REPRESENTATIONS OF UNARY ALGEBRAS IN UNARS

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A *unary F-algebra* is a universal algebra $\mathbf{A} = (A; F)$ with a carrier A on which each $f \in F$ induces a unary operation $a \mapsto a^f$. If $F = \{f\}$ is a one-element set, then $\mathbf{A} = (A; f)$ is called a *unar*, and then we usually write a' instead of a^f. If $(B; ')$ is a unar, then a mapping $(b, k) \mapsto b^k$ of $B \times N$ into B is defined by: $b^0 = b$, $b^{k+1} = (b^k)'$, where $b \in B$, $k \in N$ (N is the set of non-negative integers). Let $l: f \mapsto |f|$ be a mapping of F into N, $\mathbf{A} = (A; F)$ be an F-algebra and $\mathbf{B} = (B; ')$ be a unar. A mapping $\varphi: A \to B$ is called an *l-homomorphism of* $\mathbf{A}$ into $\mathbf{B}$ if

$$\varphi(x^f) = (\varphi(x))^{|f|}$$

for any $x \in A$ and $f \in F$. If α is a cardinal number such that $\text{Card}\,\varphi(A) \leqslant \alpha$ for every l-homomorphism $\varphi: \mathbf{A} \to \mathbf{B}$ and the equality $\alpha = \text{Card}\,\psi(A)$ holds for at least one l-homomorphism $\psi: \mathbf{A} \to \mathbf{B}$, then we say that α is the *l-order* of $\mathbf{A}$, and we write $\|\mathbf{A}\|_l = \alpha$ or, simply, $\|\mathbf{A}\| = \alpha$. If $\|\mathbf{A}\| = 1$, then $\mathbf{A}$ is said to be *l-singular*.

Some properties concerning l-orders, or l-singularity of unary algebras are shown in this paper. Namely, we show that almost all the results obtained in [2] for semigroup orders of universal algebras have corresponding analogies for unary algebras.

1. SINGULAR UNARY ALGEBRAS

Consider first the case when $\text{Card}\,F \geqslant 2$. Let f, g be two different elements of F such that $m = |f| \geqslant |g| = n$. Suppose that A is a non-empty set and e a fixed element of A. Let $\mathbf{A} = (A; F)$ be a unary F-algebra such that

$$(\forall x \in A) \quad x^f = x, \; x^g = e\,.$$

If $\mathbf{B} = (B; ')$ is a unar and $\varphi: \mathbf{A} \to \mathbf{B}$ an l-homomorphism, then we have:

$$\varphi(x) = \varphi(x^f) = (\varphi(x))^m = ((\varphi(x))^n)^{m-n} = (\varphi(x^g))^{m-n} = (\varphi(e))^{m-n} =$$
$$= (\varphi(e^g))^{m-n} = ((\varphi(e))^n)^{m-n} = (\varphi(e))^m = \varphi(e^f) = \varphi(e),$$

for any $x \in A$. Therefore $\mathbf{A}$ is an l-singular algebra.

Thus we have proved the following proposition:

1.1. *If* Card $F \geqslant 2$, *then any non-empty set A is the carrier of an l-singular algebra.* ∎

Assume again that Card $F \geqslant 2$, and let f, g, m, n be as above. Let A be a non-empty set and e an object such that $e \notin A \times N$. Let $\mathbf{A}^*$ be an F-algebra with the carrier $A^* = \{e\} \cup A \times N$ such that:

(i) $e^g = e^f = e$,

(ii) $(x, k+1)^g = e, \ (x, k+1)^f = (x, k)$ for any $x \in A, \ k \in N$.

Let $\varphi: A^* \to B$ be an l-homomorphism from $\mathbf{A}^*$ into a unar $\mathbf{B}$. Then we have:

$$\varphi(x, k) = \varphi((x, k+1)^f) = (\varphi(x, k+1))^m = ((\varphi(x, k+1))^n)^{m-n} =$$
$$= ((\varphi((x, k+1)^g))^{m-n} = (\varphi(e))^{m-n} = ((\varphi(e))^n)^{m-n} =$$
$$= (\varphi(e))^m = \varphi(e^f) = \varphi(e),$$

for every $x \in A, \ k \in N$. This implies that $\mathbf{A}^*$ is an l-singular algebra.

Now, we can show the following proposition.

1.2. *If* Card $F \geqslant 2$, *then every F-algebra is a subalgebra of an l-singular F-algebra.*

Namely, a unary F-algebra $\mathbf{A} = (A; F)$ can be embedded as a subalgebra in an algebra $\mathbf{A}^*$ defined as above. ∎

It remains the case when $F = \{f\}$ is a one-element set. Then, if $|f| = n$, we say „an *n-singular unar*" instead of „an l-singular unar".

