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COMPATIBLE SUBASSOCIATIVES

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The notion of compatible n-subsemigroup of an n-semigroup, introduced in [1], and almost all the results on compatibility obtained there, can be generalized for J-subassociatives of a J-associative in a straightforward way. In this paper we shall consider these questions for J-associatives in some details.

§1. Preliminaries

Let $\underline{A} = (A;F)$ be an algebra with the carrier A and a nonempty set of finitary operations, $F = F_2 \cup F_3 \cup \dots \cup F_n \cup \dots$, where F_n consists of the n-ary operations of F. If $f \in F_{n+1}$ and $f:(x_o,x_1,\dots,x_n) \mapsto y$, then it is written $y = fx_o x_1 \dots x_n$.

The semigroup $\underline{A}^\wedge$ with a presentation

$$\langle A;\{a = a_o a_1 \dots a_n | a = fa_o a_1 \dots a_n \text{ in } \underline{A}\}\rangle$$

is called the universal semigroup for $\underline{A}$. Denoting by $a^\wedge$ the element of $A^\wedge$ determined by $a \in A$ and putting $^\wedge : a \to a^\wedge$, we obtain a mapping from A into $A^\wedge$. The algebra $\underline{A}$ is called a semigroup algebra if the mapping $^\wedge$ is injective.

If $\phi:\underline{A} \to \underline{A}'$ is a homomorphism, then there exists a unique homomorphism $\phi^\wedge:\underline{A}^\wedge \to \underline{A}'^\wedge$ such that $\phi^\wedge(a^\wedge) = \phi(a)$ for any $a \in A$. Clearly, if ϕ is an epimorphism (isomorphism), then $\phi^\wedge$ is also an epimorphism (isomorphism), but it may happen ϕ to be a monomorphism and $\phi^\wedge$ not to be such one (Ex. 1), §3). A monomorphism $\phi:\underline{A} \to \underline{A}'$ is said to be compatible if $\phi^\wedge:\underline{A}^\wedge \to \underline{A}'^\wedge$ is also a monomorphism. And, a subalgebra $\underline{B}$ of $\underline{A}$ is said to be compatible in $\underline{A}$ if the embedding monomorphism $\varepsilon:\underline{B} \to \underline{A}$ is compatible.

The subject of this paper are compatible subassociatives of an associative. Namely, an F-algebra $\underline{A} = (A;F)$ is called an F-associative if it satisfies all the identities that hold in the class of semigroup F-algebras, i.e. if the general associative law holds in $\underline{A}$. An F-associative is called an F-group if (A,f) is an n-group for each $f \in F_n$. It is well known that any F-group is a semigroup F-algebra ([2]).

In studying associatives, it is convenient to consider the submonoid $J = J_F$ of the additive monoid of nonnegative integers generated by the set $\{n-1 \mid F_n \neq \emptyset\}$. If d_F is the greatest common divisor of the elements of J_F, then the following result holds: Every F-associative is a semigroup associative if and only if $d_F \in J_F$, and then an F-associative is in fact a (d_F+1)-semigroup. We note also that the associative law implies that for each $n \in J_F$ we have an "associative product"

$$[\,]: (x_o,x_1,\ldots,x_n) \to [x_o x_1 \ldots x_n]$$

in an F-associative $\underline{A}$, where $[x_o] = x_o$. This is the reason why an F-associative is called a J-associative and the operational symbols are not used. The notions: J-subassociative, J-subgroup, ideal of a J-associative have usual meaning.

A J-associative is said to be cyclic if it is generated by one of its elements. The structure of cyclic J-associatives is described in [5].

§2 Properties of compatible subassociatives

Denote by $\mathcal{L}(A)$ the set of all J-subassociatives of a J-associative A and by $\mathcal{C}(A)$ the set of all compatible J-subassociatives of A. The following statements hold:

2.1. $B \in \mathcal{C}(A) \Leftrightarrow B^\wedge$ is a subsemigroup of $A^\wedge$. □

2.2. $B \in \mathcal{L}(A) \Rightarrow \mathcal{L}(B) \cap \mathcal{C}(A) \subseteq \mathcal{C}(B)$. □

2.3. $B \in \mathcal{C}(A) \Rightarrow \mathcal{C}(B) \subseteq \mathcal{C}(A)$. □

2.4. $\mathcal{C}(A)$ is inductive, i.e. if $\{B_i \mid i \in I\}$ is a chain in $\mathcal{C}(A)$, then $B = \bigcup_i B_i \in \mathcal{C}(A)$. □

2.5. If $\varphi \in \mathrm{Aut}A$, $B \in \mathcal{L}(A)$ and $C = \varphi(B)$, then

$$B \in \mathcal{C}(A) \Longleftrightarrow C \in \mathcal{C}(A). \square$$

2.6. $B \in \mathcal{L}(A)$, $A \setminus B$ is an ideal in $A \Longrightarrow B \in \mathcal{C}(A)$.

Note that the sufficient condition in 2.6 is not necessary (Ex. 4), §3). □

2.7. If G is a J-subgroup of a semigroup J-associative A, then $G \in \mathcal{C}(A)$. □

If $A = \langle a \rangle = \{a^{n+1} | n \in J\}$ is an infinite cyclic J-associative, then $A^{\wedge}$ is the free semigroup generated by a (3.1 in [5]). The theorem 4.1 od [1] is true for J-associatives too:

2.8. A J-subassociative B of an infinite cyclic J-associative A is compatible in A if and only if B is cyclic. □

Using the fact that every J-subassociative C of a finite J-group G is a J-subgroup of G, as well as 2.7 and 2.8 it can be proved the following proposition:

2.9. Let $A = \{a^{n+1} | n \in J\}$ be a finite cyclic J-associative, let P be its periodic part and C be a J-subassociative of A.

i) If $C \subseteq P$, then $C \in \mathcal{C}(A)$.

ii) Let $C \nsubseteq P$ and let k be the least integer such that $b = a^{k+1} \in C$. If there exists $q \in J$ such that $c = a^{q+1} \in C$, $k+1 \nmid q+1$ and $q < s$, then $C \notin \mathcal{C}(A)$.

