

RESEARCH ARTICLE

VECTOR VALUED SEMIGROUPS

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An (n,m) -semigroup $A[\]$ is an associative mapping $[\]:A^n \rightarrow A^m$. Thus, an $(n,1)$ -semigroup is an n -semigroup, and more specially a $(2,1)$ -semigroup is a semigroup. We show that each (n,m) -semigroup $A[\]$ can be embedded in a semigroup S generated by A such that $[a_1 \dots a_n] = a_1 \dots a_n$ for any $a_1, \dots, a_n \in A$. The classes of cancellative (n,m) -semigroups, (n,m) -groups and commutative (n,m) -semigroups are also considered. A number of known results for n -semigroups hold in the class of (n,m) -semigroups as well, but some different situations arise in the case $m \geq 2$. For example, any semigroup is a covering semigroup of an $(n,1)$ -semigroup but this is not true if $m \geq 2$; it is also well known that any nonempty set is the carrier of an n -group, but it is not known whether this is true for (n,m) -groups when m is not a divisor of n .

1. Let A be a nonempty set, n,m two positive integers and

$$[\]: (x_1, \dots, x_n) \mapsto [x_1 \dots x_n]$$

a mapping from the n -th cartesian power A^n of A into A^m . Throughout the paper it will be assumed that $k = n-m > 0$. We say that A is an (n,m) -groupoid.

If $0 \leq i \leq k$ and $[x_{i+1} \dots x_{i+n}] = (y_1, \dots, y_m)$ then the "product"

$$[x_1 \dots x_i y_1 \dots y_m x_{i+n+1} \dots x_{2k+m}]$$

will be denoted by

$$[x_1 \dots x_i [x_{i+1} \dots x_{i+n}]^{x_{i+n+1}} \dots x_{2k+m}] \cdot$$

$A[]$ is called an (n,m) -semigroup if the following equation is an identity on $A[]$

$$[[x_1 \dots x_n]^{x_{n+1}} \dots x_{2k+m}] = [x_1 \dots x_i [x_{i+1} \dots x_{i+n}]^{x_{i+n+1}} \dots x_{2k+m}] \quad (1.1)$$

for each $i \in \{1, \dots, k\}$.

It can be easily seen that in an (n,m) -groupoid there exist "continued products" $\Pi(x_1, \dots, x_{sk+m})$ of a length $sk+m$ for each $s \geq 1$. If $A[]$ is an (n,m) -semigroup, then two continued products $\Pi'(x_1, \dots, x_{sk+m}), \Pi''(x_1, \dots, x_{sk+m})$ of the same length $sk+m$ are equal, i.e. "the general associative law" holds. Therefore we can omit "inside brackets", i.e. a continued product $\Pi(x_1, \dots, x_{sk+m})$ will be written in the following form

$$[x_1 \dots x_{sk+m}]_s \text{ or } [x_1 \dots x_{sk+m}].$$

Thus we can state the following proposition.

1.1. If $A[]$ is an (n,m) -semigroup, then $A[]_s$ is an $(sk+m,m)$ -semigroup for each $s \geq 1$.

Examples. 1) An $(n,1)$ -semigroup is an n -semigroup, and a $(2,1)$ -semigroup is a usual binary semigroup.

2) Let A and B be two nonempty sets.

a) If $a = (a_1, \dots, a_m)$ is a fixed element of A^m , then the identity $[x_1 \dots x_n] = a$ defines a constant (n,m) -semigroup.

b) $[x_1 \dots x_n] = (x_1, \dots, x_m)$ defines a left zero (n,m) -semigroup on A .

c) An (n,m) -semigroup can be built on $A \times B$ by:

$$[(x_1, y_1) \dots (x_n, y_n)] = ((x_1, y_{k+1}), \dots, (x_m, y_n)).$$

3) Let (A, \cdot) be a (binary) semigroup and $n=tm$. Then the t -th cartesian power of the m -th cartesian power of the given semigroup determines an (n,m) -semigroup. Namely, in this case we have:

$$[x_1 \dots x_n] = (x_1 x_{m+1} \dots x_{(t-1)m+1}, x_2 \dots x_{(t-1)m+2}, \dots, x_m x_{2m} \dots x_{tm}).$$

