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EMBEDDING UNOIDS IN SMEIGROUPS BELONGING TO SOME REGULAR VARIETIES OF SEMIGROUPS

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Any (finite) unoid $\mathbf{A} = (A; \Omega)$ is embeddable in a (finite) semigroup $S(.)$, in the sense that $\psi(\omega(a)) = \overline{\omega}\,\psi(a)$ for every $\omega \in \Omega$, $a \in A$ and for some mapping $\overline{\omega} \to \bar{\omega}$ from Ω into S and some injection $\psi : A \to S$ ([1] [2], [7]). Hence the class of all (finite) subunoids of (finite) semigroups coincides with the class of all (finite) unoids. The purpose of this note is to investigate the classes of subunoids of semigroups from some regular variety defined by regular identities, i.e. the identities which have the same symbols on the both sides of the equality. (Specialy, we are dealing with the problem whether these classes are varieties (they are surely quasivarieties ([6], pg. 254)).

0. DENOTATION, DEFINITIONS AND PRELIMINARY RESULTS

Let Z be an arbitrary nonempty set. Denote by Z^* the set of all finite sequences of elements in Z. For $u \in Z^*$ denote by $d(u)$ the lenght of the sequence u, by $c(u)$ the set of elements from Z occuring in u and by $u(i)$ the i-th element from right to left occuring in u and by (i) u the i-th element from left to right occuring in u. Also, if $z \in Z$, $d_z(u)$ will stand for the number of occurrences of the symbol z in u.

An identity $\xi = \eta$ in an arbitrary signature of operators is said to be a regular identity if $c(\xi) = c(\eta)$ and it is a balanced identity if (it is regular and) $d_x(\xi) = d_x(\eta)$ for any operator or variable symbol x. A variety of arbitrary algebras is said to be a regular (balanced) variety if it is defined by regular (balanced) identities.

Let K be a variety of semigroups and Ω be a signature of unar operators. We shall use $K(\Omega)$ to denote the class of all Ω-unoids that are subunoids of the semigroups in K. Let Σ_i (Σ_q) be the complete system of iden tities (quasiidentities) in K. The following lemma gives the pattern for „translateing" this system into the complete system $\Sigma_i(\Omega)$ ($\Sigma_q(\Omega)$) of identities (quasiidentities) in the quasivariety $K(\Omega)$.

LEMMA 0.1. ([5]) Let φ be an open Ω-formula and $\overline{\varphi}$ be obtained from φ by substituting different operators in Ω by different variables which are also different from the other variables occuring in φ. Hence $\overline{\varphi}$ can be treatted as a semigroup-theoretic formula and $\overline{\varphi}$ is valid in a class of semigroups K iff φ is valid in the class of Ω-algebras $K(\Omega)$::.

With respect to this lemma, the problem whether $K(\Omega)$ is a variety is equivalent to the problem whether $\Sigma_q(\Omega)$ is a consequence of $\Sigma_t(\Omega)$.

Remark. Notice that for card $(\Omega) \geqslant$ }-{, the class of all varieties $K(\Omega)$, for K passing through the set of all semigroup-varieties, is a proper subclass of the class of all varieties of Ω-unoids (the cardinality of the first class is less then }-{ and the cardinality of the second is $2^{card\,(\Omega)}$, [6], pg. 351).

The following theorem is deduced from the corresponding fact con cerning the embedding of arbitrary algebras in semigroups. The theorem shows that in order to prove that the class $K(\Omega)$ is a proper quasivariety for a regular variety of semigrops K, we can reduce our attention to the „small" signatures.

THEOREM 0.2. ([4], [5]) Let K be a regular variety of semigroups, $\Omega \subseteq \Omega'$ and $K(\Omega)$ is a proper quasivariety. Then $K(\Omega')$ is a proper quasi-variety too. : :

1. THE MAIN RESULTS AND SOME EXAMPLES

THEOREM 1.1. Let K be a balanced variety of semigroups. Then $K(\{\omega\})$ conicides with the variety of all $\{\omega\}$-unoids.

Remark. A hipothesis is that an analogous assertion is valid for the class of regular varieties of semigroups.

The following theorems will be ilustrated each by two examples of semigroup-varieties related to them.

THEOREM 1.2. Let K be a regular variety of semigroups defined by identities $\xi = \eta$ such that $\xi(1) = (\eta(1)(= x)$ and $d_x(\xi)$, $d_x(\eta) \geqslant 2$. Then for any signature Ω, $K(\Omega)$ is a variety.

Examples.

a) $xyxyx = xyx$

b) $x^i = x^j$, $i,j \geqslant 2$.

