}$, i.e. λ would not be a monomorphism. If $s \notin B$, then the same argument as above shows that there are infinitely many elements c^{y+1} which are equal to c^{s+1} and this implies that λ is not a monomorphism. Thus, if λ is a monomorphism, then (3.4) holds.

Conversely, assume that (3.4) is satisfied. If $x, y \in J$, $x \neq y$ and $c^{x+1} = \lambda(a^{x+1}) = \lambda(a^{y+1}) = c^{y+1}$, then $x, y \geqslant s$ and $x \equiv y \pmod{m}$. By (3.4), we have $x, y \in B$ and $x \equiv y \pmod{m}$, which, according to (3.2), imply $x \,\alpha\, y$, i.e. $a^{x+1} = a^{y+1}$. Thus λ is a monomorphism. □

As a cosnequence of **3.3.** and **3.4.** we have the following:

3.5. Corollary. *If the index of α is 0, then J_α is a semigroup associative and the universal covering of J_α is the cyclic group C_m with m elements, where m is the modulus of α.* □

We also note that J_α in this case is a J-group ([1]); for a J-associative A is a J-group iff $A^\wedge$ is a group.

4. *Some examples*

We shall consider here some examples.

1. Let $J = \{0, 2, 4\} \cup \{n \in \mathbf{N} \mid n \geqslant 6\}$.

a) If $B = J \setminus \{0\}$ and m is any positive integer, then there exists only one congruence α with the characteristic (B, m) and then the corresponding associative J_α is a semigroup associative with $m+1$ elements.

b) If $B=J\setminus\{0, 2, 4, 6, 9\}$ and m is any positive integer, then the congruences with the characteristic (B, m) yield non-semigroup J-associatives (since 9 will belong to a finite α-class for any such conrguence α, and $s_\alpha < 9$).

2. Let $J=\{0, 6, 9, 10, 12, 15, 16, 18, 19, 20, 21, 22\}\cup\{n\in\mathbf{N}|n\geqslant 24\}$ $B=J\setminus\{0, 6, 9, 10, 12\}$ and $m=2$. The following two congruences α and β with the same characteristic $(B, 2)$ can be formed:

$$p^\alpha=\{p\} \text{ for } p\in J\setminus B,\ 15^\alpha=\{x\in B|\,x\equiv 15 \pmod 2\}=15^\beta,$$

$$16^\alpha=\{x\in B|\,x\equiv 16 \pmod 2\}=16^\beta,\ q^\beta=\{q\} \text{ for } q=0,\ 6,\ 9$$

$$\text{and } 10^\beta=\{10, 12\} \text{ (here: } s_\alpha=15,\ s_\beta=10).$$

The corresponding J-associative $J_\alpha=\{a, a^7, a^{10}, a^{11}, a^{13}, a^{16}, a^{17}\}$ is a semigroup one, and $J_\beta=\{a, a^7, a^{10}, a^{11}, a^{16}, a^{17}\}$ is a non-semigroup J-associative.

3. Let $J=\{0, 5, 9, 10, 11\}\cup\{n\in\mathbf{N}|\,n\geqslant 14\}$, $B=J\setminus\{0, 5, 9, 11, 17, 18\}$ and $m=3$. In order to obtain all the congruences on $J(+)$ with the characteristic $(B, 3)$, we have to divide the set $\{5, 9, 11, 17, 18\}$ into classes mod 3: $\{5, 11, 17\}$ and $\{9, 18\}$. A simple check shows that the set $\{9, 18\}$ can not be an α-class (namely, $9\ \alpha\ 18$ would imply $18\ \alpha\ 36$, which is contradictory and so 9 and 18 will form one-element α-classes.

There are five possibilities for making different classes from the set $\{5, 11, 17\}$:

α_1 : $\{p\}$ for $p\in J\setminus B$;

α_2 : $\{p\}$ for $p=0, 9, 17, 18$ and $\{5, 11\}$;

α_3 : $\{p\}$ for $p=0, 9, 11, 18$, and $\{5, 17\}$;

α_4 : $\{p\}$ for $p=0, 5, 9, 18$ and $\{11, 17\}$;

α_5 : $\{p\}$ for $p=0, 9, 18$ and $\{5, 11, 17\}$.

Putting $10^{\alpha_i}=\{x\in B|\,x\equiv 10 \pmod 3\}$,

$14^{\alpha_i}=\{x\in B|\,x\equiv 14 \pmod 3\}$,

$15^{\alpha_i}=\{x\in B|\,x\equiv 15 \pmod 3\}$,

$i=1, \ldots, 5$, we obtain five congruences $\alpha_1, \ldots, \alpha_5$ on $J(+)$ with the same characteristic (B, 3) and the corresponding J-associatives J_{α_i} are non-semigroup J-associatives (since $s_{\alpha_i} < 18$ for all $i=1, \ldots, 5$).

(We note that, in the above examples, the greater common divisor d of the nonzero elements of J is 1, but this restriction is not essential. Namely if $d > 1$, then by dividing the elements of J by d, we shall obtain a subsemigroup $J'(+)$ of $\mathbf{N}^\circ(+)$, with $d' = 1$ and $J'(+) \cong J(+)$.)

REFERENCES

[1] *Ѓ. Чупона*: За асоцијативите; МАНУ, Прилози I—1 (1969), 9—20.

[2] *Ѓ. Чупона*: Асоцијативи со кратење; Год. зб. ПМФ-Скопје, 19 (1969), 5—14.

[3] N. *Celakoski*: On Semigroup Associatives; Maced. Acad. of Sc. and Arts; Contributions IX—2 (1977), 5—19.

[4] *Д. Димовски*: Адитивни полугрупи на цели броеви; МАНУ, Прилози IX—2 (1977), 21—26.

[5] *A. H. Clifford, G.B. Preston*: The algebraic theory of semigroups; vol. 1, 1961, Providence

РЕЗИМЕ

ЗА ЦИКЛИЧНИТЕ АСОЦИЈАТИВИ

Наум ЦЕЛАКОСКИ

Нека J е потполугрупа од $\mathbf{N}^\circ$ (адитивната полугрупа на ненегативните цели броеви), таква што $0 \in J$ и J содржи повеќе од еден елемент. Една алгебра $\mathbf{A} = (A; (f_n) n \in J)$, каде што f_n е $(n+1)$-арна операција на множеството A, се вика *J-асоцијатив* ако во $\mathbf{A}$ е исполнет општиот асоцијативен закон, т.е. (1.1).

Работава се надоврзува на [3]. Овде се разгледуваат J-асоцијативи што се генерирани од некој свој елемент и се наречуваат *циклични* J-асоцијативи. Се покажува дека бесконечниот цикличен J-асоцијатив е слободен, па значи, тој е полугрупен (в. и [3], **4.4**) и дека неговта слободна покривка е $\mathbf{N}(+)$ (адитивната полугрупа на природните броеви).

Потоа се покажува дека секој цикличен J-асоцијатив е определен, до изоморфизам, со некоја конгруенција на $J(+)$ и дека секоја конгруенција α на $J(+)$ определува цикличен J-асоцијатив, означен со J_α. За главен резултат во рботава ја сметаме теоремата **3.4**, во која се дава критериум за полугрупност на конечните циклични J-асоцијативи. Резултатите од работава даваат едноставен начин за наоѓање примери на конечни циклични асоцијативи; на крајот се наведуваат неколку такви примери.

Faculty of Mathematics,
Skopje.