Proceedings of the Symposium
n-ARY STRUCTURES, Skopje 1982

POST THEOREMS FOR UNARY ALGEBRAS

Ǵ.Čupona, N.Celakoski, A.Samardžiski

The well-known Post Coset Theorem ([1]) states that any n-group (Q,f) is embeddable into a group (G,o) in such a way that f is the restriction of o^{n-1} on Q. Several generalizations of this Theorem are known (ex. [2], [3] and [4]). It is assumed in all of them that the arities of the corresponding operations are larger than 1. We show in this paper that almost all known "Post Theorems" have corresponding "unary translations".

1. Embeddings of unary algebras in unars

An algebra $\underline{B} = (B;')$, where B is a nonempty set and $':x \to x'$ is a transformation of B, is called a <u>unar</u> ([5]). A mapping $(x,n) \mapsto x^n$ of $B \times N$ into B (where N is the set of nonnegative integers) can be defined in the following way:

$$x^0 = x, \qquad x^{n+1} = (x^n)'. \tag{1.1}$$

Let F be a nonempty set of elements called <u>unary operators</u>. If A is a nonempty set and if each operator $f \in F$ is interpreted as a unary operation $f_{\underline{A}}:x \mapsto x^f$ on A (i.e. $f_{\underline{A}}$ is a transformation of A), then we say that $\underline{A}$ is a (unary) <u>F-algebra</u> with a carrier A. Let $L: f \to |f|$ be a mapping of F into the set N of nonnegative integers. (Throughout the paper it is assumed that any appearing F-algebra is considered with a mapping $L:F \to N$.) A mapping $\phi:A \to B$ is called an <u>L-homomorphism</u> of an F-algebra $\underline{A} = (A;F)$ into a unar $\underline{B} = (B;')$ if

$$(\forall x \in A,\ f \in F) \quad \phi(x^f) = \phi(x)^{|f|}. \tag{1.2}$$

An F-algebra $\underline{A}$ is called an *ℓ-subunar* of a unar $\underline{B}$ iff there exists an injective ℓ-homomorphism of $\underline{A}$ into $\underline{B}$. Thus, $\underline{A}$ is an ℓ-subunar of a unar iff there exists a unar $(B;')$ such that

$$A \subseteq B \quad \text{and} \quad (\forall x \in A,\ f \in F) \quad x^f = x^{|f|}. \qquad (1.3)$$

A description of the class of ℓ-subunars of unars will be given below.

First we define the notion of universal ℓ-enveloping unar. Namely, if $\underline{A}$ is an F-algebra with the carrier A, then the unar $\underline{A}^{\ell}$ with the following presentation (in the class of unars)

$$\langle A;\ \{b = a^{|f|} \mid b = a^f,\ a \in A,\ f \in F\}\rangle \qquad (1.4)$$

is called the *universal ℓ-enveloping unar* of $\underline{A}$.

Now we will give a more explicit description of $\underline{A}^{\ell}$.

Define a transformation of $A \times N$ by $(a,n)' = (a,n+1)$. Then the algebra $\underline{C} = (A \times N;')$ is the *free unar* on A. If $b = a^f$ in $\underline{A}$, then we write

$$(b,n) \vdash (a,n+|f|), \quad (a,n+|f|) \dashv (b,n).$$

Denote by $\vdash\!\dashv$ the disjunction of $\vdash$ and $\dashv$, i.e. $\vdash\!\dashv$ is the symmetric extension of $\dashv$. Let $\approx$ be the reflexive and transitive extension of $\vdash\!\dashv$, i.e. $\approx$ is defined in the following way:

$$u \approx v \iff (\exists p \ge 0,\ u_o,\dots,u_p)\ u = u_o,\ v = u_p,$$
$$(i \ge 1 \Rightarrow u_{i-1} \vdash\!\dashv u_i).$$

Then $\approx$ is an equivalence relation on $A \times N$ and $u \approx v \Rightarrow u' \approx v'$, i.e. $\approx$ is a congruence on $\underline{C} = (A \times N;')$. The factor unar $\underline{C}/\approx$ is in fact the universal ℓ-enveloping unar $\underline{A}^{\ell}$ of $\underline{A}$.

The following proposition gives another description of the universal ℓ-enveloping unar $\underline{A}^{\ell}$.

