

Algebraic conference
NOVI SAD 1981

POLYNOMIAL SUBALGEBRAS
G. Čuřona and S. Markovski

A polynomial subalgebra of an algebra $\underline{A} = (A, \mathcal{O})$ is a subset B of the carrier of the algebra which is closed under the polynomials belonging to a set of \mathcal{O} -polynomials. In this paper polynomial subalgebras are considered, together with a few properties and examples. A special attention is given to the polynomial subalgebras of the algebras belonging to a variety.

1. Throughout the paper \mathcal{O} and \mathcal{O}' will be two sets of operational symbols and $X = \{x_1, x_2, \dots, x_n, \dots\}$ will be the set of individual variables. By \mathcal{O}_X will be denoted the set of all \mathcal{O} -polynomials, i.e. $\mathcal{O}_X = \text{Term}(\mathcal{O})$. If $p \in \mathcal{O}_X$ and if each variable that occurs in p is in the set $\{x_1, \dots, x_n\}$, then we will usually write $p = p(x_1, \dots, x_n)$. Let $\wedge: \mathcal{O} \rightarrow \mathcal{O}'_X$ be a mapping such that if $f \in \mathcal{O}(n)$, then $f^\wedge = f^\wedge(x_1, \dots, x_n)$. The mapping \wedge induces a mapping from \mathcal{O}_X into \mathcal{O}'_X (denoted with the same symbol \wedge) defined by: (i) $x^\wedge = x$, for each $x \in X$ and (ii) $f \in \mathcal{O}(n)$, $p = fp_1 \dots p_n \Rightarrow p^\wedge = f^\wedge(p_1^\wedge, \dots, p_n^\wedge)$.

Let \underline{A} be an \mathcal{O} -algebra, \underline{A}' an \mathcal{O}' -algebra and $\phi: A \rightarrow A'$ a mapping such that $\phi(f_{\underline{A}}(a_1, \dots, a_n)) = f_{\underline{A}'}(\phi(a_1), \dots, \phi(a_n))$ for each $f \in \mathcal{O}(n)$ and $a_1, \dots, a_n \in A$. The mapping ϕ in this case will be called a \wedge -homomorphism from \underline{A} into \underline{A}' . Moreover, if $A \subseteq A'$ and if the embedding of \underline{A} into \underline{A}' is a \wedge -homomorphism, then \underline{A} is said to be a \wedge -subalgebra of \underline{A}' . (We will sometimes say polynomial homomorphism (polynomial subalgebra) instead of \wedge -homomorphism (\wedge -subalgebra).)

If \underline{A}' is an \mathcal{O}' -algebra, then an \mathcal{O} -algebra $\wedge \underline{A}'$ by the same carrier A' is defined by: $f_{\wedge \underline{A}'}(a'_1, \dots, a'_n) = f_{\underline{A}'}^\wedge(a'_1, \dots, a'_n)$, for each $f \in \mathcal{O}(n)$ and $a'_1, \dots, a'_n \in A'$. We say that $\wedge \underline{A}'$ is induced from \underline{A}' by \wedge .

Let \mathcal{C}' be a class of \mathcal{O}' -algebras and \mathcal{C} be a class of \mathcal{O} -algebras. Then by $\hat{\mathcal{C}}'$ will be denoted the class of \mathcal{O} -algebras which are \wedge -subalgebras of \mathcal{O}' -algebras belonging to \mathcal{C}' , and by \mathcal{C}^\wedge the class of \mathcal{O}' -algebras \underline{A}' such that all \wedge -subalgebras of \underline{A}' are in \mathcal{C} . We say that a pair $(\mathcal{C}, \mathcal{C}')$ is \wedge -compatible if each algebra $\underline{A} \in \mathcal{C}$ is a \wedge -subalgebra of an algebra $\underline{A}' \in \mathcal{C}'$ such that $\wedge \underline{A}' \in \mathcal{C}$.

The following properties give some connections between \mathcal{C} , \mathcal{C}' , \mathcal{C}^\wedge and $\hat{\mathcal{C}}'$.

1°. (a) If \mathcal{C} is a class of \mathcal{O} -algebras and \mathcal{C}' a class of \mathcal{O}' -algebras, then: $\wedge(\mathcal{C}^\wedge) \subseteq \mathcal{C}$, $\mathcal{C}' \subseteq (\hat{\mathcal{C}}')^\wedge$.

