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n-SUBGROUPOIDS OF CANCELLATIVE GROUPOIDS

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The classes of $n(t)$-subgroupoids of cancellative groupoids are considered in this paper, where t is a groupoid term of some special form. It is known that these classes are quasivarieties ([2], [3] p. 274). We give here a description of the axiom sets of that quasivarieties.

1. *n-subgroupoids of groupoids.* Let $X=\{x_1, x_2, \ldots, x_n, \ldots\}$ be a set of variables, and let $*$ be a binary operation symbol. Define the notion of a $*$-*term* (or, briefly, of a *term*) in the following inductive way:

(i) every variable is a term,

(ii) if t_1, t_2 are terms, then the string $* t_1 t_2$ is a term,

(iii) a string of elements of the set $X \cup \{*\}$ is a term iff it is obtained by a finite application of (i) and (ii).

Define a *subterm* of a term t by this inductive way: t is a subterm of itself, and if $t = t't''$, then t', t'' and any subterm of t', t'' are subterms of t too. By $t = t_1(\ldots, t_2, \ldots)$ we denote that the term t_2 is a subterm of t.

If t is a term, then $t(y_1, y_2, \ldots, y_k)$ denotes that $\{y_1, y_2, \ldots, y_k\}$ is the set of variables which appear in t.

Let A be a set and let $t(y_1, \ldots, y_k)$ be a $*$-term. If every appearence of a variable y_i is changed by an element $a_i \in A$, then we get a string $t(a_1 \ldots, \ldots, a_k)$ of elements of the set $A \cup \{*\}$, called a $*$-*word* (or a *word*) over the set A. We say that the word $t(a_1, \ldots, a_n)$ has the form of the term $t(y_1, \ldots, \ldots, y_k)$. We define a subword of a word in the same way as we have defined a subterm of a term. Denote by $W(A)$ the set of all $*$-words over the set A. Clearly, $A \subseteq W(A)$.

Define an operation $\cdot$ on the set $W(A)$ by

$$(\forall u, v \in W(A)) \quad u \cdot v = * uv.$$

In this way $\mathbf{F}_A = (W(A), \cdot)$ becomes a grupoid, which is isomorphic to the free groupoid generated by the set A.

For any groupoid $\mathbf{G} = (G, \circ)$ and any $*$-word $u = t(a_1, \ldots, a_k)$ we define $u_G = t_G(a_1, \ldots, a_k)$ to be that element of G which is obtained from u by replacing any operation symbol $*$ by the operation $\circ$ of $\mathbf{G}$.

A universal algebra $\mathbf{A} = (A, f)$ with one n-ary operation f $(n \geqslant 1)$ is said to be an n-groupoid.

Consider an n-grupoid (A, f) and let $t = t(x_1, \ldots, x_n)$ be a $*$-term which is not a variable. The n-groupoid (A, f) is said to be an *$n(t)$-subgroupoid of a groupoid* $(G, \circ)$ iff $A \subseteq G$ and for all $a_1, \ldots, a_n \in A$

$$f(a_1, \ldots, a_n) = t_G(a_1, \ldots, a_n). \tag{1.1}$$

The $*$-term t is called a *representation* of the n-ary operation f. If the representation t of f is given, instead of $n(t)$-subgroupoid we will say simply n-subgroupoid.

Suppose that t is a given representation of the operation f of the n-groupoid (A, f). Let $P = \{u \in W(A) \mid u$ has not a subword of the form $t(a_1, \ldots, a_n)$, where $a_1, \ldots, a_n \in A\}$. Define an operation $\bullet$ on P by

$$u, v \in P, \quad *uv \in P \Rightarrow u \bullet v = *uv,$$

$$u, v \in P, \quad *uv = t(a_1, \ldots, a_n) \Rightarrow u \bullet v = f(a_1, \ldots, a_n).$$

Since t contains exactly n distinct variables, $(P, \bullet)$ is a groupoid. As t is not a variable, we have that $A \subset P$, and (1.1) is satisfied by the definition of $(P, \bullet)$. We say that $(P, \bullet)$ is the *(universal) t-covering groupoid* of the n-groupoid (A, f), and denote it by $\mathbf{A}^{\wedge} = (A^{\wedge}, \bullet)$.