2. Here we will show that each (n,m) -semigroup is embeddable in a semigroup.

Let S be a semigroup and A a subset of S (i.e. of the carrier of the semigroup) with the following properties:

- (I) $(\forall a_1, \dots, a_n \in A) (\exists b_1, \dots, b_m \in A) a_1 \cdots a_n = b_1 \cdots b_m$
 (II) $(\forall a_\nu, b_\nu \in A) (a_1 \cdots a_m = b_1 \cdots b_m \Rightarrow a_1 = b_1, \dots, a_m = b_m)$
 (III) $(\forall a_\nu, b_\lambda \in A) (i < m \ \& \ a_1 \cdots a_i = b_1 \cdots b_p \Rightarrow$
 $\Rightarrow i = p \ \& \ a_1 = b_1, \dots, a_i = b_i).$

Then, an (n,m) -semigroup $A[]$ can be obtained by:

$$(IV) \ a_1 \cdots a_n = (b_1, \dots, b_m) \Leftrightarrow a_1 \cdots a_n = b_1 \cdots b_m.$$

We will say that $A[]$ is an (n,m) -subsemigroup of S , and if, moreover, S is generated by A , then S is called a covering semigroup of $A[]$.

It is assumed in the following statements that S is a covering semigroup of a given (n,m) -semigroup $A[]$, and S_* is the subset of S defined by:

$$S_* = \{a_1 \cdots a_i \mid i \geq m, a_\nu \in A\}. \quad (2.1)$$

We will say that S_* is a strong covering of $A[]$.

- 2.1. S_* is an ideal in S , and $S = S_*$ iff $m = 1$.
2.2. S is a group iff $m = 1$ and $A[]$ is an n -group.
2.3. If $m \geq 2$, then S is commutative only if $|A| = 1$, and then S is a cyclic semigroup with an index m and a period which is a divisor of k .
2.4. If S and T are covering semigroups of $A[]$, then $S \cong T$ iff $S_* \cong T_*$.

We note that the propositions 2.1, 2.3, and 2.4 are obvious; the same is true for 2.4 if $m \geq 2$, and the case $m = 1$ is considered in [10; Th. 2] and [1; Th. 2.2]. We also note that the assumption $m \geq 2$ is essential in 2.3, for there exist n -semigroups admitting only commutative coverings [11; §2].

We will show now that each (n,m) -semigroup $A[]$ admits a covering semigroup. Namely, let A^\wedge be the semigroup with the following presentation:

$$\langle A; \{a_1 \cdots a_m = b_1 \cdots b_n \mid (a_1, \dots, a_m) = b_1 \cdots b_n \text{ in } A[]\} \rangle. \quad (2.2)$$

Clearly,

$$a, b \in A \Rightarrow (a = b \text{ in } A^\wedge \Rightarrow a = b \text{ in } A),$$

and thus it can be assumed that A is a generating subset of A^\wedge . It is also clear that the conditions (I), (III) and (IV) are satisfied. And, it is easy to see that if $a_\nu, b_\lambda \in A$, $t \neq m$ and $a_1 \cdots a_m = b_1 \cdots b_t$ in A^\wedge , then there is an $s \geq 1$ such that $t = sk + m$ and $(a_1, \dots, a_m) = [b_1 \dots b_t]$ in the given (n, m) -semigroup; this implies that condition (II) is also satisfied. Thus we can state the following propositions.

2.5. If $A[]$ is an (n, m) -semigroup, then A^\wedge is a covering semigroup of $A[]$.

2.6. If S is a covering semigroup of an (n, m) -semigroup $A[]$, then there exists a unique homomorphism $f: A \rightarrow S$, such that $f(a) = a$ for each $a \in A$.

The last proposition suggests that A^\wedge is the universal covering of $A[]$.

It is not difficult to give explicit descriptions of universal semigroups of the (n, m) -semigroups given in the example 1.2).

