Proceedings od the Symposium
n-ARY STRUCTURES, Skopje 1982

ON COMMUTATIVE n-SEMIGROUPS

Pano Kržovski

The purpose of this paper is to show that the well known characteristic of commutative semigroups [1] could be generalized for the class of n-semigroups, for $n>2$.

1. Some definitions

An algebra S with an associative n-ary operation

$$(x_1,x_2,\dots,x_n) \to x_1x_2\dots x_n$$

is said to be an n-semigroup. An n-semigroup S is said to be *commutative* if the following identity

$$x_1x_2\dots x_n = x_{i_1}x_{i_2}\dots x_{i_n}$$

holds for every permutation $(i_1,i_2,\dots,i_n)$ of the integers $1,2,\dots,n$. S is said to be *idempotent* if $x^n=x$ for all $x\in S$. A commutative and idempotent n-semigroup S is called an *n-semilattice* if the following identity equalities, hold:

$$x_1^{i_1}x_2^{i_2}\dots x_k^{i_k} = x_1^{j_1}x_2^{j_2}\dots x_k^{j_k},\ i_\nu,j_\nu>0,$$

where $i_1+i_2+\dots+i_k = j_1+j_2+\dots+j_k = n$. Putting $(x\le y \Leftrightarrow xy^{n-1}=x)$, one obtains that S is a partialy ordered set.

If a and b are elements of a commutative n-semigroup S, we say that a *divides* b, and write $a|b$, if there exist $u_1,u_2,\dots,u_{n-1}\in S$ such that $u_1u_2\dots u_{n-1}a = b$.

It is easy to see that the following statements for a commutative n-semigroup are true:

1.1. (i) If $a|b$ and $b|c$, then $a|c$.

(ii) If $a_r|b_r$ then $a_1a_2\dots a_{k(n-1)+1}|b_1b_2\dots b_{k(n-1)+1}$.

(iii) If $b|c^{r(n-1)+1}$, then $b^{k(n-1)+1}|(c^{r(n-1)+1})^{k(n-1)+1}$.

2. Decomposition of commutative n-semigroups

We define a relation η on any commutative n-semigroup as follows:

$$a\,\eta\,b \iff \text{each of the elements } a \text{ and } b \text{ devides some power of the other} \qquad (1)$$

2.1. **The relation η defined by (1) on a commutative n-semigroup S is the least n-semilattice congruence.**

Proof. Obviosly, the relation η is reflexive and symmetric. Let $a\,\eta\,b$ and $b\,\eta\,c$; then, there exist integers r,s and elements $u_1,\dots,u_{n-1}, v_1,\dots,v_{n-1} \in S$ such that

$$u_1\dots u_{n-1}a = b^{r(n-1)+1}, \quad v_1\dots v_{n-1}b = c^{s(n-1)+1}. \qquad (2)$$

By the second equality, using the first one, we obtain

$$(v_1^{\,r(n-1)+1}\dots v_{n-1}^{\,r(n-1)+1}u_1)u_2\dots u_{n-1}a = c^{[rs(n-1)+r+s)](n-1)+1}$$

Similarly, c divides some power of a, and thus $a\,\eta\,c$.

Now we will show that η is the least n-semilattice congruence.

Let $a\,\eta\,b$, i.e. $u_1\dots u_{n-1}a=b^{r(n-1)+1}$ for some $r\in N$ and some $u_1,\dots,u_{n-1}\in S$, and let $z_1,\dots,z_{n-1}$ be any elements of S. Then

$$u_1\dots u_{n-1}(az_1\dots z_{n-1}) = b^{r(n-1)+1}z_1\dots z_{n-1},$$

i.e.

$$az_1\dots z_{n-1}|b^{r(n-1)+1}z_1\dots z_{n-1}. \qquad (3)$$

Therefore

$$(bz_1\dots z_{n-1})^{r(n-1)+1} = (b^{r(n-1)+1}z_1\dots z_{n-1})\,(z_1^{\,(r-1)(n-1)+1})\dots(z_{n-1}^{\,r(n-1)}z_1)$$

and thus we have

$$b^{r(n-1)+1}z_1\dots z_{n-1}|(bz_1\dots z_{n-1})^{r(n-1)+1}. \qquad (4)$$

From (3) and (4), as a consequence of 1.1, we obtain

$$az_1\dots z_{n-1}|(bz_1\dots z_{n-1})^{r(n-1)+1} \text{ for some } r\in N;$$

similarly $bz_1\dots z_{n-1}|(az_1\dots z_{n-1})^{s(n-1)+1}$ for some $s\in N$, from what follows:

