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THE MAXIMAL SEMILATTICE DECOMPOSITION OF AN n-SEMIGROUP

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The purpose of this paper is to generalize the notion of the maximal semilattice decomposition of a semigroup to n-ary case.

1. Some definitions. Let S be an n-semigroup i.e. an algebra S whit an associative n-operation

$$(x_1,x_2,\ldots,x_n) \to x_1x_2\ldots x_n$$

S is called an n-semilattice if S is commutative, idempotent and satisfies the following identity

$$x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k} = x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k}$$

where $i_1+i_2+\ldots+i_k = j_1+j_2+\ldots+j_k = n$, $i_\nu, j_\nu > 0$.

A congruence α on an n-semigroup S is called a semilattice congruence if S/α is an n-semilattice.

An ideal I of S is said to be completely simple iff

$$x_1x_2\ldots x_n \in I \Leftrightarrow x_1 \in I \text{ or } x_2 \in I \text{ or } \ldots \text{ or } x_n \in I.$$

A subset F of S is a filter in S iff $I = S \setminus F$ is a completely simple ideal.

2. Characterisation of semilattice congruences whit completely simple ideals.

2.1. Let Σ be the set of all completely simple ideals in S. Then the relation α defined by

$$x\alpha y \Leftrightarrow (\forall I \in \Sigma)(x,y \in I \text{ or } x,y \in I)$$

is a semilattice congruence.

Proof. Since the elements of Σ are completely simple ideals, one easily obtains that α is a congruence on S; so it remains to show that α is a semilattice congruence. Let $I \in \Sigma$ and $x_1,x_2,\ldots,x_n \in$
Since I is a completely simple ideal we have that

$$x^n \in I \Leftrightarrow x \in I;\ x_1x_2\ldots x_n \in I \Leftrightarrow x_{i_1}x_{i_2}\ldots x_{i_n} \in I$$

where $\nu \to i_\nu$, is a permutation of $\{1,2,\ldots,n\}$;

$$x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k} \in I \Leftrightarrow x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k} \in I,$$

$$i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k = n,$$

which implies

$$x^n \alpha x;\quad x_1x_2\ldots x_n x \alpha x_{i_1}x_{i_2}\ldots x_{i_n};$$

$$x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}\ \alpha\ x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k}$$

i.e. α is a semilattice congruence. □

Let us denote the congruence α of 1.1. by α_Σ. We shall show now that the converse of 1.1. is also true:

<u>2.2</u>. If α is a semilattice congruence on S, then there is a family Σ of completely simple ideals in S such that $\alpha = \alpha_\Sigma$.

<u>Proof</u>. Let α be a semilattice congruence on S and let us associate to each element $x \in S$ the subset F_x of S defined by

$$F_x = \{y \in S \mid x\alpha x^{n-1}y\}.$$

The set F_x is nonempty and a filtre in S. Namelly it is clear that $x \in F_x$. If $u_1, u_2, \ldots, u_n \in F_x$, then we have that

$$x\alpha x^{n-1}u_n \alpha x^{n-2}(x^{n-1}u_{n-1})u_n$$

so we get

$x\alpha x^{(n-1)(n-1)}u_1u_2\ldots u_n$, so $u_1u_2\ldots u_n \in F_x$. Conversely let $u_1u_2\ldots u_n \in F_x$. Then

$$x\alpha x^{n-1}u_1u_2\ldots u_n \alpha x^{n-1}u_1u_2\ldots u_n u_n^{n-1} \alpha x u_n^{n-1} \alpha x^{n-1}u_n,$$

i.e. $u_n \in F_x$. Since $x^{n-1}u_1u_2\ldots u_n \alpha x^{n-1}u_{i_1}u_{i_2}\ldots u_{i_n}$ where $\nu \to i_\nu$, is a permutation of $\{1,2,\ldots,n\}$ we get $u_1, u_2, \ldots, u_n \in F_x$, i.e. F_x is a filtre.

Put $I_x = S \setminus F_x$ and let $\Sigma_\alpha = \{I_x \mid x \in S\}$. So Σ_α is a set of completely simple ideals in S. We shall show that $\alpha = \alpha_{\Sigma_\alpha}$.

