

SOME PROPERTIES OF PRESENTATIONS OF GROUPS WITH ONE DEFINING RELATOR

Janeva Biljana

In this paper two properties of presentations of groups with one defining relator will be given.

The first property shows that the testing of the conditions of the Lyndon's theorem ([2]) for presentations of groups with one defining relator could be diminished. Namely, if the symmetrized set R of defining relator r satisfies the condition $C'(\lambda)$, $1/3 \leq \lambda < 1/2$, then R does not satisfy the condition $T(5)$.

In the second property sufficient conditions are given for transforming a presentation of a group with one defining relator r to a presentation of the same group with one defining relator s , such that $|r| > |s|$.

1. Preliminaries

Let $\langle A; R \rangle$ be a presentation of a group G , where A is a set of generating symbols, and R is a set of defining relators, such that every r in R is freely reduced.

Definition 1: A set R of defining relators is *symmetrized* if the following conditions are satisfied:

(i) $r \in R \Rightarrow r^{-1} \in R$

(ii) $r \in R \Rightarrow$ every cyclic permutation r^* of r is also in R .

Notes: 1) We will use the symbol R to denote a symmetrized set of defining relators.

2) If a set S of defining relators is given, one could „construct“ the symmetrized set R of S by adding all the inverse words of the words in S and all the cyclic permutations of the elements in S and their inverses.

3) If $\langle A; S \rangle$ is a presentation of the group G and R is the symmetrized set of the set S , then $\langle A; R \rangle$ is a presentation of the same group G .

Definition 2. The set R of defining relators satisfies the condition $C'(\lambda)$ ($0 \leq \lambda \leq 1$, and λ is a real number) if in the free reduction of the pro-

duct $r_1 r_2$ of any two elements $r_1, r_2 \in R$, such that $r_1 \neq r_2^{-1}$ *, a subword a from r_1 is cancelled, then $|a| < \lambda |r_1|$ **).

Definition 3. The set R of defining relators satisfies the condition $T(q)$ (q is a natural number) if for each natural number h , $3 \leq h < q$, and every sequence $r_1, \dots, r_h \in R$ such that no neighbouring pair is an inverse pair, at least one of the products $r_1 r_2, r_2 r_3, \dots, r_h r_1$ is freely reduced.

2. Some properties of presentations of groups with one defining relator

Proposition 1. Let $\langle A; r \rangle$ be a presentation of the group G , where r is a defining relator such that $|r| \geq 5$, and let R be the symmetrized set of the defining relator r . If R satisfies $C'(\lambda)$ for $1/3 \leq \lambda < 1/2$, and there is no $\mu < \lambda$ such that $C'(\mu)$ is satisfied, then R does not satisfy $T(5)$.

Proof: R is the symmetrized set of one defining relator r , $|r| \geq 5$, so if R satisfies $C'(\lambda)$, $1/3 \leq \lambda < 1/2$, then r , or a cyclic permutation of r , must contain a subword s , $|s| = \lambda |r| = k \geq 2$, s appears in r twice, or both s and s^{-1} are subwords of r .

We can assume that the subword s appears in r from the first symbol from the left, because R contains all the cyclic permutations of r .

Case I. s appears twice in r

I a) The second appearance of s is from the second symbol from the left of the word r .

Let the second appearance of s be from the second symbol from the left of the word r . Then $s \equiv a^\varepsilon k$, $a \in A$, $\varepsilon = \pm 1$, $k \geq 2$, and the words $r_1 \equiv \equiv a^\varepsilon k a^\varepsilon t$ and $r_2 \equiv a^\varepsilon k t a^\varepsilon$ are in R , where t is a subword of r , and t does not begin with a^ε , $r_1 \neq r_2$ (In the contrary t must begin with the symbol a^ε).

$r_3 \equiv a^\varepsilon(k-1) t a^\varepsilon$ is a cyclic permutation of r_2 , thus $r_3, r_3^{-1} \in R$.

The sequence $r_2, r_3^{-1}, r_3, r_2^{-1}$ does not satisfy the condition $T(5)$.

I b) the second appearance of the word s in r is at least from the third symbol from the left.

In this case

$$r \equiv st \equiv mnt, \quad s \equiv mn, \quad |m| \geq 2$$

where $t \equiv pq$ and $s \equiv np$, and m, n, p, q, t are subwords of r .