We have shown in [3] that if $(A; f)$ is a unar and $n \geqslant 1$, then there exists a unar $(B; ')$ such that $A \subseteq B$ and $a^f = a^n$, for any $a \in A$. This implies the following result:

1.3. *If* $n \geqslant 1$, *then a unar* $(A; f)$ *is n-singular iff* Card $A = 1$. ∎

We recall (see, for example, [5]) that if a relation $\sim$ is defined in a unar $(A; f)$ by

$$x \sim y \Leftrightarrow (\exists p, q \in N) \ x^{f^p} = y^{f^q},$$

then a congruence is obtained, and if $(B; ')$ is the corresponding factor-unar, then we have: $b' = b$, for any $b \in B$. Then the canonical mapping

$$\text{nat}_\sim : a \mapsto b \quad (a \in b)$$

is a 0-homomorphism of $(A; f)$ in $(B; ')$. Assume now that φ is an arbitrary *0-homomorphism* from $(A; f)$ into a unar $(C; *)$, i.e. $\varphi(a^f) = \varphi(a)$ for every $a \in A$. This implies that:

$$x \sim y \Rightarrow \varphi(x) = \varphi(y), \quad \text{i.e.} \sim \subseteq \ker \varphi.$$

A $\sim$-equivalence class is called a *connected class* of $(A;f)$ and the unar is *connected* iff there exists only one connected class.

Thus we have the following result:

1. 4. *The* 0-*order of a unar is the number of its connected classes. (Therefore, a unar is* 0-*singuldr iff it is connected.)* ∎

As a corollary we obtain the following two propositions:

1. 5. *Let A be a non-empty set and α a cardinal number such that* $0 < \alpha \leqslant \operatorname{Card} A$. *Then there is a unar $(A;f)$ with the* 0-*order α. Therefore, any non-empty set is the carrier of a* 0-*singular unar.)* ∎

1. 6. *Let* **B** *be a subunar of a unar* **A**. *If α is the* 0-*order of* **A** *and β is the* 0-*order of* **B**, *then* $\beta \leqslant \alpha$. *(Thus, every subunar of a* 0-*singular unar is a* 0-*singular unar.)* ∎

From the above results it follows that neither of the propositions **1. 1**, **1. 2** hold for n-singular unars if $n > 0$. As concerns the 0-singularity, we have the same situation with **1. 2**, but **1. 1** is satisfied.

2. UNARY F-ALGEBRAS WITH ARBITRARY UNARY ORDERS

Let $\mathbf{A} = (A; F)$ be a unary F-algebra and let $l: F \rightarrow N$ be an arbitrary mapping. Denote by $F(A)$ the set $\{x^f \mid x \in A,\ f \in F\}$, and put $B = \{e\} \cup (A \setminus F(A))$, where $e \notin A \setminus F(A)$. If a unar $\mathbf{B} = (B;')$ is defined by $(\forall\, x \in B)\, x' = e$ and $\varphi: A \rightarrow B$ is defined by

$$\varphi(x) = \begin{cases} x & \text{if } x \in A \setminus F(A), \\ e & \text{if } x \in F(A), \end{cases}$$

then an l-homomorphism from **A** into **B** is obtained. Thus:

2. 1. *If* $\mathbf{A} = (A; F)$ *is an arbitrary F-algebra and* $l: F \rightarrow N$ is *an arbitrary mapping, then the following inequality is satisfied:*

$$\|\mathbf{A}\| \geqslant \operatorname{Card}(A \setminus F(A)) + 1. \blacksquare \tag{2. 1}$$

As a consequence from (2. 1) we obtain that:

2. 2. *If* $\mathbf{A} = (A; F)$ *is l-singular for some l, then it is surjective,* $F(A) = A$. ∎

Let **A** be a subalgebra of a unary F-algebra $\mathbf{A}^*$ and let $\varphi: \mathbf{A}^* \rightarrow \mathbf{B}$ be such an l-homomorphism that $\operatorname{Card} \varphi(A^*) = \|\mathbf{A}^*\|$. Then the restriction φ_A of φ on A is an l-homomorphism as well and this implies that:

$$\begin{aligned} \|\mathbf{A}^*\| &= \operatorname{Card} \varphi(A^*) = \operatorname{Card} \varphi_A(A) + \operatorname{Card} \varphi(A^* \setminus A) \\ &\leqslant \|\mathbf{A}\| + \operatorname{Card}(A^* \setminus A). \end{aligned}$$