3. Assuming that \underline{C} is a class of semigroups, it is desirable to find a set of classes $\{\underline{C}(n, m) \mid n > m \geq 1\}$ such that $\underline{C}(2, 1) = \underline{C}$ and $\underline{C}(n, m)$ is a class of (n, m) -semigroups for each pair (n, m) . The class $\underline{C}(n, m)$ can be defined by an axiom system $\phi(n, m)$ in such a way that $\phi(2, 1)$ is an axiom system of \underline{C} . Also, $\underline{C}(n, m)$ can be the class of (n, m) -semigroups $A[]$ such that either the (strong) universal covering $A^\wedge(A_\star)$ is in \underline{C} , or some (strong) covering $S (S_\star)$ of A is in \underline{C} , or every (strong) covering of $A[]$ is in \underline{C} . Here we will consider only the cases when \underline{C} is the class of all cancellative semigroups, groups, commutative semigroups - respectively.

An (n, m) -semigroup $A[]$ is called cancellative if it satisfies the following condition:

$$(\forall a \in A^k, x, y \in A^m) [a x] = [a y] \text{ or } [x a] = [y a] \Rightarrow x = y. \quad (3.1)$$

Almost all the results on cancellative n -semigroups proved in [11; §3] and [9; III.1] hold for cancellative (n, m) -semigroups as well. Let us state one of them.

3.1. An (n, m) -semigroup $A[]$ is cancellative iff some covering semigroup of $A[]$ is cancellative.

Proof. It is clear that if some covering semigroup of $A[\]$ is cancellative, then $A[\]$ is also cancellative.

Assume now that $A[\]$ is a cancellative (n,m) -semigroup. Denote by F the semigroup freely generated by A , i.e. $F = A^+$, and define a relation \equiv on F by:

$$u \equiv v \Leftrightarrow (\exists a \in F) [au] = [av]. \quad (3.2)$$

Then, \equiv is a congruence on F , and F/\equiv is a cancellative covering of $A[\]$. Namely, F/\equiv is the universal cancellative covering of $A[\]$. We are not giving a detailed proof, for it would be an obvious generalization of the proof of the corresponding special result for cancellative n -semigroups given in [9; III.1].

An (n,m) -semigroup $A[\]$ is called an (n,m) -group if the following statement is satisfied:

$$(\forall a \in A^k, b \in A^m) (\exists x, y \in A^m) [ax] = b, [ya] = b. \quad (3.3)$$

We have a similar situation in the class of (n,m) -groups as in the class of cancellative (n,m) -semigroups, i.e. a lot of known results on n -groups which can be found in 4, 11 and 1 also hold for (n,m) -groups. Below we will only state some of them.

3.2. If $A[\]$ is an (n,m) -semigroup then the following statements are equivalent:

- (a) $A[\]$ is an (n,m) -group;
- (b) there is an $s \geq 1$ such that $A[\]_s$ is an $(sk+m,m)$ -group;
- (c) $A[\]_s$ is an $(sk+m,m)$ -group for each $s \geq 1$;
- (d) A^*_s is a group.

3.3. An (n,m) -semigroup $A[\]$ is an (n,m) -group iff there exist (n,m) -operations

$$[/]: (x_1, \dots, x_n) \mapsto [x_1 \dots x_m / x_{m+1} \dots x_n]$$

$$[\backslash]: (x_1, \dots, x_n) \mapsto [x_{m+1} \dots x_n \backslash x_1 \dots x_m]$$

on A such that:

$$(\forall x \in A^k, y \in A^m) [x[x\backslash y]] = y, [[y/x]x] = y. \quad (3.4)$$

3.4. If the semigroup in the example 1.3) is cancellative (a group) then $A[\]$ is a cancellative (n,m) -semigroup (an (n,m) -group).

From 3.4 it follows that if m is a divisor of n , then any nonempty set is the carrier of an (n,m) -group. The situation is not so simple if m is not a divisor of n . To illustrate this statement, let us prove the following proposition.

3.5. If a $(3,2)$ -semigroup $A[]$ satisfies the following condition:

$$(\forall a, b \in A^2) (\exists x, y \in A) [a x] = b, [y a] = b,$$

then $|A| = 1$.

Proof. Let $a \in A^2$ and $e, f \in A$ be such that $[e a] = a$, $[a f] = a$. Then, in a usual way, we can show that:

$$(\forall b \in A^2) [e b] = b, [b f] = b,$$

which will imply that $(\forall x \in A) (x, f) = [e x f] = (e, x)$.