(b) The equation $\wedge(\mathcal{C}^\wedge) = \mathcal{C}$ holds iff each \mathcal{O} -algebra $\underline{A} \in \mathcal{C}$ is a \wedge -subalgebra of an \mathcal{O}' -algebra \underline{A}' such that each \wedge -subalgebra of \underline{A}' is in \mathcal{C} .

(c) The equation $(\hat{\mathcal{C}}')^\wedge = \mathcal{C}'$ holds iff \mathcal{C}' contains any \mathcal{O}' -algebra \underline{A}' such that every \wedge -subalgebra \underline{A} of \underline{A}' is \wedge -subalgebra of $\underline{A}'' \in \mathcal{C}'$.

2°. If $(\mathcal{C}, \mathcal{C}')$ is a \wedge -compatible, then $\mathcal{C} \subseteq \hat{\mathcal{C}}'$.

3°. If \mathcal{C}' is a quasivariety of \mathcal{O}' -algebras, then $\hat{\mathcal{C}}'$ is also a quasivariety of \mathcal{O} -algebras. ([8], p. 274).

We note that there are known infinite many varieties of \mathcal{O}' -algebras \mathcal{C}' such that $\hat{\mathcal{C}}'$ is a proper quasivariety. This suggests to look for a description of the set of varieties \mathcal{C}' of \mathcal{O}' -algebras such that $\hat{\mathcal{C}}'$ to be also a variety of \mathcal{O} -algebras.

4°. Let \mathcal{C}' be a variety of \mathcal{O}' -algebras and \underline{A} be an \mathcal{O} -algebra. Let \underline{F}' be the \mathcal{O}' -algebra which is freely generated by \underline{A} in \mathcal{C}' and let ρ be the least congruence on \underline{F}' such that:

$$a = f_{\underline{A}}(a_1, \dots, a_n) \text{ in } \underline{A} \Rightarrow a \rho f_{\underline{F}'}(a_1, \dots, a_n).$$

Then $\underline{A} \in \hat{\mathcal{C}}'$ if the following condition is satisfied:

$$a, b \in \underline{A} \Rightarrow (a \rho b \Rightarrow a = b).$$

5^o. Let $\mathcal{C}' = \text{Var}_{\mathcal{O}'}, \Sigma'$ be a variety of \mathcal{O}' -algebras defined by a set of identities Σ' . Denote by $\langle \Sigma' \rangle$ the set of identities which are consequences from Σ' , i.e. which hold in all \mathcal{O}' -algebras belonging to \mathcal{C}' , and denote by $\wedge \Sigma'$ the set of \mathcal{O} -identities $p \equiv q$ such that $p \wedge \equiv q \wedge \in \langle \Sigma' \rangle$. Then $\wedge \mathcal{C}'$ is a variety iff $\wedge \mathcal{C}' = \text{Var} \wedge \Sigma'$. And, if $\wedge \mathcal{C}'$ is the variety of all \mathcal{O} -algebras then $\wedge \Sigma'$ consists of trivial identities, i.e. the identities of the form $p \equiv p$, where $p \in \mathcal{O}_X$.

6^o Let $\mathcal{C} = \text{Var}_{\mathcal{O}}, \Sigma$, $\mathcal{C}' = \text{Var}_{\mathcal{O}', \Sigma'}$ be such that $\mathcal{C} \subseteq \wedge \mathcal{C}'$. Denote by Σ'' the following set of \mathcal{O} -identities:

$$\{p \wedge \equiv q \wedge \mid p \equiv q \in \Sigma\} \cup \Sigma',$$

and let $\mathcal{C}'' = \text{Var}_{\mathcal{O}}, \Sigma''$. Then the pair $(\mathcal{C}, \mathcal{C}')$ is \wedge -compatible iff $\mathcal{C}' \subseteq \mathcal{C}''$.

7^o. If \mathcal{C}' is an axiomatizable class of \mathcal{O}' -algebras, then $\wedge \mathcal{C}'$ can be defined by a system of open formulas. ([7]).

8^o. Let Σ' be a class of \mathcal{O}' -identities satisfying the following condition:

(**) If u', v' are finite sequences on $\mathcal{O}' \cup X$, $p' \in \mathcal{O}'_X$ and if there is a $q' \in \mathcal{O}'_X$ such that $u' p' v' \equiv q' \in \langle \Sigma' \rangle$, then there is a $q'' \in \langle \Sigma' \rangle$ such that $u' x v' \equiv q'' \in \langle \Sigma' \rangle$, where x is a variable which does not occur in $u' p' v'$.