<u>1.1</u>. The mapping $\Lambda : a \mapsto a^{\approx}$ (where $a^{\approx}$ is the $\approx$-equivalence class containing a) is an ℓ-homomorphism of $\underline{A}$ into A^{ℓ}, and if $\phi : \underline{A} \to \underline{B}$ is an arbitrary ℓ-homomorphism of $\underline{A}$ into a unar $\underline{B}$, then there exists a unique homomorphism $\phi^{\ell} : \underline{A}^{\ell} \to \underline{B}^{\ell}$ such that $\phi = \phi^{\ell} \Lambda$.

(In other words, Λ is a universal ℓ-nomomorphism.)

Now assume that $\underline{A}$ is an ℓ-subunar of a unar $\underline{B}$, i.e. there is an injective ℓ-homomorphism $\phi : \underline{A} \to \underline{B}$. If $a_1, a_2 \in A$ and $a_1^{\approx} = a_2^{\approx}$, i.e. $\Lambda(a_1) = \Lambda(a_2)$, then

$$\phi(a_1) = \phi^{\ell}\Lambda(a_1) = \phi^{\ell}\Lambda(a_2) = \phi(a_2)$$

and this implies $a_1 = a_2$. Therefore we have:

$$a_1, a_2 \in A \Longrightarrow (a_1 \approx a_2 \Longrightarrow a_1 = a_2). \qquad (1.5)$$

Conversely, if (1.5) is satisfied, then Λ is an injective ℓ-homomorphism of $\underline{A}$ into $\underline{A}^{\ell}$.

We can assume in this case that $A \subseteq A^{\ell}$ by putting $a = a^{\approx}$ for any $a \in A$, and we say that $\underline{A}^{\ell}$ is the <u>universal ℓ-covering unar</u> of $\underline{A}$.

Thus we proved the following proposition.

<u>1.2</u>. An F-algebra is an ℓ-subunar of a unar iff (1.5) is satisfied. □

It is also clear that:

<u>1.3</u>. If $\underline{A}$ is an ℓ-subunar of a unar and if $f \in F$ is such that $|f| = 0$, then $(\forall x \in A)\ \ x^f = x$. □

Now consider the case when $F = \{f\}$ (F consists of one operator only), and let $|f| = n > 0$. Then it can be easily shown that the condition (1.5) is satisfied, which will imply that (A;f) is an ℓ-subunar (or, more specific, an n-subunar) of a unar. Thus we have the following proposition:

<u>1.4</u>. If (A;f) is a unar and n a positive integer, then there exists a unar (B;´) such that $A \subseteq B$ and $x^f = x^n$ for any $x \in A$. □

We note that a direct proof of 1.4 is more convenient. Namely, let $B = A \cup A \times \{1,2,\dots,n-1\}$ and let $u \mapsto u'$ be a transformation of B defined by

$$x' = (x,1),\ (x,n-1)' = x^f,\ (x,i)' = (x,i+1)$$

for any $x \in A$ and $i \in \{1,2,\dots,n-2\}$. Then $(B;')$ has the desired property. Namely, the unar $(B;')$ is the universal ℓ-covering of $(A;f)$, and it is called the universal n-covering of $(A;f)$.

Consider again the general case. Let $\underline{A}$ be a unary F-algebra and let F^* be the free monoid on F, i.e. the monoid of finite sequences (including the empty sequence e) on F. Then a unary F-algebra $\underline{A}^* = (A;F^*)$ with the same carrier A can be defined by

$$(\forall x \in A,\ f_1,\dots,f_r \in F)\ x^e = x,\ x^{f_1\dots f_r} = (x^{f_1\dots f_{r-1}})^{f_r} \quad (1.6)$$

And, if $\ell : f \mapsto |f|$ is a mapping of F into N, then it can be extended to a mapping $\ell^* : f \mapsto |f|^*$ of F^* into N in the following way:

$$|e|^* = 0,\ |f_1\dots f_r|^* = |f_1| + \dots + |f_r|. \quad (1.7)$$

(Namely, ℓ^* is the unique homomorphism from F^* into N, which extends ℓ.) Further on we will often write $|f|$ instead of $|f|^*$.

It is easy to show that the following properties hold.