The existence of $(k+m,m)$ -groups for any pair of positive integers k, m will be shown in 4.3.

An (n,m) -semigroup $A[]$ is called commutative if the following identity is satisfied

$$[x_1 \dots x_n] = [x_{i_1} \dots x_{i_n}],$$

for any permutation $v \mapsto i_v$ of $N_n = \{1, 2, \dots, n\}$. In a usual way we obtain:

3.6. If $A[]$ is a commutative (n,m) -semigroup, then for each $s \geq 1$ $A[]_s$ is also commutative.

By 2.3, if $m \geq 2$ and $|A| \geq 2$, no covering semigroup of an (n,m) -semigroup $A[]$ is commutative. But, if $A[]$ is a commutative (n,m) -semigroup, then some covering semigroups of $A[]$ do have corresponding properties of commutativity. First, let us say that a semigroup S is t -commutative if products with at least t factors are invariant under permutations.

Assume now $A[]$ to be a commutative (n,m) -semigroup, and consider the semigroup S with the presentation $\langle A; \Lambda' \cup \Lambda'' \rangle$, where

$$\Lambda' = \{a_1 \dots a_m = b_1 \dots b_n \mid (a_1, \dots, a_m) = b_1 \dots b_n \text{ in } A[]\},$$

$$\Lambda'' = \{a_0 \dots a_m = a_0 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_m \mid a_v \in A, 0 \leq i \leq m-1\}.$$

It is clear that S is an $m+1$ -commutative semigroup and that it satisfies the conditions 2(I), 2(III), 2(IV); and it is easy to show

that 2(II) also holds. Therefore:

3.7. If $A[\]$ is a commutative (n,m) -semigroup, then there exists an $m+1$ -commutative covering semigroup of $A[\]$.

The following proposition is also clear.

3.8. If S is an $m+1$ -commutative covering of an (n,m) -semigroup $A[\]$, then S_* and $A[\]$ are commutative, and A^m is in the centre of S .

4. An (n,m) -groupoid $A[\]$ induces an algebra $(A; [\]_1, \dots, [\]_m)$ with m n -ary operations defined in the following way:

$$[x_1 \dots x_n] = (y_1, \dots, y_m) \Leftrightarrow (\forall i \in N_m) [x_1 \dots x_n]_i = y_i. \quad (4.1)$$

We say that the algebra obtained is the component algebra of the given (n,m) -groupoid. In an obvious way, axiom systems for the classes of component algebras of (n,m) -semigroups, cancellative (n,m) -semigroups, (n,m) -groups, commutative (n,m) -semigroups respectively can be formulated. We will state the corresponding proposition only for (n,m) -semigroups.

4.1. An (n,m) -groupoid $A[\]$ is an (n,m) -semigroup iff the corresponding component algebra satisfies the following identity:

$$\begin{aligned} & [[x_1 \dots x_n]_1 \dots [x_1 \dots x_n]_m x_{n+1} \dots x_{2k+m}]_i = \\ & = [x_1 \dots x_j [x_{j+1} \dots x_{j+n}]_1 \dots [x_{j+1} \dots x_{j+n}]_m \dots x_{2k+m}]_i \end{aligned} \quad (4.2)$$

for each $i \in N_m, j \in N_k$.

Thus, the class of (n,m) -semigroups can be considered as a variety of algebras with m n -ary operations. The same is also true for the class of commutative (n,m) -semigroups, and the class of cancellative (n,m) -semigroups is equivalent with a quasivariety of algebras. The axiom system of component algebras corresponding to (n,m) -groups involves existential quantifiers, but (as in the binary and the n -ary case) the class of (n,m) -groups can be described by a variety of algebras. Namely, by 3.3 an (n,m) -group can be defined as a "vector valued algebra" $(A; [\], [\ \backslash], [/])$ such that the identities (1.1) and (3.4) are satisfied. Therefore we can state the following proposition.