$$az_1 \ldots z_{n-1} \,\eta\, bz_1 \ldots z_{n-1}.$$

Obviously, the congruence η is idempotent and commutative. Since

$$(a_1^{i_1} \ldots a_k^{i_k})^n = a_1^{j_1} \ldots a_k^{j_k} (a_1^{ni_1-j_1} \ldots a_k^{ni_k-j_k})$$

it follows that

$$a_1^{j_1} \ldots a_k^{j_k} \mid (a_1^{i_1} \ldots a_k^{i_k})^n.$$

Similarly:

$$a_1^{i_1} \ldots a_k^{i_k} \mid (a_1^{j_1} \ldots a_k^{j_k})^n.$$

It remains to show that η is contained in any n-semilattice congruence.

Let ρ be any n-semilattice congruence on S and let $a \,\eta\, b$, i.e. (2). Then

$$b \,\rho\, b^{r(n-1)+1} = u_1 \ldots u_{n-1} a, \text{ i.e. } b \,\rho\, u_1 \ldots u_{n-1} a.$$

Similarly we get that $a \,\rho\, v_1 \ldots v_{n-1} b$. Thus we obtain

$$a \,\rho\, (v_1 \ldots v_{n-1} b) \,\rho\, (b^n v_1 \ldots v_{n-1}) = bv_1 \ldots v_{n-1} b^{n-1} \rho\, ab^{n-1} \rho$$

$$\rho\, (a^{n-1} b) \,\rho\, (a^{n-1} a u_1 \ldots u_{n-1}) \,\rho\, (a^n u_1 \ldots u_{n-1}) \,\rho\, (au_1 \ldots u_{n-1}) \,\rho\, b$$

Thus $a \,\rho\, b$, and we conclude that $\eta \subseteq \rho$.

We shall say that a commutative n-semigroup S is <u>archimedean</u> if, for any two elements of S, each of them divides some power of the other.

<u>2.2</u>. Every commutative semigroup S is an n-semilattice Y of archimedean semigroups S_α $(\alpha \in Y)$.

<u>Proof</u>. Let S be a commutative semigroup and let η be the relation on S defined by (1). By 2.1, S/η is a maximal n-semilattice decomposition on S. We shall show that every class S_α $(\alpha \in Y)$ is archimedean.

If $a, b \in S_\alpha$, then $a \,\eta\, b$, which means that the equalities (2) hold.

Thus we obtain

$$b^{n-1}u_1\dots u_{n-1}a = b^{(r+1)(n-1)+1}$$

and

$$b^{n-1}u_\nu | b^{(r+1)(n-1)+1} \text{ for every } \nu=1,2,\dots,n-1.$$

Since $b|b^{n-1}u_\nu$, we conclude that $b\,\eta\, b^{n-1}u_\nu$ for every $\nu=1,2,\dots,n-1$, and we have

$$(u_1b^{n-1})\dots(u_{n-1}b^{n-1}) = b^{(r+n-1)(n-1)+1}.$$

Similarly we can show that

$$b|a^{(s+n-1)(n-1)+1} \text{ for some } s\in N.$$

3. Commutative and separative n-semigroups

An n-semigroups S is cancellative if the following quasi-identities

$$x_1\dots x_{i-1}ax_{i+1}\dots x_n = x_1\dots x_{i-1}bx_{i+1}\dots x_n \Rightarrow a = b,$$

hold for all $i=1,2,\dots,n$. S is said to be separative if for any $x,y\in S$,

$$x^n = x^{i-1}yx^{n-i},\quad y^b = y^{i-1}xy^{n-i} \Rightarrow x = y$$

for all $i=1,2,\dots,n$. ([7]). A congruence ρ on a commutative n-semigroup is separative if S/ρ is separative.

Define a relation σ in S by

$a\,\sigma\, b$ if and only if there exists an integer r such that

$$ab^{r(n-1)} = b^{r(n-1)+1},\quad ba^{r(n-1)} = a^{r(n-1)+1} \tag{5}$$

3.1. If there exist integers r,s such that

$$ab^{r(n-1)} = b^{r(n-1)+1},\quad ba^{s(n-1)} = a^{s(n-1)+1},$$

then $a\,\sigma\, b$.

Proof. Let $s<r$. Multiplying the identity $ab^{s(n-1)} = b^{s(n-1)+1}$ by $b^{(r-s)(n-1)}$ we obtain $ab^{r(n-1)} = b^{r(n-1)+1}$.