1.5. $\underline{A}$ is an ℓ-subunar of a unar iff $\underline{A}^*$ is an ℓ^*-subunar of a unar.□

1.6. Let $\underline{A}$ be an F-algebra, $\underline{B} = (B;')$ a unar and $\phi : A \to B$ a mapping. Then ϕ is an ℓ-homomorphism iff ϕ is an ℓ^*-homomorphism. □

1.7. The universal ℓ-enveloping unar of $\underline{A}$ and the universal ℓ^*-enveloping unar of $\underline{A}^*$ coincide. □

1.8. The set $J = \{|f|^* \mid f \in F^*\}$ is a submonoid of the additive monoid of nonnegative integers and it is generated by the set $\{|f| \mid f \in F\}$. □

2. The quasivariety of ℓ-subunars of unars

As a consequence of a more general result (see, for example, [6], p. 274 or [7]), it follows that the class of ℓ-subunars of unars is a quasivariety of F-algebras. The proposition 1.2 gives a description of this quasivariety, and below we will give a more explicit description of the set of quasiidentities which determines this quasivariety.

Namely, by 1.2, an F-algebra $\underline{A}$ is an ℓ-subunar of a unar iff: $a,b \in A \Rightarrow (a \approx b \Rightarrow a = b)$. By the definition of the relation $\approx$ it follows that: $a \approx b$ iff there exist $a_1, a_2, \ldots, a_p \in A$ and $f_1, f_2, \ldots, f_{2p} \in F^*$ such that the following relations are satisfied (we write $|f|$ instead of $|f|^*$, as it was mentioned before 1.5):

$$a = a_1^{f_1},\ a_1^{f_2} = a_2^{f_3}, \ldots, a_{p-1}^{f_{2p-2}} = a_p^{f_{2p-1}},\ a_p^{f_{2p}} = b; \quad (2.1)$$

$$\begin{aligned} &|f_1| = |f_2| + m_1, \\ &|f_3| + m_1 = |f_4| + m_2, \qquad\qquad (m_\nu \geq 0) \quad (2.2)\\ &\cdots\cdots\cdots\cdots \\ &|f_{2p-3}| + m_{p-2} = |f_{2p-2}| + m_{p-1}, \\ &|f_{2p-1}| + m_{p-1} = |f_{2p}|, \end{aligned}$$

i.e.

$$\sum_{\nu=1}^{2i} (-1)^{\nu+1}|f_\nu| \geq 0,\ |f_{2p}| = \sum_{\nu=1}^{2p-1} (-1)^{\nu+1}|f_\nu| \quad (2.3)$$

$$(i \in \{1,2,\ldots,p\}).$$

We note that if $f_\nu = e$ for some ν (e is the empty sequence on F), then we can "shorten" the sequence $f_1, f_2, \ldots, f_{2p}$, i.e. (2.1). For example, if $f_2 = e$, then we can consider the sequence $f_1, f_3, f_4, \ldots, f_{2p}$ and then

$$a = a_2^{f_3 f_1},\ a_2^{f_4} = a_3^{f_5}, \ldots, a_p^{f_{2p}} = b. \quad (2.1')$$

Thus we have the following proposition:

2.1. Let $\underline{A}$ be an F-algebra. $\underline{A}$ is an ℓ-subunar of a unar iff it satisfies any quasiidentity of the following form:

$$x = x^{f_1} \& x_1^{f_2} = x_2^{f_3} \& \dots \& x_p^{f_{2p}} = y \Longrightarrow x = y, \qquad (2.4)$$

where $f_\nu \in F^*$ are such that (2.2) and (2.3) hold. ▯

Now, we will find the set of identities which hold in every ℓ-subunar of a unar.

Let $\underline{A}$ be an ℓ-subunar of a unar $(B;')$. If $f \in F$ is such that $|f| = 0$, then for any $a \in A$ we have

$$a^f = a^{|f|} = a^0 = a.$$

Also, if $f_1, \dots, f_r, g_1, \dots, g_s \in F$ are such that

$$|f_1| + \dots + |f_r| = |g_1| + \dots + |g_s|, \qquad (2.5)$$

then for any $a \in A$ we have

$$a^{f_1 \dots f_r} = a^{|f_1| + \dots + |f_r|} = a^{|g_1| + \dots + |g_s|} = a^{g_1 \dots g_s}.$$

Conversely, assume that

$$x^{f_1 \dots f_r} = x^{g_1 \dots g_s} \qquad (2.6)$$

is an identity in the class of ℓ-subunars of unars. Consider the unar $(N;')$, where $x' = x+1$, and define an F-algebra $\underline{N}$ with a carrier N by putting

$$(\forall x \in N,\ f \in F) \quad x^f = x + |f|. \qquad (2.7)$$

Then, clearly, $\underline{N}$ is an ℓ-subunar of the unar $(N;')$ and therefore we have

$$|f_1| + \dots + |f_r| = 0^{f_1 \dots f_r} = 0^{g_1 \dots g_s} = |g_1| + \dots + |g_s|,$$

i.e. (2.5) is satisfied.