4.2. An (n,m) -semigroup $A[\]$ is an (n,m) -group iff there is an algebra

$$(A; [\]_1, \dots, [\]_m, {}^1[\ \backslash], \dots, {}^m[\ \backslash], [/]^1, \dots, [/]^m) \quad (4.3)$$

with $3m$ n -ary operations such that $(A; [\]_1, \dots, [\]_m)$ is the component algebra of $A[\]$, and the following identities are satisfied:

$$x_i = [x_{m+1} \dots x_n \]^1 [x_{m+1} \dots x_n \ x_1 \dots x_m] \dots [x_{m+1} \dots x_n \ x_1 \dots x_m]]_i, \quad (4.4)$$

$$x_i = [[x_1 \dots x_m / x_{m+1} \dots x_n] \]^1 \dots [x_1 \dots x_m / x_{m+1} \dots x_n]]^m x_{m+1} \dots x_n]_i, \quad (4.5)$$

for each $i \in N_m$.

The algebra (4.3) is called the component algebra of the (n,m) -group $A[\]$, or better of the (n,m) -group $(A; [\], [\ \backslash], [\ /])$. Therefore any algebra (4.3) which satisfies all the identities (4.2), (4.4) and (4.5) induces an (n,m) -group, and conversely.

The usual notions of homomorphisms, direct products, free (n,m) -semigroups, and others, can be defined directly, but we do not give explicit definitions, for they are meaningful in the class of component algebras.

Proposition 4.2 can be used in proving the existence of nontrivial (n,m) -groups.

4.3. If $n > m \geq 1$, then there exists a nontrivial (n,m) -group.

Proof. By 3.2 we can assume that $n = m+1$.

Consider the variety of algebras (4.3) which satisfy all the identities (4.2), (4.4) and (4.5). The existence of nontrivial algebras in this variety is equivalent to the statement that an identity $x_1 = x_2$ is not a consequence of the defining set of identities of the variety.

Assume that $x_1 = p$ is a consequence of the set of identities (4.2), (4.4) and (4.5), where p is a term in the corresponding first order language. By induction it can be shown that p must have one of the following forms:

(a) p is x_1

(b) p is $[[p_{11} \dots p_{1m} / q_1] \]^1 \dots [p_{m1} \dots p_{mm} / q_m]]^m q]_i$

(c) p is $[q \]^1 [q_1 \backslash p_{11} \dots p_{1m}] \dots [q_m \backslash p_{m1} \dots p_{mm}]]_i,$

where $x_i = p_{ii}$, $q_v = q_\lambda$, $q_v = q$ are consequences of the mentioned set of identities, for any v, λ . This completes the proof.

5. We will state some problems concerning the class of (n,m) -semi-

groups, assuming that $m \geq 2$.

5.1. Although we gave some examples of (n,m) -semigroups, the problem of the existence of "good" examples of (n,m) -semigroups is open. For example, we know only one kind of nontrivial (n,m) -groups (the class of free (n,m) -groups) when m is not a divisor of n .

5.2. It is not true that each semigroup is a covering semigroup of an (n,m) -semigroup, and thus we have the problem of giving corresponding description of (strong) covering semigroups for (n,m) -semigroups belonging to convenient classes.

5.3. We do not know any convenient description of the complete system of identities that hold in the class of component algebras of (n,m) -semigroups.

5.4. Is it true that any commutative and cancellative (n,m) -semigroup is an (n,m) -subsemigroup of a commutative (n,m) -group?

5.5. Is it true that every (n,m) -semigroup is an (n,m) -subsemigroup of an $(m+1,m)$ -semigroup?

5.6. Cohn-Rebane's theorem ([2; IV.4] or [8; 12]), and Gluskin-Hosszu's theorem ([4], [7] or [6; II.5]) are well known. Are the corresponding generalizations true?

5.7. Does there exist nontrivial (n,m) -quasigroups ([3]) which are also (n,m) -groups?

5.8. If $i_\nu, j_\lambda > 0$, $s, t \geq 0$, then we have an (n,m) -semigroup identity of the following form

$$[x_{i_1} \dots x_{i_{sk+m}}] = [x_{j_1} \dots x_{j_{tk+m}}], \quad (5.1)$$

and a variety of (n,m) -semigroups can be defined by a set of (n,m) -semigroups identities. Any identity (5.1) induces a semigroup identity, and thus each variety of (n,m) -semigroups induces a variety of semigroups. What are the connections between the corresponding varieties?

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