Therefore the following proposition holds:

2. 3. *If* **A** *is a subalgebra of a unary F-algebra* $\mathbf{A}^* = (A^*; F)$, *then the following inequality is satisfied:*

$$\|\mathbf{A}^*\| \leqslant \|\mathbf{A}\| + \mathrm{Card}\,(A^* \setminus A) \tag{2.2}$$

for any mapping $l: F \to N$. ∎

Now we will show that every F-algebra **A** is a subalgebra of an F-algebra $\mathbf{A}^*$ such that the equality holds in (2. 2).

2. 4. *Let* **A** *be a unary F-algebra and* α *an arbitrary cardinal number. There is an F-algebra* $\mathbf{A}^* = (A^*; F)$ *such that* **A** *is a subalgebra of* $\mathbf{A}^*$ *and the following equalities are satisfied:*

$$\alpha = \mathrm{Card}\,(\mathbf{A}^* \setminus A),\ \|\mathbf{A}^*\| = \|\mathbf{A}\| + \alpha.$$

Proof. We can assume that $\alpha > 0$, for if $\alpha = 0$, then there is nothing to prove. Let C be a set disjoint with A such that $e \in C$ and $\mathrm{Card}\, C = \alpha$. Let $A^* = A \cup C$ and let $\mathbf{A}^* = (A^*; F)$ be defined in the following way:

(i) **A** is a subalgebra of $\mathbf{A}^*$;

(ii) $(\forall x \in C,\ f \in F)\ x^f = e$.

Then, by **2. 3**, we have: $\|\mathbf{A}^*\| \leqslant \|\mathbf{A}\| + \alpha$.

Let $\varphi: A \to B$ be an l-homomorphism such that $B \cap C = \emptyset$ and $\mathrm{Card}\ \varphi: (A) = \|\mathbf{A}\|$. Define a unar $\mathbf{B}^* = (B \cup C; ')$ such that:

(iii) **B** is a subunar of $\mathbf{B}^*$;

(iv) $(\forall x \in C)\ x' = e$.

Extend the mapping φ to a mapping $\psi: A^* = A \cup C \to B \cup C = B^*$ *by*

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in A \\ x & \text{if } x \in C. \end{cases}$$

Then $\psi: \mathbf{A}^* \to \mathbf{B}^*$ is an l-homomorphism such that $\psi(A^*) = \psi(A) \cup C$ and therefore we obtain: $\|\mathbf{A}^*\| \geqslant \mathrm{Card}\,\psi(A^*) = \mathrm{Card}\,\varphi(A) + \alpha = \|\mathbf{A}\| + \alpha$. This, finally, implies that $\|\mathbf{A}^*\| = \|\mathbf{A}\| + \alpha$, which completes the proof. ∎

Further on the algebra $\mathbf{A}^*$ obtained in the proof of the previous proposition will be denoted by $\mathbf{A}(C)$.

Now we can generalize the proposition **1. 2.**

2. 5. *If* $\alpha\,(\neq 0)$ *is a given cardinal and* $\mathrm{Card}\, F \geqslant 2$, *then every F-algebra* **A** *is a subalgebra of an F-algebra* $\mathbf{A}^{**}$ *such that* $\|\mathbf{A}^{**}\| = \alpha$.

Proof. By **1.2**, **A** is a subalgebra of an l-singular algebra $\mathbf{A}^*$. If $\alpha = 1$, then $\mathbf{A}^*$ is the desired algebra, and thus we can assume that $\alpha > 1$. Let C be a non-empty set such that $A \cap C = \emptyset$ and $1 + \operatorname{Card} C = \alpha$. Then $\mathbf{A}^*$ is a subalgebra of $\mathbf{A}^*(C)$ and by **2.4.** we have

$$||\mathbf{A}^*(C)|| = ||\mathbf{A}^*|| + \operatorname{Card} C = 1 + \operatorname{Card} C = \alpha . \blacksquare$$

The proposition **1.1** can be generalized as well. Assume that $\operatorname{Card} F \geqslant 2$ and that α is a given cardinal such that $1 \leqslant \alpha \leqslant \operatorname{Card} A$. Let $A = \mathbf{A}^* \cup C$, $A^* \cap C = \emptyset$ and $1 + \operatorname{Card} C = \alpha$. By **1.1**, there is an l-singular algebra $\mathbf{A}^* = (A^*; F)$. If $\mathbf{A} = \mathbf{A}^*(C)$, then by **2.4** we have

$$||\mathbf{A}|| = ||\mathbf{A}^*|| + \operatorname{Card} C = 1 + \operatorname{Card} C = \alpha .$$