3.2. The relation σ defined by (5) in a commutative n-semigroup S is a minimal separative congruence in S.

Proof. Evidently, σ is reflexive and symmetric. Let $a\,\sigma\,b$ and $b\,\sigma\,c$ $(a,b,c\in S)$, then there exist integers s and r such that

$$ab^{r(n-1)} = b^{r(n-1)+1},\quad ba^{r(n-1)} = a^{r(n-1)+1},$$

$$bc^{s(n-1)} = c^{s(n-1)+1},\quad cb^{s(n-1)} = b^{s(n-1)+1}.$$

Denote by k the integer $r+s+rs(n-1)$. Then

$$ac^{k(n-1)}=ac^{(r+s+rs(n-1))(n-1)}=ab^{r(n-1)}(c^{s(n-1)+1})^{r(n-1)}c^{s(n-1)}=$$

$$= a(bc^{s(n-1)+1})^{(n-1)}c^{s(n-1)}=ab^{r(n-1)}(c^{s(n-1)})^{r(n-1)+1}=$$

$$= b^{s(n-1)+1}(c^{s(n-1)})^{r(n-1)+1}=(bc^{s(n-1)})^{r(n-1)+1}=$$

$$= (c^{s(n-1)+1})^{r(n-1)+1}=c^{k(n-1)+1}.$$

Similarly we prove that $ca^{k(n-1)}=a^{k(n-1)+1}$. Now we will prove that σ is a congruence. Let $a\,\sigma\,b$, i.e. $ab^{r(n-1)}=b^{r(n-1)}$, $ba^{r(n-1)}=b^{r(n-1)+1}$ for some $r\in N$ and let $z_1,\dots,z_{n-1}$ be arbitrary elements of S; then

$$(az_1\dots z_{n-1})(bz_1\dots z_{n-1})^{r(n-1)}=ab^{r(n-1)}z_1^{\,r(n-1)+1}\dots z_{n-1}^{\,r(n-1)+1}=$$

$$= b^{r(n-1)+1}z_1^{\,r(n-1)+1}\dots z_{n-1}^{\,r(n-1)+1}=(bz_1\dots z_{n-1})^{r(n-1)+1}$$

Similarly we prove that

$$(bz_1\dots z_{n-1})(az_1\dots z_{n-1})^{r(n-1)}=(az_1\dots z_{n-1})^{r(n-1)+1}$$

and we obtain

$$az_1\dots z_{n-1}\,\sigma\,bz_1\dots z_{n-1}.$$

It remains to prove that σ is separative. Let $b^{n-1}a\,\sigma a^{n-1}b\,\sigma\,\sigma a^n\sigma b^n$; then there exist integers r and s such that

$$(a^{n-1}b)(a^n)^{r(n-1)}=(a^n)^{r(n-1)+1},\quad (b^{n-1}a)(b^n)^{s(n-1)}=(b^n)^{s(n-1)+1},$$

which implies that

$$ba^{(nr+1)(n-1)}=a^{(nr+1)(n-1)+1},\quad ab^{(ns+1)(n-1)}=b^{(ns+1)(n-1)+1}.$$

According to 3.1 we obtain $a\,\sigma\,b$.

The proof will be completed when we show that σ is contained in every separative congruence ρ on S.

Let $a \sigma b$, say $ab^{r(n-1)}=b^{r(n-1)+1}$, $ba^{r(n-1)}=a^{r(n-1)+1}$. Let k be any positive integer such that

$$ab^{k(n-1)} \rho\, b^{k(n-1)+1} \rho\, ba^{k(n-1)} \rho\, a^{k(n-1)+1} \qquad (6)$$

We will show by induction that (6) holds for k=1.

$(ab^{(k-1)(n-1)})^n = (ab^{(k-1)(n-1)-1})^{n-1} ab^{k(n-1)} \rho\, (ab^{(k-1)(n-1)-1})^{n-1}$,

$b^{k(n-1)+1} = (ab^{(k-1)(n-1)})^{n-1} b^{(k-1)(n-1)+1}$,

$(b^{(k-1)(n-1)+1})^n = b^{k(n-1)+1} (b^{(k-1)(n-1)})^{n-1} \rho$

$\rho\, ab^{k(n-1)} (b^{(k-1)(n-1)})^{n-1} = ab^{(k-1)(n-1)} (b^{(k-1)(n-1)+1})^{n-1}$

Putting $ab^{(k-1)(n-1)} = x$, $b^{(k-1)(n-1)+1} = y$, we have

$$x^n \rho\, x^{n-1}y, \quad y^n \rho\, xy^{n-1}$$

Since ρ is separative we get $x \rho y$, i.e. $ab^{(k-1)(n-1)} \rho\, b^{(k-1)(n-1)+1}$ and similarly we show that $a^{(k-1)(n-1)+1} \rho\, ba^{(k-1)(n-1)}$. Therefore (6) holds for k-1. By induction, (6) holds for k=1, i.e. $ab^{n-1} \rho\, b^n$ and $ba^{n-1} \rho\, a^n$, whence $a \rho b$.