THEOREM 1.3. Let K be a regular variety defined by identities $\xi = \eta$ such that $\xi(1) = \eta(1)(= x)$ and $d_x(\xi) = 1 = d_x(\eta)$. Then for every signature Ω, $K(\Omega)$ is a variety.

Examples.

a) $xyz = xyxz$ (left-distributive semigroups, [3])

b) $x^i z = x^j z$, $i, j \geqslant 1$.

We allow the semigroup-theoretic terms appearing in Theorem 1.4., 1.5. and 1.6. to be empty words, unless otherwise stated.

THEOREM 1.4. Let K be a regular variety defined by an identity of the type $\xi \xi_1 = \xi \xi_1^n$ $(n \geqslant 1)$. Then for every signature Ω, $K(\Omega)$ is a variety

Examples.

a) $x = x^k$, $k \geqslant 1$

b) $xyz = xyzyz$

THEOREM 1.5. Let K be a regular variety defined by an identity of the type $\xi x = \xi x \eta x$. Then for every signature Ω, $K(\Omega)$ is a variety.

Examples.

a) $xy = xy^i x^j y^k$, $j \geqslant 0$, $i, k \geqslant 1$

b) $xyz = xyzxz$

THEOREM 1.6. Let K be a regular variety defined by an identity of the type $\xi x = \eta_1 x \eta_2$, whereas

(1) $x \notin c(\xi) \cup c(\eta_1)$

(2) $(1)\xi = (1)\eta_1 x$

(3) $c(\eta_1) \neq c(\xi)$.

Then for every signature Ω containing at least three operators, $K(\Omega)$ is a proper quasivariety.

Examples.

a) $xyz = xzyz$ (right-distributive semigroups, [3])
b) $xyz = xzy$

2. PROOFS OF THE THEOREMS

2.1. A GENERAL APPROACH

Let Ω be an arbitrary signature consisting of unar operators only, $\mathbf{A} = (A; \Omega)$ an Ω-unoid and K a variety of semigroups. Denote by $F(.)$ the free semigroup generated by the set $A \cup \Omega$ (proposing that $A \cap \Omega = \emptyset$). Define an equivalence relation $\approx$ on $F(.)$ as follows:

S. Markovski's critical remarks helped me improveing and correcting the formulations and proofs of the theorems.

for $u, v \in F$ put $u \square v$ and $v \square u$ if

a) there exists a subsequence ωa ($\omega \in \Omega$, $a \in A$) of u such that u is graphicaly equal (denoted by $\equiv$) to $u'\omega a u$ and $v = u'bu''$ ($b \in A$), whereas $\omega(a) = b$ in **A**, or

b) $u = v$ is valid in the free semigroup $F_k(.)$ in K, generated by the set $A \cup \Omega$.

Define $\approx$ to be the equivalence extension of the relation $\square$.

Relation $\approx$ is a congruence on $F(.)$.

The cover of **A** in the variety K is defined to be the semigroup $F(.) / \approx = D(.)$.

In order to prove that $K(\Omega)$ is a variety it is sufficient to prove that every unoid **A** belonging to the variety $VK(\Omega)$ (the variety defined by the all identities valid in $K(\Omega)$) is a subunoid of its cover in K[1]). To check that define value, denoted by [ ], as a partial mapping from F_k into A by

$$[\omega_1 \omega_2 \dots \omega_s a] = \omega_1 \omega_2 \dots \omega_s (a)$$

for every $\omega_1, \omega_2, \dots, \omega_s \in \Omega$ and $a \in A$. (Notice that $\omega_1 \dots \omega_s a$ is a representative of the set of elements equal to it in F_k (.)).

The verification of the fact that [ ] is well defined is direct and we omit it.

For $a \in A$ we have immediately $[a] = a$. Hence, what is left to be checked is whether

(*) for any $a, b \in A$, $a \approx b$ implies $[a] = [b]$.

Namely, (*) implies that the mapping $a \to a^{\approx}$ is an injection and, on the other hand, it is obvious that $(\omega a)^{\approx} = \omega . a^{\approx}$. Hence, **A** would be a subunoid of its cover in K and $K(\Omega)$ would coincide with the variety $VK(\Omega)$.

In the proofs of the theorems 1.1.—1.5. we shall follow this general idea and just check the relation (*), taking over the notation of this paragraphe.

2.2. PROOFS OF THEOREMS 1.1.—1.5.

Let $a, b \in A$ and $a \approx b$ via $a \equiv u_1, u_2, \dots, u_k \equiv b$ (i.e. $u_i \square u_{i+1}$ for every $i = 1, 2, \dots, k-1$).