Then $\wedge \text{Var}_{\mathcal{O}', \Sigma'}$ is a variety of \mathcal{O} -algebras ([5]).

2. Now, we will state some results concerning special classes of algebras, which will throw better look on the properties 1^o-8^o.

1) Let $\underline{\text{Sem}}$ be the variety of semigroups. If $\mathcal{O}' = \{.\} = \mathcal{O}'(2)$ and if $p(x_1, \dots, x_n) \in \mathcal{O}'_X$, then by the associative law an (2) identity of the form $p \equiv x_{i_1} x_{i_2} \dots x_{i_k}$ holds in $\underline{\text{Sem}}$, where $i_v \in \{1, 2, \dots, n\}$. Thus, we can assume that if \mathcal{C} is a variety of semigroups, then $\mathcal{C} = \text{Var} \Sigma$, where Σ is a set of identities of the forms $x_{i_1} \dots x_{i_k} \equiv x_{j_1} \dots x_{j_l}$, where $i_v, j_\lambda \in \{1, 2, \dots\}$, including the identity $x_1(x_2 x_3) = (x_1 x_2)x_3$.

The following result is known as Cohn-Rebane's theorem ([1] page 185):

If \underline{A} is an \mathcal{O} -algebra, then there is a semigroup \underline{S} and a mapping $f \mapsto \bar{f}$ of \mathcal{O} into S such that $A \subseteq S$ and $f_{\underline{A}}(a_1, \dots, a_n) = \bar{f}a_1 \dots a_n$ for each $f \in \mathcal{O}(n)$ and all $a_1, \dots, a_n \in A$. Then we say that \underline{A} is an \mathcal{O} -subalgebra of the semigroup \underline{S} . If \mathcal{C}' is a class of semigroups, then by $\mathcal{C}'(\mathcal{O})$ will be denoted the class of \mathcal{O} -algebras which are \mathcal{O} -subalgebras of semigroups belonging to \mathcal{C}' . Thus, the Cohn-Rebane's theorem can be formulated as follows:

1.1) $\underline{\text{Sem}}(\mathcal{O})$ is the variety of all \mathcal{O} -algebras.

We will state some other results. First, we will give some definitions. If $p \in \mathcal{O}_X$ and if $b \in X \cup \mathcal{O}$, then $|p|_b$ is the number of occurrences of the symbol b in p . Also, by $\underline{\text{Absem}}$ we denote the variety of commutative semigroups, and by $\underline{C}_{r,m}$ the variety $\underline{\text{Absem}}(x^r = x^{r+m})$, where r and m are positive integers. Then we have:

1.2) $\underline{A} \in \underline{\text{Absem}}(\mathcal{O})$ if \underline{A} satisfies any identity $p \equiv q$, where $p, q \in \mathcal{O}_X$ are such that $|p|_b = |q|_b$, for each $b \in \mathcal{O} \cup X$ ([10]).

1.3) $\underline{C}_{r,m}(\mathcal{O})$ is a variety iff $r=1$ or $\mathcal{O} = \mathcal{O}(1)$. ([6]).

We note that, if $\mathcal{O}(o) = \emptyset$, then 1.1) and 1.2) are consequences from \mathcal{S}^o . If in 1.1) or 1.2) we have $\mathcal{O}(o) = \emptyset, \mathcal{O} \setminus \mathcal{O}(o) \neq \emptyset$ (or in 1.3) $\mathcal{O} \neq \emptyset$), then the condition $(**)$ of \mathcal{S}^o is not satisfied.

2) Let $\mathcal{O} = \{f\} = \mathcal{O}(n), \mathcal{O}' = \{\cdot\} = \mathcal{O}'(2)$ and $f^{\wedge} = x_1 x_2 \dots x_n$. If \mathcal{C}' is a class of groupoids, then \mathcal{C}'^{\wedge} is denoted by $\mathcal{C}'(n)$. Also, $\underline{\text{Sem}}(xyz = xyxz, xyz = xzyz), \underline{\text{Sem}}(xyz = xyxz), \underline{\text{Sem}}(x^r = x^{r+m})$ will be denoted respectively by: $\underline{D}, \underline{D}^{\wedge}, \underline{P}_{r,m}$. And, $\underline{\text{Sem}}_n$ is the class of n -semigroups, i.e. algebras with an associative n -ary operation.