Thus we have proved the following proposition:

2.2. An identity holds in every ℓ-subunar of a unar iff it has a form (2.6) when (2.5) is satisfied, or

$$x^f = x \quad \text{when} \quad |f| = 0. \square \qquad (2.8)$$

An F-algebra is called an <u>ℓ-unar</u> if it satisfies all the identities that hold in the class of all ℓ-subunars of unars, i.e. $x^f = x$ when $|f| = 0$ and

$$x^{f_1 \dots f_2} = x^{g_1 \dots g_s} \quad \text{when} \quad |f_1|+\dots+|f_r| = |g_1|+\dots+|g_s|.$$

Let F^* and ℓ^* be defined as in Section 1 and let F' be a subset of F^* such that

$$\{|f| \;\big|\; f \in F'\}$$

is a generating subset of the additive monoid J (cf. 1.8). Then, by ℓ' is denoted the restriction of ℓ^* on F' and by $\underline{A}'$ the corresponding F'-algebra $\underline{A}^*|F'$.

It is easy to see that the following propositions hold.

2.3. An F-algebra $\underline{A}$ is an ℓ-unar iff $\underline{A}^*$ is an ℓ^*-unar.

2.4. If $\underline{A}$ is an ℓ-unar, then $\underline{A}'$ is an ℓ'-unar, and if in 1.5, 1.6, 1.7 we replace $\underline{A}^*$ by $\underline{A}'$ and ℓ^*, F^* by ℓ', F' respectively, then the corresponding statements 1.5′, 1.6′, 1.7′ are also true. □

Now we can give a description of the class of mappings $\ell : F \to N$ such that every ℓ-unar is an ℓ-subunar of a unar.

First, if $|f| = 0$ for all $f \in F$, then any ℓ-unar is an ℓ-subunar of a unar. Namely, in this case, an ℓ-unar $\underline{A}$ is an ℓ-subunar of any unar $(B;')$ such that $A \subseteq B$, and the free unar $(A \times N;')$ on A is the universal ℓ-covering unar of $\underline{A}$.

Thus we have to consider the case when $|f| > 0$ for some $f \in F$. Assume that the least positive integer p of $\{|f| \;|\; f \in F\}$ is a divisor of any number of this set. Then, if $F' = \{f'\}$, where $|f'| = p$, we have that p is a generator of J. Thus $\underline{A}$ is an ℓ-subunar of a unar iff $\underline{A}' = (A;f')$ has the corresponding property; but then, by 1.4, $\underline{A}'$ is a p-subunar or a unar and therefore $\underline{A}$ has the same property.

It remains the case when there exists $g \in F$ such that p is not a divisor of $q = |g|$; assume that q is the least integer with this property. We will show that there is an ℓ-unar which is not an ℓ-subunar of any unar.

Let $A = \{a,b,c\}$ be a set with three elements. An F-algebra with the carrier A can be defined as follows:

$$f \in F,\ |f| = 0 \Rightarrow (\forall x \in A)\quad x^f = x,$$

$$f \in F,\ |f| = q \Rightarrow a^f = b^f = a,\ c^f = b,$$

$$f \in F,\ |f| \neq 0,\ q \Rightarrow (\forall x \in A)\quad x^f = a$$

(where q is as above). It is easy to see that $\underline{A}$ is an ℓ-unar. This algebra is not an ℓ-subunar of a unar, for if it were, when $|f| = p$, $|g| = q$ and $p \nmid q$, then we would have

$$b = c^g = c^q = (c^p)^{q-p} = (c^f)^{q-p} = (a^f)^{q-p} =$$

$$= (a^p)^{q-p} = a^q = a^g = a,$$

which is a contradiction.

Thus we proved the following proposition:

2.5. Let F and $\ell : F \to N$ be given. The class of ℓ-unars coincides with the class of ℓ-subunars of unars (i.e. the class of ℓ-subunars of unars is a variety) iff there exists a $g \in F$ such that $|g|$ is a divisor of $|f|$ for all $f \in F$.