A cyclic permutation r_1 of r is

$$r_1 \equiv ntm$$

*) $w \equiv v$ denotes that w and v are graphically equal words

**) $|w|$ denotes the length of the word w

Let a and b be the first two symbols of s . Let us examine the following cyclic permutations of the relator r :

$$r' \equiv bs'ta \quad \text{and} \quad r_1' \equiv bn'tma$$

where $s \equiv abs'$, $n \equiv abn'$.

The sequence $r', r_1'^{-1}, r', r_1'^{-1}$ does not satisfy the condition T (5).

Case II. Both s and s^{-1} appear in r .

Let both s and s^{-1} appear in r . Then

$$r \equiv st_1s^{-1}t_2, \quad |s| \geq 2.$$

Let $s \equiv av$, where a and b are generating symbols or their inverses, and v be a subword of r .

Then the words

$$r_1 \equiv brt_1v^{-1}b^{-1}a^{-1}t_2a \quad \text{and}$$

$$r_2 \equiv a^{-1}t_2avt_1v^{-1}b^{-1}$$

are in R , and the sequence r_1, r_2, r_1, r_2 does not satisfy the condition T(5).

Proposition 2: Let R be the symetrized set of the defining relator r , and let there exist two symbols a, b , $a \neq b$, such that satisfy the property:

if k_a is the number of elements of R in which a appears as the first symbol from the left, and k is the number of elements of R that begin with the word ab , then $2k > k_a$.

Then the presentation of the group G with one defining relator r could be transformed into a presentation of the same group, with one defining relator s , such that $|r| > |s|$.

Proof: Let $\langle A; r \rangle$ be a presentation of the group G and the defining relator r satisfy the condition in the statement of the proposition. Let a, b be two different symbols, as mentioned above.

Using the Tietze transformations (T3) and (T4) ([1]) we transform the presentation $\langle A; r \rangle$ of the group G in the following way:

We add a new symbol c , and a new defining relator abc^{-1} (using Tietze transformation (T3)).

$$\langle \{c\} \cup A; r, abc^{-1} \rangle \quad (1)$$

is a presentation of the same group G .

Let us express the symbol a by b and c and using Tietze transformation (T4) remove the generating symbol a and the relation abc^{-1} from the presentation (1) of the group G .

$$\langle \{c\} \cup (A \setminus \{a\}); s \rangle \quad (2)$$

is a presentation of the group G , where s is a word received from r by replacing the symbol a with the word ab^{-1} and freely reducing the received word.

The number of elements of R that begin with the word ab is k , thus in a cyclic permutation of r the subword ab or its inverse appear k times. Substituting the symbol a in the subwords ab , or a^{-1} in its inverse, by cb^{-1} , or its inverse, respectively, and freely reducing the received word s' , the number of symbols in s' is $|r| - k$. The number of elements of R that begin with the symbol a is k_0 , k times in the subword ab , thus, there are $k_0 - k$ elements of R that begin by the symbol a not followed by the symbol b . Substituting the symbol a by cb^{-1} in s' , a new freely reduced word s is received. The number of symbols in s is $|s'| + k_0 - k$, or $|r| - k + k_0 - k$. Using the inequality $2k > k_0$, we get $|r| + k_0 - 2k < |r|$, i.e. $|r| < |s|$.

3. Conclusion

In this paper two properties of presentations of groups with one defining relator are given. The first shows that the testing of the conditions of the Lyndon's theorem could be diminished, and the second gives a sufficient condition for transforming a presentation of the group with one defining relator into a presentation of the same group with one defining relator, but with the smaller length than the first one.

In some cases, for example when an investigation of presentations of groups with one defining relator for different lengths of the defining relator is needed, testing the conditions of the second property of this paper, the number of the investigated cases could be reduced.

LITERATURE:

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НЕКОИ ОСОБИНИ НА ПРЕТСТАВУВАЊА НА ГРУПИ СО ЕДЕН ДЕФИНИРАЧКИ РЕЛАТОР

Јанева БИЉАНА

Резиме

Во овој труд дадени се две особини на претставувања на групи со еден дефинирачки релатор. Првата покажува дека испитувањето на условите на теоремата на Линдон може да се намали, имено ако симетризираното множество на дефинирачкиот релатор го задоволува условот $C'(\lambda)$, $1/3 \leq \lambda \leq 1/2$, тоа не го задоволува условот $T(5)$. Втората особина дава доволен услов за трансформирање на претставување на група со еден дефинирачки релатор r во претставување на истата група со еден дефинирачки s , така што $|r| > |s|$.