Thus we have the following proposition:

2.6. *Let* $\operatorname{Card} F \geqslant 2$ *and let* A *be a non-empty set. If* α *is a cardinal number such that* $1 \leqslant \alpha \leqslant \operatorname{Card} A$, *then there is an* F*-algebra* $\mathbf{A} = (A; F)$ *such that* $||\mathbf{A}|| = \alpha$. $\blacksquare$

If $F = \{f\}$ is a one-element set and if $|f| = n \geqslant 1$, then neither of the propositions **2.5**, **2.6** hold, for then the n-order of a unar is the usual order of the unar. And, if $|f| = 0$, then by **1.5** and **1.6** the proposition **2.6** is satisfied as well, but **2.5** does not hold.

The l-order of an F-algebra is closely connected with the *l-universal unar* $\mathbf{A}^\wedge$ for the given algebra $\mathbf{A} = (A; F)$. Namely, $\mathbf{A}^\wedge$ is the unar with the following presentation:

$$< A; \{b = a^{|f|} \mid b = a^f \text{ in } \mathbf{A}\} > \tag{2.3}$$

in the class of unars. A more explicit construction of $\mathbf{A}^\wedge$ can be found in [3]. If $a, b \in A$ and a, b define the same element in $\mathbf{A}^\wedge$, then we write $a \approx b$ and we say that a and b are *equivalent*. Now we can state the following proposition:

2.7. *The relation* $\approx$ *is an equivalence on* A *and* $||\mathbf{A}|| = \operatorname{Card}(A/\approx)$. *(Therefore,* **A** *is l-singular iff* $(\forall a, b \in A)\ a \approx b$.*)* $\blacksquare$

We note that if $|f| = 0$ for each $f \in F$, then $\approx$ is the congruence on **A** generated by $\{(a, a^f) \mid a \in A\}$.

3. UNARY ORDERS OF J-UNARS

Let J be a subsemigroup of the additive semigroup of positive integers and let A be a non-empty set. If $(a,n) \mapsto an$ is a mapping from $A \times J$ into A satisfying the following condition

$$(\forall a \in A, \quad m, n \in J) \quad a(m+n) = (am)n , \tag{3.1}$$

then we say that $\mathbf{A} = (A; J)$ is a *J-unar*. (In other words, a *J*-unar is a *right J-system* [1; 11. 1].) A *J*-unar can be also considered as a unary *F*-algebra, where

$$F = \{f_n \mid n \in J\} \text{ and } (\forall a \in A, \quad n \in J) \ a^{f_n} = an.$$

If we define a mapping $l: F \to N$ by $l(f_n) = n$, we can speak of the notion of the *l*-order of a *J*-unar. In this case we will say „*unary order of* $\mathbf{A}$" instead of „*l*-order of $\mathbf{A}$"; and the meaning of the notion „*a singular J-unar*" will be clear.

We need some results on additive semigroups of positive integers.

3. 1. *Let J be an additive semigroup of positive integers.*

(*i*) *There exists a uniquely determined minimal generating subset* $K = \{n_1, \ldots, n_k\}$ *of J, which is called the basis of J.*

(*ii*) *If d is the largest common divisor of the numbers in K, then there exists a* $t \in N$ *such that* $t + \nu d \in J$, *for any* $\nu \geqslant 0$. (*If* t_0 *is the minimal number with that property, then the set* $R(J) = \{t_0 + \nu d \mid \nu \in N)$ *is called the regular part of J. The basis of* $R(J)$ *will be denoted by* $P = \{m_1, \ldots, m_p\}$.) ([4]) ∎

The universal unar $\mathbf{A}^{\wedge}$ for a *J*-unar $\mathbf{A}$ is defined as in the previous section. Therefore $\operatorname{Card}(A/\approx)$ is the unary order of the *J*-unar $\mathbf{A}$.