3. Injective and surjective J-unars

As in Section 1, the set $J = \{|f| \mid f \in F\}$ is a submonoid of the additive monoid N. Further on we will assume that $|f| > 0$ for some $f \in F$ (therefore J is infinite) and d will denote the greatest common divisor of the numbers in J. Then (cf. ex. [8]) there exists an integer $t \in J$ such that

$$R(J) = \{t + \nu d \mid \nu \in N\} \subseteq J;$$

if t is the least positive integer with this property, then we call R(J) the *regular part* of J.

Let $\underline{A}$ be an ℓ -unar and $f,g \in F^*$ be such that $|f| = |g|$. Then $x^f = x^g$ for any $x \in A$ and thus $\underline{A}$ induces a mapping

$$(x,n) \longmapsto x^{[n]}$$

of $A \times J$ into A with the following properties

$$(\forall x \in A,\ m,n \in J)\ x^{[0]} = x,\ (x^{[m]})^{[n]} = x^{[m+n]}. \qquad (3.1)$$

By this reason we will say a "J-unar" instead of " ℓ-unar" and a "J-subunar of a unar" instead of " ℓ-subunar of a unar"; we will sometimes write (A;J) to indicate that $\underline{A}$ is a J-unar.

Let $\underline{A}$ be a J-unar and $a,b \in A$ be such that $a \approx b$. Then by 2.1, there exist $n_1,n_2,\ldots,n_{2p} \in J$ and $a_1,a_2,\ldots,a_p \in A$ such that

$$a = a_1^{[n_1]},\ a_1^{[n_2]} = a_2^{[n_3]},\ldots,a_{p-1}^{[n_{2p-2}]} = a_p^{[n_{2p-1}]},\ a_p^{[n_{2p}]} = b, \qquad (3.2)$$

where:

$$\begin{aligned} n_1 &= n_2 + m_1, \\ n_3 + m_1 &= n_4 + m_2, \\ n_5 + m_2 &= n_6 + m_3, \\ &\cdots\cdots\cdots\cdots \\ n_{2p-1} + m_{p-1} &= n_{2p} \end{aligned} \qquad (3.3)$$

and $m_\nu \geq 0$. Clearly, from (3.3) it follows that d is a divisor of $m_1,\ldots,m_{p-1}$ and thus, if t belongs to $R(J)$, then $t+m_\nu$ is also in $R(J)$ for any $\nu \in \{1,2,\ldots,p\}$.

We will show that $a^{[t]} = b^{[t]}$. Assume, for technical reasons, that p=4. Then we have:

$$\begin{aligned} a^{[t]} &= (a_1^{[n_1]})^{[t]} = a_1^{[n_2+m_1+t]} = (a_1^{[n_2]})^{[m_1+t]} = (a_2^{[n_3]})^{[m_1+t]} = \\ &= a_2^{[n_3+m_1+t]} = a_2^{[n_4+m_2+t]} = (a_2^{[n_4]})^{[m_2+t]} = (a_3^{[n_5]})^{[m_2+t]} = \\ &= a_3^{[n_5+m_2+t]} = a_3^{[n_6+m_3+t]} = (a_3^{[n_6]})^{[m_3+t]} = (a_4^{[n_7]})^{[m_3+t]} = \end{aligned}$$

$$= a_4^{[n_7+m_3+t]} = a_4^{[n_8+t]} = (a_4^{[n_8]})^{[t]} =$$

$$= b^{[t]}.$$

Thus we proved the following proposition:

3.1. If $\underline{A}$ is a J-unar and if t is in the regular part R(J) of J, then

$$a \approx b \Longrightarrow a^{[t]} = b^{[t]}. \square \qquad (3.4)$$

A J-unar $\underline{A}$ is said to be injective if $x \mapsto x^{[n]}$ is an injective mapping for all $n \in J$, i.e. $\underline{A}$ satisfies any quasi-identity

$$x^{[n]} = y^{[n]} \Longrightarrow x = y, \qquad (3.5)$$

where $n \in J$.

We note that it is enough to assume that (3.5) is satisfied for one $n \in J$, $n \neq 0$. Namely, then we have $x^{[sn]} = y^{[sn]} \Longrightarrow x = y$ for any $s \in N$ and therefore if $m \in J$, then $x^{[mn]} = y^{[mn]} \Longrightarrow x = y$. Now, if $a^{[m]} = b^{[m]}$, then $a^{[sm]} = b^{[sm]}$ for any $s \in N$ and so $a^{[mn]} = b^{[mn]}$, which implies that $a = b$.