(Here, $s = \min\{n \in J | (\exists m \in J)\ m \neq n,\ a^{n+1} = a^{m+1}\}$, and the periodic part of A is $P = \{x | x \in A,\ x = a^{n+1}$ for infinitely many $n \in J\}$.)

§3. Examples

Below we give four examples which can be also found in [1], p.p. 26, 28. Ternary associatives, i.e. J-associatives with $J = \{2k | k \geq 0\}$ in all of them are considered.

1) Let $A = \{a,b,c\}$, $B = \{a,b\}$ and a ternary operation be defined on A by:

$$[ccc] = b \text{ and } [xyz] = a \text{ if } \{a,b\} \cap \{x,y,z\} \neq \emptyset.$$

Then A is a J-associative and B is a J-subassociative of A. The free coverings A^ and B^ are given by the following multiplication tables:

A^:

	a	b	c	α	β	γ
a	α	α	α	a	a	a
b	α	α	β	a	a	a
c	α	β	γ	a	a	b
α	a	a	a	α	α	α
β	a	a	a	α	α	α
γ	a	a	b	α	α	β

B^:

	a	b	u	v
a	u	u	a	a
b	u	v	a	a
u	a	a	u	u
v	a	a	u	u

$|A^\wedge| = 6, \quad |B^\wedge| = 4.$

The extension $\varepsilon^\wedge$ of the embedding monomorphism $\varepsilon : B \to A$ is not a monomorphism, for $\varepsilon^\wedge(u) = \varepsilon^\wedge(v) = \alpha$ but $u \neq v$. Thus $B \notin \mathcal{C}(A)$.

2) Let $A = \{a,b,c,d,e\}$ and a ternary operation $[\ ]$ be defined on A by:

$$\{x,y,z\} \cap \{c,d,e\} \neq \emptyset,\ (x,y,z) \neq (e,e,e) \implies [xyz] = c,$$

$$x,y,z \in \{a,b\} \implies [xyz] = a$$

and $[eee] = d$. Then A is a J-associative, $B = \{a,b\}$ and $C = \{c,d\}$ are two isomorphic J-subassociatives and

$$A^\wedge = \{a,b,c,d,e,aa,bb,cc,ee,be,eb,de\},\ |A^\wedge| = 12,$$

(aa=ab=ba, cc=ac=ca=ad=da=ae=ea=bc=cb=cd=dc=dd=ec=ce, de=ed);

$$B^\wedge = \{a,b,aa=ab=ba,bb\}, \quad |B^\wedge| = 4;$$

$$C^\wedge = \{c,d,cc=cd=bc,dd\}, \quad |C^\wedge| = 4.$$

Therefore $B \in \mathcal{C}(A)$ and $C \notin \mathcal{C}(A)$, for cc=dd in $A^\wedge$ but cc≠dd in $C^\wedge$.

Thus isomorphism, in general, do not preserve the compatibility.

3) The set $A = \{1',1'',3,5,7,\dots\}$ with the ternary operation $[xyz] = \psi(x) + \psi(y) + \psi(z)$, where the mapping $\psi : A \to \mathbb{N}$ is defined by $\psi(1') = 1 = \psi(1'')$, $\psi(a) = a$ for all $a \neq 1', 1''$, is a ternary semigroup, i.e. J-associative and $B = \{1',3,5,\dots\}$, $C = \{1'',3,5,\dots\}$ are J-subassociatives. The free coverings $A^\wedge$, $B^\wedge$, $C^\wedge$ of A,B,C, respectively, are given by:

$A^{\wedge} = \{1', 1'', (1',1'), (1',1''), (1'',1'), (1'',1'')\} \cup \{3,4,5,6,\ldots\}$,

$B^{\wedge} = \{1', (1',1'), 3,4,5,6,\ldots\}$,

$C^{\wedge} = \{1'', (1'',1''), 3,4,5,6,\ldots\}$,

where

$$1^i * 1^j = (1^i, 1^j), \quad 1^i * (1^j, 1^k) = 3 = (1^j, j^k) * 1^i,$$

$$(1^i, 1^j) * (2+k) = 4+k = (2+k) * (1^i, 1^j),$$

$$1^i * (2+k) = 3+k = (2+k) * 1^i.$$

Thus $B, C \in \mathcal{C}(A)$.

The intersection $D = B \cap C$ is also a subassociative of A, but it is not compatible in A; namely, $\varepsilon^{\wedge}(3*5) = \varepsilon^{\wedge}(5*3) = 8$, but $3*5 \neq 5*3$ in D.

4) Consider the additive semigroup of positive integers, $\mathbb{N}(+)$, as a ternary semigroup A, $[xyz] = x+y+z$. The set $B = \{2k+1 \mid k=0,1,2,\ldots\}$ is a ternary subsemigroup of A and $B^{\wedge} \cong A^{\wedge} \cong \mathbb{N}(+)$. Thus the extension $\varepsilon^{\wedge}: B^{\wedge} \to A^{\wedge}$ of the embedding $\varepsilon: B \to A$ is a monomorphism, i.e. $B \in \mathcal{C}(A)$, but $A \setminus \mathbb{N} = 2\mathbb{N}$ is not an ideal in A.

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