Let $y\alpha z$, $I \in \Sigma_\alpha$ and $y \notin I_x$. Therefore $y \in F_x$ i.e. $x\alpha x^{n-1}y$. Since $x^{n-1}y\alpha x^{n-1}z$ we have that $z \in F_x$, i.e. $z \notin I_x$. We have thus shown that $\alpha \subseteq \alpha_{\Sigma_\alpha}$. Conversely, let $x\alpha_{\Sigma_\alpha} y$; then $x \in F_x$

implies $y \in F_x$, i.e. $x\alpha x^{n-1}y$. For the same reason $y \in F_y$ implies $y\alpha y^{n-1}x$. But since α is a semilattice congruence, we have

$$x^{n-1}y \,\alpha\, y^{n-1}x \quad \text{and} \quad x \,\alpha\, y. \square$$

Let us note that:

2.3. If Σ_1 and Σ_2 are sets of completely simple ideals and $S \notin \Sigma_1$, $S \notin \Sigma_2$, then $\alpha_{\Sigma_1} = \alpha_{\Sigma_2}$ if and only if $\Sigma_1 = \Sigma_2$. $\square$

3. The least semilattice congruence.

It is clear that the intersection η of all semilattice congruences is a semilattice congruence. So:

3.1. $x\eta y$ iff for every completely simple ideal I in S $x,y \in I$ or $x,y \notin I$. $\square$

Now we shall give another description of η. Let us denote by $N(x)$ the minimal filtre in S containing x, i.e. $N(x)$ is the filtre generated by x.

A direct consequence of 3.1. and the definition of $N(x)$ is

3.2. $x\eta y \iff N(x) = N(y)$. $\square$

The classes of the congruence η are called η-classes. If $x \in S$, then the η-class which contains x is denoteed by N_x. With this notations we have that:

3.3. I) $N_{x_1x_2\ldots x_n} = N_{x_{i_1}x_{i_2}\ldots x_{i_n}}$, where $i_1, i_2, \ldots, i_n$ is a permutation of $\{1,2,\ldots,n\}$.

II) $N_{x^n} = N_x$.

III) $N_{x_1^{i_1}x_2^{i_2}\ldots x_k^{i_k}} = N_{x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k}}$, where $i_1 + i_2 + \ldots + i_k = j_1 + j_2 + \ldots + j_k = n$.

IV) N_x is a subsemigroup of S. $\square$

As in the binary case S is said to be η-simple iff S has no proper completely simple ideals.

For the n-ary case, and in a similar way as in the binary case, we can prove some analogous properties for the semilattice decomposition, a part of wich is formulated below.

A constructive way for abtaining N(x), which has an inductive nature, is given with the following statement:

<u>3.4</u>. Let x be an element in S. Let $N_1(x) = \{x, x^n, x^{2(n-1)+1}, \ldots, x^{k(n-1)+1} \ldots\}$ and let $N_{n+1}(x)$ be the n-semigroup generated by all elements y in S such that $N_n(x) \cap J(y) \neq \emptyset$, where $J(y) = y \cup S^{n-1}y \cup \cup S^{n-2}yS \cup \ldots \cup yS^{n-1} \cup S^{n-1}yS^{n-1}$. Then $N(x) = \bigcup_{n=1}^{\infty} N_n(x)$. □

<u>3.5</u>. If I is an ideal of some η-class of an n-semigroup S, then I has no proper completely simple ideals.

<u>Proof</u>. Let S be an n-semigroup, $z \in S$ and I an ideal of N_z. It will suffice to prove that I is the only filtre of I. Let F be a filtre of I, $a \in F$ and let

$$T = \{x \in S \mid a^{2n-2}x \in F\}.$$

We shall show that T is a filtre of S. Let $x_1, x_2, \ldots, x_n \in T$; then $a^{2n-2}x_i \in F$ for $i = 1,2,\ldots,n$. By the inclusion $F \subseteq I \subseteq N_z$ we have that $N_{a^{2n-2}x_i} = N_{a^{n-1}x_i} = N_{x_i a^{n-1}} = N_z$ which implies $a^{2n-2}x_i$, $x_i a^{2n-2} \in I$. Since $a^{2n-2}x_i$, $a \in F$ it follows that $(a^{2n-2}x_i)a^{2n-2} = a^{2n-2}(x_i a^{2n-2}) \in F$ and $x_i a^{2n-2} \in F$. N_z is an n-subsemigroup of S, so $(a^{n-1}x_1x_2a^{n-1})a^{n-2} \in N_z$ which implies