As a corollary of 3.1 we obtain:

3.2. If $\underline{A}$ is an injective J-unar, then it is a J-subunar of a unar. $\square$

A J-unar $\underline{A}$ is said to be surjective if

$$(\forall x \in A)(\exists y \in A,\ n \in N)\quad n \neq 0,\quad x = y^{[n]}. \qquad (3.6)$$

Clearly, if there is an $n \in J$, $n \neq 0$, such that the mapping $x \mapsto x^{[n]}$ is surjective, then the J-unar $\underline{A}$ is surjective. In a similar way as the corresponding property of J-associatives (see [4]) it can be shown that the converse is also true.

3.3. If $\underline{A}$ is a surjective J-unar, then for any $n \in J$ the mapping $x \mapsto x^{[n]}$ is surjective. $\square$

Now we will give an example of a surjective J-unar which is not a J-subunar of a unar.

Assume that $d \notin J$. Let t be the least positive integer in the regular part R(J) of J. Consider the free J-unar $B = (C \cup D) \times J$ on $C \cup D$, where $C = \{c_0, c_1, \ldots\}$, $D = \{d_0, d_1, \ldots\}$ are disjoint infinite sets, and $(b,m)^{[n]} = (n, m+n)$.

Define a relation $\sim$ on $(C \cup D) \times J$ by:

$$(c_0, t) \sim (d_0, t), \quad (c_i, 0) \sim (c_{i+1}, t), \quad (d_i, 0) \sim (d_{i+1}, t)$$

for any $i \in N$. Let $\equiv$ be the minimal congruence on $\underline{B}$ generated by $\sim$. Then the factor J-unar $\underline{A} = \underline{B}/\equiv$ is surjective, but it is not a J-subunar of a unar. Namely, we have $(c_o, t) \equiv (d_o, t)$, but $(c_o, t+d) \not\equiv (d_o, t+d)$.

We note that the above J-unar $\underline{A}$ can be described better as a J-unar given by the following presentation

$$C \cup D;\ c_o^{[t]} = d_o^{[t]},\ c_o = c_1^{[t]},\ c_1 = c_2^{[t]}, \ldots, d_o = d_1^{[t]},\ d_1 = d_2^{[t]}, \ldots \ .$$

This example shows that a surjective J-subunar of a unar satisfies some additional conditions, which will be considered below.

Assume that $\underline{A}$ is a J-subunar of a unar $(B;\,')$ and let $m,n,p \in N$ be such that $m,n,m+p,n+p \in J$. If $c,d \in A$ and $c^{[m]} = d^{[n]}$, then

$$c^{[m+p]} = c^{m+p} = (c^m)^p = (d^n)^p = d^{n+p} = d^{[n+p]}.$$

Therefore we have the following proposition:

<u>3.4</u>. If $\underline{A}$ is a J-subunar of a unar, and if $m,n,p \in N$ are such that $m,n,m+p,n+p \in J$, then the following quasiidentity is satisfied:

$$x^{[m]} = y^{[n]} \Rightarrow x^{[m+p]} = y^{[n+p]}. \ \square \qquad (3.7)$$

Before we give a characterisation of surjective J-subunars of unars, we will establish a lemma.

<u>3.5</u>. Let $\underline{A}$ be a J-unar which satisfies any quasiidentity of the form (3.7) and let $a, a_1, \ldots, a_p, b \in A$, $n_1, n_2, \ldots, \ldots, n_{2p} \in J$, $m_1, m_2, \ldots, m_{p-1} \in N$ be such that (3.2) and (3.3) are satisfied. If, in addition, $m_1 + n_{2i+1} \in J$ for any $i \in \{1, \ldots, p-1\}$, then a=b.

Namely, by (3.2), (3.3), (3.7) and the fact that $m_1 + n_{2i+1} \in J$, we have:

$$a = a_1^{[n_1]} = a_1^{[n_2+m_1]},$$
$$a_1^{[n_2+m_1]} = a_2^{[n_3+m_1]} = a_2^{[n_4+m_2]},$$
$$a_2^{[n_4+m_2]} = a_3^{[n_5+m_3]} = a_3^{[n_6+m_3]},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{p-1}^{[n_{2p-2}+m_{p-1}]} = a_p^{[n_{2p-1}+m_{p-1}]} = a^{[n_{2p}]} = b,$$

i.e. a=b. ▯

Now we can prove the following theorem:

<u>3.6</u>. A surjective J-unar $\underline{A}$ is a J-subunar of a unar iff $\underline{A}$ satisfies all the quasiidentities (3.7).