3.3. A commutative n-semigroup S which archimedean components S_α $(\alpha \in Y)$ are cancellative is separative.

Proof. Let S be a commutative n-semigroup such that every archimedean component S_α is cancellative. Let $a,b \in S$ such that $a^n=a^{n-1}b$, $b^n=ab^{n-1}$. Let $a \in S_\alpha$, $b \in S_\beta$; then

$$a^{n-1}b \in S^{n-1}_{\alpha\beta} = S_\alpha, \quad b^{n-1}a \in S^{n-1}_{\alpha\beta} = S_\beta,$$

so that $\alpha=\beta$. Since S_α is cancellative we obtain a=b.

The converse of 3.3 is also true. First we will prove the following proposition.

3.4. Let S be a commutative separative n-semigroup. If $a,b \in S$ are such that

$$ab^{r(n-1)} = b^{r(n-1)+1}, \quad ba^{r(n-1)} = a^{r(n-1)+1}$$

for some $r,s \in N$, then a=b.

Proof. By 3.1, $a \sigma b$. Since S is separative, the identy relation ι on S is separative. By 3.2 $\sigma \leq \iota$ and hence a=b.

3.5. If a commutative n-semigroup is separative, then its archimedean components are cancellative.

Proof. Let S be a commutative separative n-semigroup and let S_α be an archimedean component of S. Since S is separative, then S_α is sepàrative. We will show that S_α is cancellative. Let $a,b,c_1,c_2,\dots,c_{n-1}$ be elements of S_α such that

$$ac_1c_2\dots c_{n-1} = bc_1c_2\dots c_{n-1}.$$

Since S is archimedean then for a and c_1 there exist elements $u_{11},u_{12},\dots,u_{1n-1} \in S$ and integer r_1 such that

$$c_1u_{11}u_{12}\dots u_{1n-1} = a^{r_1(n-1)+1}$$

Similarly for a and $c_2,\dots,$ a and c_{n-1} there exist $u_{21},u_{22},\dots,u_{2,n-1},\dots,u_{n-11},u_{n-12},\dots,u_{n-1n-1} \in S$ and integers $r_2,\dots,r_{n-1}$ such that

$$c_2u_{21}u_{22}\dots u_{2n-1} = a^{r_2(n-1)+1}$$

$$\dots\dots\dots\dots\dots\dots$$

$$c_{n-1}u_{n-11}u_{n-12}\dots u_{n-1n-1} = a^{r_{n-1}(n-1)+1}$$

So we have

$$ac_1c_2\dots c_{n-1}u_{11}u_{12}\dots u_{1n-1}\dots u_{n-11}u_{n-12}\dots u_{n-1n-1}=$$
$$= a^{(r_1+r_2+\dots+r_{n-1}+1)(n-1)+1} =$$
$$= bc_1c_2\dots c_{n-1}u_{11}u_{12}\dots u_{1n-1}\dots u_{n-11}u_{n-12}\dots u_{n-1n-1}=$$
$$= ba^{(r_1+r_2+\dots+r_{n-1}+1)(n-1)}$$

Denoting $r_1+r_2+\dots+r_{n-1}+1$ by k, we have

$$a^{k(n-1)+1} = ba^{k(n-1)}.$$

Similarly we can show that

$$b^{s(n-1)+1} = ab^{s(n-1)} \quad \text{for some } s \in N.$$

By 3.4, a=b.

R E F E R E N C E S

[1] Tamura, T. and Kimura N.: On decomposition of a commutative semigroup, Kodai Math. Sem. Rep. 1954 (109-112)

[2] Clifford, A.H. and Preston, G.B.: The algebraic theory of semigroups; vol. I, 1961, Providence

[3] Колесников О.В.: Вполне регулярные n-полугруппы; "Алгебра и Теория чисел" Вып. 2. 1977 (93-107

[4] Колесников О.В.: Полиадические полугруппы и Математика, 1977, 9 А 217 ДЕП

[5] Трпеновски Л.Б.: Антикомутативни n-групоиди; Годишен зборник на Електро-машински факултет во Скопје кн. 1(1967)

[6] Kržovski, P.: The maximal semilattice decomposition of an n-semigroup: Algebraic Conference, Novi Sad 1981

[7] Kržovski, P.: On separative n-semigroups; Математички билтен. 1979-1980 (49-60)

Faculty of Mathematics,
Skopje, P.O.Box 504, Yugoslavia