$$N_{a^{n-1}x_1x_2a^{n-1}a^{n-2}} = N_{a^{n-1}x_1x_2} = N_z,$$

and finally $a^{2n-3}x_1x_2 \in I$. Since F is a filtre, then

$$a(a^{2n-3}x_1x_2)a^{3n-n} = (a^{2n-2}x_1)(x_2a^{2n-2})a^{n-2} \in F$$

implies $a^{2n-3}x_1x_2 \in F$. By induction, if follows that T is an n-subsemigroup.

Let $x_1x_2\ldots x_n \in T$. By the inclusion $F \subseteq I \subseteq N_z$ we have that $a, a^{2n-2}x_1x_2\ldots x_n \in N_z$ and $N(a) \subseteq N(a^{n-1}x_1) \subseteq N(a^{n-2}x_1x_2) \subseteq \ldots \subseteq N(ax_1x_2\ldots x_{n-1}) \subseteq N(a^{n-2}x_1x_2\ldots x_n) = N(a) = N(z)$.

So we have shown that

$$a^{n-1}x_i,\ x_i a^{n-1},\ a^{n-2}x_1x_2,\dots,ax_1x_2\dots x_{n-1}\in N_z.$$

Since J is an ideal it follows that

$$a^{2n-2}x_i,\ x_i a^{2n-2},\ a^{2n-3}x_1x_2,\dots,a^n x_1x_2\dots x_{n-1}\in I.$$

We have that $a^{2n-2}x_1x_2\dots x_n$, $a\in F$, so

$$(a^{2n-2}x_1x_2\dots x_n)a^{2n-2} = a^{n-2}(a^n x_1x_2\dots x_{n-1})(x_n a^{2n-2})\in F$$

which implies $a^n x_1x_2\dots x_{n-1}$, $x_n a^{2n-2}\in F$. But then

$$a^{2n-2}(x_n a^{2n-2}) = (a^{2n-2}x_n)a^{2n-2}\in F \text{ and so } a^{2n-2}x_n\in F,$$

i.e. $x_n\in T$. By reapeting this procedure with $a, a^n x_1x_2\dots x_{n-2}x_{n-1}\in F$ we get $x_{n-1}\in T$. Thus T is a filtre.

Ie is clear that $F\subseteq T\subseteq I$. Let $x\in T\cap I$. Then $a^{2n-2}x\in F$. Since F is a filtre, it follows that $x\in F$. But from $a\in N_z\cap T$ it follows that $N_z\subseteq T$. So $T\cap I = I$ and finally $F = I$.□

As a consequence of 3.5 we conclude that

<u>3.6</u>. Every n-semigroup is an n-semilattice of η-simple n-semigroups.□

<u>3.7</u>. If I is a completely simple ideal of an n-semigroup S and if $I\cap N_x = \emptyset$, then $I\cap N_x$ is completely simple.□

The following is a consequence of 3.7.

<u>3.8</u>. Every completely simple ideal of an n-semigroup S is a union of η-classes.□

If Y_s denotes the set of all η-classes of an n-semigroup S, then the following holds:

<u>3.9</u>. If I is a completely simple ideal of an n-semigroup S, then $J = \{N_x\in Y_s \mid x\in I\}$ is a completely simple ideal in Y_s. Conversely, if J is a completely simple ideal in Y_s, then $I = \{x\in S \mid N_x\in J\}$ is a completely simple ideal in S.□

R E F E R E N C E S

[1] Tamura T., and Kimura N: On decomposition of a commutative semigroup. Kodai Math. Sem. Rep. 4. 109-112 (1954)

[2] Yamada M: On the greates semilattice decomposition of a semigroup. Kodai Math. Sem. Rep. 59-62 (1955)

[3] Petrich M: The maximal semilattice decomposition of a semigroup. Math. Zeitschnift 85, 68-82 (1964)