2.1) $\underline{\text{Sem}}(n) = \underline{\text{Sem}}_n$.

2.2) $\underline{P}_{r,m}(n)$ is a variety iff $r=1$ or $n-1$ is a divisor of m .

2.3) $\underline{C}_{r,m}(n)$ is a variety for all r, m, n .

2.4) $\underline{D}(n)$ is a variety for every n .

2.5) $\underline{D}^l(n)$ is a proper quasivariety for every $n \geq 3$.

2.6) Let Σ' be a set of semigroup identities $p \equiv q$ such that

$$|p|_i \equiv |q|_i \pmod{n-1} \quad (***)$$

for each $i=1,2,\dots$, where $n \geq 3$, and let $\underline{C}' = \underline{\text{Sem}}(\Sigma')$. Then $\underline{C}'(n)$ is a variety. (We note that this result is a corollary from $\underline{\delta}^0$; and, conversely, if a variety $\underline{C}' = \underline{\text{Sem}}(\Sigma')$ satisfies the condition (***) of $\underline{\delta}^0$, then (***) is satisfied for every identity $p \equiv q \in \Sigma'$.)

The above results are proved in the papers [3], [4], [5], [9]. Some of the results in 1) and 2) suggest the following conjecture: If \underline{C}' is a variety of semigroups such that $\underline{C}'(\underline{O})$ is a variety of \underline{O} -algebras for every \underline{O} , then $\underline{C}'(n)$ is a variety of n -semigroups for every $n \geq 2$.

3) If \underline{R} is a ring, then by 1.1) there is a semigroup \underline{S} and a pair of elements $a, b \in S$ such that $x+y = axy$, $x \bullet y = bxy$ (" \bullet " is the multiplication in the ring \underline{R}). But, if \underline{S} is a semigroup with at least two elements, and if the operations $+$ and \bullet defined on \underline{S} by: $x+y = axy$, $x \bullet y = bxy$, where $a, b \in S$, then $(S; +, \bullet)$ is never a ring. This example shows that it can happen a pair $(\underline{C}, \underline{C}')$ to be not \wedge -compatible, although $\underline{C} \subseteq \wedge \underline{C}'$. In [2] there are given several examples of such noncompatible pairs. We note that in each of the examples 1.1)-1.3), 2.1)-2.4) we have a compatible pair of varieties.

4) Now we will finish our considerations by an example of a variety $\underline{C}' = \text{Var} \Sigma'$ such that $\wedge \underline{C}'$ is not a variety although $\wedge \Sigma'$ does not contain non trivial identities. Namely, let $\underline{O}' = \underline{O}'(2) = \{.\}$, $\underline{O} = \underline{O}(3) = \{f\}$, and $f \wedge = (x_1 x_2) x_3$. If $\Sigma' = \{(((x_1 x_2) x_1) x_2) x_1 = ((x_1 x_1) x_1) (x_2 x_2)\}$, then $\wedge \Sigma'$ does not contain nontrivial identities, but $\wedge \text{Var} \Sigma'$ is a proper subclass of the class of ternary groupoids (i.e. algebras with a ternary operation).

R E F E R E N C E S

- [1] Cohn P.M.: Universal Algebra, New York 1965
- [2] Čupona G.: Za teorematata na Kon-Rebane, God. Zb. 20, 1970, 15-34
- [3] Čupona G.: n-subsemigroups of semigroups satisfying the law $x^r = x^{r+m}$, God. Zb. Mat. fak. 30, 1979, 5-14
- [4] Čupona G., Celakoski N.: Polyadic Subsemigroups of Semigroups, Algebraic Conference, Skopje 1980, 131-152
- [5] Čupona G., Markovski S.: Smestuvanje na univerzalni algeabri, God. Zb. 25-26, 1976, 15-34
- [6] Čupona G., Crvenković S., Vojvodić G.: Subalgebras of commutative semigroups satisfying the law $x^r = x^{r+m}$ Zbor.rad.PMF Novi Sad br. 11,1981(to appear)
- [7] Los J.: On the extending of models (I), Fund. Math. 42, 1955, 38-54
- [8] Maljcev A.I.: Algebraičeskie sistemi, Moskva 1970
- [9] Markovski S.: Za distributivnite polugrupi, God. Zb. Mat. fak. 30, 1979, Skopje
- [10] Rebane Ju. K.: O predstavlennii universalnih algebar v komutativnih polugruppah, Sib. Mat. Žurn. 7, 1966, 878-885