<u>Proof</u>. Assume that $\underline{A}$ is a J-unar which satisfies all the quasiidentities (3.7). Let $a, b \in A$ be such that $a \approx b$. Then there exist $a_1, \ldots, a_p \in A$, $n_1, \ldots, n_{2p} \in J$, $m_1, \ldots, m_{p-1} \in N$ such that (3.2) and (3.3) are satisfied. Let t be an element of the regular part R(J) of J. By <u>3.3</u>, there exist $b_1, \ldots, b_p$ such that $a = b^t$, and thus we have:

$$a = b_1^{[n_1+t]},\ b_1^{[n_2+t]} = b_2^{[n_3+t]}, \ldots, b_{p-1}^{[n_{2p-2}+t]} = b_p^{[n_{2p-1}+t]},\ b^{[n_{2p}]} = b$$

and

$$n_1 + t = n_2 + t + m_1,$$
$$n_3 + t + m_1 = n_4 + t + m_2,$$
$$n_5 + t + m_2 = n_6 + t + m_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$n_{2p-1} + t + m_{p-1} = n_{2p} + t$$

and also $n_{2i+i}+t+m_i \in J$. Now, by 3.5 we obtain a=b, and this (with 3.4) completes the proof of 3.6. □

We note that the necessary condition (3.7) for a J-unar to be a J-subunar of a unar is not sufficient, which is seen by the following:

Example. Let $J = \langle 3,4 \rangle = \{3,4,6,7,8,\ldots\}$. Consider the J-unar $\underline{A}$ generated by the set $A = \{a,b,c\}$ under the conditions:

$$a^{[3]} = b^{[4]} \qquad a^{[6+\nu]} = b^{[7+\nu]}$$
$$b^{[3]} = b^{[4]} \qquad b^{[6+\nu]} = c^{[7+\nu]} \qquad (\nu=0,1,2,\ldots).$$

Clearly, (3.7) is satisfied, but $\underline{A} = (A;J)$ is not a J-subunar of a unar. (Namely, $a^{[4]} \neq c^{[6]}$; if $\underline{A}$ were a J-subunar of a unar (B;g), then it could be easy to obtain the equality $a^{[4]} = c^{[6]}$, a contradiction.)

A J-unar is said to be bijective if it is both injective and surjective. By 3.2, a bijective J-unar is a J-subunar of a unar. Moreover, the following proposition holds.

3.7. Let $\underline{A}$ be a bijective J-unar and let d be the greatest common divisor of the numbers in J. (It is also assumed that $J \neq \{0\}$). Then there exists a bijective unar $(A;')$ such that

$$(\forall x \in A,\ n \in J) \quad x^{[n]} = x^{n/d}.$$

To prove this proposition assume that t is in R(J). Then t+d is also in R(J). Let $a \in A$. Then there exists exactly one $b \in A$ such that $b^{[t]} = a$. Thus we can define a' by:

$$a' = b^{[t+d]}.$$

It is easily seen that $a^k = b^{[t+kd]}$ for any $k \in N$ and thus we have

$$a^{n/d} = b^{[t+n]} = (b^{[t]})^{[n]} = a^{[n]}$$

which completes the proof. □

REFERENCES

[1] Post E.L.: Polyadic groups; Trans.Amer.Math.Soc. Vol. 48 (1940), 208-350

[2] Čupona G.: On associative congruence (in Macedonian); Bilten DMF na SR Makedonija, 13 (1962), 5-10

[3] Čupona G.: On associatives (in Macedonian); MANU, Contributions I-1 (1969), 9-20

[4] Celakoski N.: On semigroup associatives; MANU. Contributions IX-2 (1977), 5-19

[5] Skornjakov L.A.: Unars; Coll.Math. Soc.Janos Bolyai, 29. Univ.algebra, Esztergom (Hungary), 1977, 735-743

[6] Mal'cev A.I.: Algebraičeskie sistem'i (in Russian); Moskva 1970

[7] Markovski S.: On quasivarieties of generalized subalgebras; Algebraic conference Skopje 1980, 125-129

[8] Dimovski D.: Additive semigroups of integers (in Macedonian); MANU, Contributions IX-2 (1977), 21-26

Faculty of Mathematics, P.O.Box 504,
University of Skopje, Yugoslavia