Thus, we have proved

Theorem 1.1. *Every n-groupoid is an n-subgroupoid of its universal t-covering groupoid.* ∎

Notice the following property of the t-covering groupoid $\mathbf{A}^{\wedge}$:

$$u_1 \bullet v_1 = u_2 \bullet v_2 \Rightarrow (u_1 = u_2,\ v_1 = v_2 \text{ in } \mathbf{F}_A) \text{ or } (u_1 \bullet v_1 = f(a_1, \ldots, a_n),\ u_2 \bullet v_2 = f(b_1, \ldots, b_n) \text{ and } f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n), \text{ where } a_\nu, b_\nu \in A). \tag{1.2}$$

$$u \bullet v \in A \Leftrightarrow *uv = t(a_1, \ldots, a_n), \text{ where } a_\nu \in A. \tag{1.3}$$

2. *n-subgroupoids of cancellative groupoids.* An n-groupoid (A, f) is said to be i-*cancellative* iff

$$f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n) \Rightarrow b = c$$

for any $a_\nu, b, c \in A$. An n-groupoid (A, f) is *cancellative* (*left cancellative, right cancellative*) iff it is i-cancellative for all $i = 1, 2, \ldots, n$ (n-cancellative, 1-cancellative).

Let the representation t of the n-ary operation f have the form $t = *t_1 t_2$. Denote by $V(t_1)$ ($V(t_2)$) the set of variables which appear in the subterm t_1 (t_2). We have the following

Theorem 2.1. (i) *If $V(t_1) \subseteq V(t_2)$, then every n-groupoid is an n-subgroupoid of a right cancellative groupoid.*

(ii) *If $V(t_2) \subseteq V(t_1)$, then every n-groupoid is an n-subgroupoid of a left cancellative groupoid.*

(iii) *If $V(t_1) = V(t_2)$, then every n-groupoid is an n-subgroupoid of a cancellative groupoid.*

Proof: (i) By Theorem 1.1 it is enough to be shown that the t-covering groupoid $A^\wedge$ of any n-groupoid (A, f) is right cancellative, when $V(t_1) \subseteq V(t_2)$.

Let $u, v, w \in A^\wedge$, and $v \bullet u = w \bullet u$.

If $v \bullet u = *vu$, then $w \bullet u = *wu$, and (1.2) implies that $v = w$.

If $v \bullet u \neq *vu$, then by (1.3) we have that $w \bullet u \neq *wu$ and $v \bullet u = f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n) = w \bullet u$, where $*vu = t(a_1, \ldots, a_n)$, $*wu = t(b_1, \ldots, b_n)$ for some $a_\nu, b_\nu \in A^\wedge$. In fact, as $V(t_1) \subseteq V(t_2)$, we can suppose that $t(x_1, \ldots, x_n) = *t_1(x_1, \ldots, x_i)\, t_2(x_1, \ldots, x_n)$, i. e.

$$v = t_1(a_1, \ldots, a_i), \quad w = t_1(b_1, \ldots, b_i),$$

$$u = t_2(a_1, \ldots, a_n) = t_2(b_1, \ldots, b_n).$$

We have from the last equality that $a_1 = b_1, \ldots, a_n = b_n$, which implies that $v = w$.

(ii) In the same manner as in (i) we can prove that $V(t_2) \subseteq V(t_1)$ implies that the t-covering groupoid of any n-groupoid is left cancellative.

(iii) is a consequence of (i) and (ii). ∎

3. *$n(t)$-subgroupoids of cancellative groupoids, when t is a balanced term.* A term is said to be *balanced* if a variable appears in it at most ones. Here we will consider only such representations $t(x_1, ..., x_n)$ of n-ary operations which are balanced and the order of the appearences of the variables in the term t is natural (i.e. x_{i+1} follows x_i, for all i).

A subterm t_2 of a term t is said to be *terminal* iff either $t = *t_1 t_2$ or $t = *t_1 t_3$ and t_2 is terminal in t_3.

Further on we will use shorter notations of the strings of variables of elements of a given set in this way: y_j^k will denote the string of variables $y_j, y_{j+1}, \ldots, y_k$ $(j \leqslant k)$, a_m^n will denote the string $a_m, \ldots, a_n$ $(m \leqslant n)$ of elements of a set A, $f(a_1^n)$ will denote $f(a_1, \ldots, a_n)$ and so on.