The following two results are proved in [3]:

3. 2. *If* $a, b \in A$ *and* $m \in R(J)$, *then*

$$a \approx b \Rightarrow am = bm. \ ∎ \tag{3. 2}$$

3. 3. *If a J-unar* $(A; J)$ *is surjective, i.e.* $AJ = A$, *then* $An = A$ *for any* $n \in J$. ∎

Now it is easy to show that every singular *J*-unar is trivial. Namely, if $(A; J)$ is a singular *J*-unar, then by **2. 2** it is surjective, and by **3. 3** we have $An = A$ for any $n \in J$. The singularity also implies that $a \approx b$ for any $a, b \in A$, and this by **3. 2** implies that $am = bm$ for any $m \in R(J)$; thus, if $m \in R(J)$, the mapping $x \mapsto xm$ is a constant; on the other hand we have $Am = A$, and therefore we obtain that $\operatorname{Card} A = 1$. Thus we have proved the following proposition:

3. 4. *A J-unar* $(A; J)$ *is singular iff* $\operatorname{Card} A = 1$. ∎

Some connections between the unar order of a *J*-unar $(A; J)$ and $\operatorname{Card} A$ will be established below.

Let $(A;J)$ be a J-unar and let $K, R(J)$, P be as in **3. 1** and $Q = J\setminus R(J)$, $\operatorname{Card} Q = q$. Let α be the unary order of the given J-unar and B, C, A' be subsets of A defined as follows:

$$B = AP, \quad C = AJ\setminus B, \quad A' = A\setminus AJ.$$

If $a, b \in A$ are such that $a \approx b$, then for each $m \in P$ we have $am = bm$, and this implies that $\operatorname{Card} Am \leqslant \alpha$, i. e. $\operatorname{Card} B \leqslant \alpha p$. By **2. 1** we have that $\operatorname{Card} A' \leqslant \alpha - 1$. If $c \in C$ and if n is the maximal number of $J\setminus R(J)$ such that $c \in An$, then there is an element $a \in A'$ such that $c = an$. This implies that $\operatorname{Card} C \leqslant q(\alpha - 1)$. Finally we obtain the following relation:

$$\operatorname{Card} A = \operatorname{Card} A' + \operatorname{Card} B + \operatorname{Card} C \tag{3. 3}$$
$$\leqslant (\alpha - 1) + q(\alpha - 1) + p\alpha = \alpha(1 + p + q) - (q + 1).$$

The following propositions are obvious corollaries of (3. 3).

3. 5. *The unary order of an infinite J-unar $(A;J)$ is the cardinal of A, i. e. it is the usual order of the J-unar.* ∎

3. 6. *A J-unar $\mathbf{A} = (A;J)$ has a finite unar order α iff* $\operatorname{Card} A = \beta$ *is finite, and then we have:*

$$\beta \leqslant (p + q + 1)\alpha - (q + 1). \ ∎ \tag{3. 3'}$$

Therefore the notion of unary order of J-unars could be of interest for finite J-unars only.

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РЕЗИМЕ

ПРЕТСТАВУВАЊЕ УНАРНИ АЛГЕБРИ ВО УНАРИ

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Една универзална алгебра $\mathbf{A} = (A; F)$ со носител A и множество F од унарни оператори, такви што секој $f \in F$ индуцира унарна операција $a \mapsto a^f$ на A, се вика *унарна F-алгебра*. Ако $F = \{f\}$, тогаш $\mathbf{A} = (A; f)$ се вика *унар* и, во тој случај, обично пишуваме a' наместо a^f.

Нека $\mathbf{A} = (A; F)$ е унарна алгебра, нека $l: f \mapsto |f|$ е пресликување од F во множеството N на природните броеви и нека $\mathbf{B} = (B; ')$ е унар. Едно пресликување $\varphi: A \to B$ се вика *l-хомоморфизам* од $\mathbf{A}$ во $\mathbf{B}$ ако $\varphi(x^f) = (\varphi(x))^{|f|}$ за секој $x \in A$ и $f \in F$. Ако α е кардинален број, таков што $\mathrm{Card}\, \varphi(A) \leqslant \alpha$ за секој l-хомоморфизам φ од $\mathbf{A}$ во некој унар $\mathbf{B}$ и важи равенството $\alpha = \mathrm{Card}\, \psi(A)$ барем за еден l-хомоморфизам ψ од $\mathbf{A}$, тогаш за α велиме дека е *унарен l*-ред на $\mathbf{A}$ и пишуваме $\|\mathbf{A}\|_l = \alpha$ или, само, $\|\mathbf{A}\|$. Ако $\|\mathbf{A}\| = 1$, тогаш за F-алгебрата $\mathbf{A}$ велиме дека е *l-сингуларна*.

Во работава се испитуваат некои својства на унарните алгебри во врска со поимите l-ред и l-сингуларност. Се покажува, меѓу другото, дека скоро сите резултати, добиени во [2] за полугрупен ред на универзални алгебри, имаат соодветни аналогии за унарни алгебри.

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