Theorem 3.1. *If an n-groupoid is an n-subgroupoid of a (left, right) cancellative groupoid, then it is a (left, right) cancellative n-groupoid.*

Proof: Suppose that (A, f) is an n-groupoid, which is an $n(t)$-subgroupoid of a left cancellative groupoid $(G, \circ)$, where t is a given balanced term. Then we have (for any $a_\nu, b_\nu, c_\nu \in A$):

$$f(a_1^{n-1}, b) = f(a_1^{n-1}, c) \Rightarrow t_G(a_1^{n-1}, b) = t_G(a_1^{n-1}, c) \Rightarrow$$

$$t_{1G}(a_1^m) \circ t_{2G}(a_{m+1}^{n-1}, b) = t_{1G}(a_1^m) \circ t_{2G}(a_{m+1}^{n-1}, c) \Rightarrow$$

$$t_{2G}(a_{m+1}^{n-1}, b) = t_{2G}(a_{m+1}^{n-1}, c) \Rightarrow \cdots \Rightarrow b = c,$$

where $t(x_1^n) = * t_1(x_1^m) t_2(x_{m+1}^n), \ldots$. ∎

Theorem 3.2. *A left cancellative n-groupoid (A, f) is an n-subgroupoid of a left cancellative groupoid iff the following conditions are satisfied:*

(i) *For any terminal subterm $t_1(x_{k+1}^n)$ of the representation $t(x_1^n)$ of f, the quasiidentity*

$$f(x_1^k, y_{k+1}^n) = f(x_1^k, z_{k+1}^n) \Rightarrow f(x'^k_1, y_{k+1}^n) = f(x'^k_1, z_{k+1}^n) \tag{3.1}$$

is true in (A, f).

(ii) *If t_1 is a terminal subterm of t and if $t(x_1^n) = t_2(x_1^i, t_1(x_{i+1}^k), x_{k+1}^{n-k+i}, t_1(x_{n-k+i+1}^n))$, then the quasiidentity*

$$\begin{gathered} f(x_1^{n-k+i}, y_{n-k+i+1}^n) = f(x_1^{n-k+i}, z_{n-k+i+1}^n) \Rightarrow \\ \Rightarrow f(x'^i_1, y_{n-k+i+1}^n, x'^{n-k+i}_{i+1}) = f(x'^i_1, z_{n-k+i+1}^n, x'^{n-k+i}_{i+1}) \end{gathered} \tag{3.2}$$

is true in (A, f). (Here, $x_\nu, x_\nu', y_\nu, z_\nu$ are variables.)

Proof: It is easy to be shown that the quasiidentities (3.1) and (3.2) are true in an n-groupoid (A, f), which is an n-subgroupoid of a left cancellative groupoid $(G, \circ)$. For instance, let the hypothesis of (3.2) is satisfied in (A, f). Then, it follows from (1.1) that

$$t_{2G}(a_1^{n-k+i}, t_{1G}(b_1^{k-i})) = t_{2G}(a_1^{n-k+i}, t_{1G}(c_1^{k-i}))$$

for any $a_\nu, b_\nu, c_\nu \in A$, which give rise to

$$t_{1G}(b_1^{k-i}) = t_{1G}(c_1^{k-i}), \tag{3.3}$$

because we can cancel on the left-hand side. Now, by multiplying on the left-hand side and on the right-hand side, it follows from (3.3) that

$$t_{2G}(d_1^i, t_{1G}(b_1^{k-i}), d_{i+1}^{n-k+i}) = t_{2G}(d_1^i, t_{1G}(c_1^{k-i}), d_{i+1}^{n-k+i})$$

for any $d_\nu \in A$, i.e. we can conclude that (3.2) holds true in (A, f).

We suppose now that the conditions (i) and (ii) are satisfied.

Define a congruence on the t-covering groupoid $\mathbf{A}^\wedge$ of the n-groupoid (A,f) as follows.

If $t_1(x_p^n)$ is a terminal subword of t, then we put

$$t_1(a_p^n)\,\alpha_0 t_1(b_p^n) \Leftrightarrow (\exists\, c_1,\dots,c_{p-1})\, f(c_1^{p-1}, b_p^n) = f(c_1^{p-1}, a_p^n) \tag{3.4}$$

where a_ν, b_ν, $c_\nu \in A$.

As a consequence of (3.1) we get that the existential quantifier in (3.4) can be replaced by the universal one. This implies that the relation α_0 on $A^\wedge$ is transitive and symmetric.

Define a relation α on $A^\wedge$ by

$$u\,\alpha\, v \Leftrightarrow u = t'(a_1^i,\, t''(b_1^p),\, d_1^j),\ v = t'(a_1^i,\, t''(c_1^p), d_1^j),\ t''(b_1^p)\ \alpha_0\ t''(c_1^p)$$

where α_ν, b_ν, c_ν, $d_\nu \in A$ and t'' is a terminal subterm of t.

Finally, we put

$$u\,\beta\, v \Leftrightarrow (\exists\, u_1,\dots,u_r, r \geqslant 0)\quad u\,\alpha\, u_1\,\alpha\, u_2\,\alpha \dots \alpha\, u_r\,\alpha\, v.$$

Since the relation α is reflexive and symmetric, β is an equivalence. Let $u, v, w \in A^\wedge$, $u\,\alpha\, v$. If $u \bullet w = *uw$, then $v \bullet w = *vw$ and it is clear that $u \bullet w\,\alpha\, v \bullet w$. Suppose that $u \bullet w \in A$. It follows by (1.3) that for some $a_1,\dots,a_n \in A$, $*uw = t(a_1^n)$, where

$$t(x_1^n) = *\,t_1(x_1^i,\, t_3(x_{i+1}^{i+p}),\, x_{i+p+1}^k)\; t_2(x_{k+1}^{n-p},\, t_3(x_{n-p+1}^n)),$$

$$u = t_1(a_1^i,\, t_3(a_{i+1}^{i+p}),\, a_{i+p+1}^k),\quad v = t_1(a_1^i,\, t_3(b_{i+1}^{i+p}),\, a_{i+p+1}^k),$$

$$w = t_2(a_{k+1}^{n-p},\, t_3(a_{n-p+1}^n))$$

where $b_\nu \in A$ and

$$t_3(a_{i+1}^{i+p})\ \alpha_0\ t_3(b_{i+1}^{i+p}). \tag{3.5}$$

Now, (3.5) implies that for some $c_\nu \in A$

$$f(c_1^{n-p},\, a_{i+1}^{i+p}) = f(c_1^{n-p},\, b_{i+1}^{i+p}),$$

and from (3.2) we get

$$u \bullet w = f(a_1^n) = f(a_1^i,\, b_{i+1}^{i+p},\, a_{i+p+1}^n) = v \bullet w$$

In the same way as above one can prove that $u, v, w \in A^\wedge \Rightarrow (u\,\alpha\, v \Rightarrow$ $\Rightarrow w \bullet u\,\alpha\, w \bullet v)$, i.e. that β is a congruence on $\mathbf{A}^\wedge$.

Now, to complete the proof, we have to show that (A, f) is an *n-sub-groupoid* of $\mathbf{A}^\wedge/\beta$ and that $\mathbf{A}^\wedge/\beta$ is a left cancellative groupoid.

Let $a \in A$, $u \in A^\wedge$ and $a \alpha u$. Then a and u have a form of a variable, i.e. $u \in A$, which means that $a\ \alpha_0\ u$. So, for any $c \in A$, $f(c_1^{n-1}, a) = f(c_1^{n-1}, u)$, and this implies $a = u$. It follows that if $a, b \in A$ and $a \beta b$, then $a = b$, and we can assume that $A \subseteq A^\wedge/\beta$.

Finally, let $u, v, w \in A^\wedge$ and $u \bullet v \beta u \bullet w$. If $u \bullet v \in A$, then $u \bullet w \in A$, and $u = t_1(a_1^i)$, $v = t_2(a_{i+1}^n)$, $w = t_2(b_{i+1}^n)$, $f(a_1^n) = f(a_1^i, b_{i+1}^n)$ where $t = *t_1 t_2$, a_ν, $b_\nu \in A$. Now from the definition of α_0 we have that $v\, \alpha_0\, w$, i.e. $v \beta w$.

Suppose that $u \bullet v$ and $u \bullet w$ are not in A. Then there exists a sequence

$$*uv = *u_1 v_1 \alpha *u_2 v_2 \alpha \cdots \alpha *u_{k-1} v_{k-1} \alpha *u_k v_k = *uw$$

where $*uv$, $*u_2v_2, \ldots, *u_{k-1} v_{k-1}$, $*uw$ have a form of a term $t' = *t_1 t_2$. Let t' be not a terminal subterm of t. Then $*u_i v_i \alpha *u_{i+1} v_{i+1}$ implies $u_i \alpha u_{i+1}$, $v_i \alpha v_{i+1}$ $(i = 1, 2, \ldots, k)$, i.e $v \beta w$. If t' is a terminal subterm of t, then we will use the following property:

$$*u_i v_i \alpha *u_{i+1} v_{i+1} \Rightarrow *u_i v_i\, \alpha_0\, *u_{i+1} v_{i+1}$$

which is an easy consequence from the definitions of α_0 and α and (3.2). In such a way the transtivity of α_0 implies $*uv\, \alpha_0 *uw$, and since v and w have a form of a terminal subterm of t, we get $v\, \alpha_0\, w$, i.e. $v \beta w$. ∎

We can formulate and prove Theorem 3.2′ for right cancellative n-groupoids in an easy way by exchanging the definition of a terminal subword by an intitial subword, left by right, and reformulating the conditions (i) and (ii) in Theorem 3.2.

When cancellative n-groupoids are regarded, one can prove in the same manner as above the following

Theorem 3.3. *A cancellative n-groupoid (A,f) is an n-subgroupoid of a cancellative groupoid iff for any representation of t in the forms*

$$t(x_1^n) = t_1(x_1^k, t_4(x_{k+1}^{k+s}), x_{k+s+1}^n) = t_2(x_1^r, t_4(x_{r+1}^{r+s}), x_{r+s+1}^n)$$
$$= t_3(x_1^m, t_4(x_{m+1}^{m+s}), x_{m+s+1}^n),$$

the following quasiidentities are satisfied in (A, f):

$$f(x'^k_1, x_1^s, x'^{n-s}_{k+1}) = f(x'^k_1, y_1^s, x'^{n-s}_{k+1})\ \&\ f(x''^r_1, y_1^s, x''^{n-s}_{r+1}) = f(x''^r_1, z_1^s, x''^{n-s}_{r+1})$$
$$\Rightarrow f(x'''^m_1, x_1^s, x'''^{n-s}_{m+1}) = f(x'''^m_1, z_1^s, x'''^{n-s}_{m+1}).\ ∎$$

4. Remarks. 1. We should note that the above results are generalizations of some of the results given in [1], and that further generalizations are possible. For example, let the representation of a ternary operation f be given by $t = **x_1x_2*x_1x_3$. Then one can prove that a ternary groupoid

(A,f) is a 3-subgroupoid of a cancellative groupoid iff it is left cancellative and 2-cancellative.

2. If the representation of the unary operation f of the 1-groupoid (A,f) is a variable, then (A,f) is an 1-subgroupoid of a groupoid $(G, \circ)$ iff $A \subseteq G$.

3. If the representation t of the n-ary operation f contains k variables, $k < n$, then an n-groupoid (S,f) is an n-subgroupoid of a groupoid only if f depends actually on k variables.

4. It could be obtained results similar to that given in 3 by using balanced terms as representations of the n-ary operations in which the variables do not appear in the natural order. In that case slightly different reformulations of Theorems 3.2, 3.2′, 3.3 should be made.

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n-ПОДГРУПОИДИ ОД ГРУПОИДИ СО КРАТЕЊЕ

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Резиме

Во работава се разгледуваат обопштувања на дел од резултатите дадени во [1]. Имено, се разгледуваат $n(t)$-подгрупоидите од групоидите со кратење, каде t е групоиден терм од некој специјален облик. Познато е дека класата од $n(t)$-подгрупоидите на групоидите со кратење е квазимногукратност ([2], [3]), а овде се дава опис на таа квазимногукратност.