Proof of Theorem 1.1.

First notice that (by Lemma 0.1. and because K is a balanced variety) trivial identities are the only identities valid in $K(\Omega)$. Thus $VK(\Omega)$ coincides with the class of all Ω-unoids.

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[1]) In fact, that is a necessary condition too.

Now it is sufficient just to remark that $\omega^n(a_j) = a$ for $a_j \in c(u_j)$ and $n = d_\omega(u_j)$ $j = 1, 2, \ldots, k$). That immediately implies that $a = b$ and (*) is fulfilled. : :

Proof of Theorem 1.2 **and Theorem** 1.3.

If u_j $(1 \leqslant j < k)$ is in the domain of [ ] then u_{j+1} is in the domain of [ ] and $[u_j] = [u_{j+1}]$ (observe that the element of A occuring in u_j is only $u_j(1)$ $(j = 1, \ldots, k)$). : :

Proof of Theorem 1.4. **and Theorem** 1.5.

We shall prove that

(4) if $u_j \equiv w_1 a_j w_2$, $w_1 \in \Omega^*$, $w_2 \in (A \cup \Omega)^*$, $a_j \in A$ then $w_1(a_j) = a$ in **A** for every $j = 1, 2, \ldots, k$.

That will be suficient, because we have immediately $b \equiv u_k$ and $u_k = a$ in **A**.

To prove (4) we use an induction. For $u_1 \equiv a$ the assertion is obviously true. Propossing that u_j satisfies the condition (4), consider u_{j+1}. The less trivial part of the proof is for $u_i = u_{i+1}$ *in* F_k (.).

The rest of the proof of Theorem 1.5.:

We have two possibilities:

i) $u_j \equiv w_1 uzw_2$ and $u_{j+1} \equiv w_1 uzvzw_2$, or

ii) $u_j \equiv w_1 uzvzw_2$ and $u_{j+1} \equiv w_1 uzw_2$

whereas $z \in A \cup \Omega$ and $uz = uzvz$ is valid in F_k (.).

Consider the first case.

Let the first occurence a_j of an element of A in u_j be in w_2. Then by Lemma 0.1. and because of the defining identity

$$w_1 uzw^*(a_j) = w_1 uzvzw^*(a_j) \text{ in } \mathbf{A}$$

whereas w^* is the part of w_2 left from a_j. We see that (4) is true for u_{j+1} too.

If a_j falls left from w_2 then (4) is obviously true for u_{j+1}.

Second case:

The first occurence a_i of an element of A in u_j is not in v because $c(v) \subseteq c(u)$. If it is in w_2 proceed as in the previous case. The other cases are obvious.

The rest of the proof of the Theorem 1.4.:

Now we have $u_j \equiv w_1 v' v w_2$ and $u_{j+1} \equiv w_1 v' v^n w_2$ or $u_j \equiv w_1 v' v^n w_2$ and $u_{j+1} \equiv w_1 v' v w_2$, whereas $v' v = v' v^n$ in F_k (.). In the both cases the relation (4) is flulfilled for u_{j+1} too (proceed just like in the proof of the Theorem 1.5.). : :

2.3. PROOF OF THEOREM 1.6.

Let $\Omega = \{\omega, \tau, \sigma\}$ and let η'_1, η'_2 and ξ' be elements of Ω^* obtained from η_1, η_2 and ξ respectively by substituting variables in $c(\eta_1)$ by τ and those in $c(\xi) \setminus c(\eta_1)$ by σ ($c(\xi) \setminus c(\eta_1) \neq \varnothing$ because of the condition (3)). Consider the quasiidentity

$$\omega \eta'_1 \, y = \tau \eta'_1 \, y \rightarrow \omega\xi' y = \tau\xi' y. \tag{5}$$

It is valid in $K(\Omega)$ because if $\mathbf{A} \in K(\Omega)$ and $\omega \eta'_1 (a) = \tau \eta'_1 (a)$ for some $a \in A$, then we have: $\omega \, \xi' (a) = \overline{\omega}. \xi' . \, a = \overline{\omega} \, . \, \eta'_1 \, . \, a \, . \, \eta''_2 = \overline{\tau}. \, \eta'_1 \, . \, a. \, \eta''_2 = \overline{\tau} \, \xi'. \, a = \tau \, \xi' (a)$, whereas η''_2 is obtained from η'_2 by substituting the occurences of variable x in η'_2(if any) by an element a.

We shall prove that (5) is not valid in the variety $VK(\Omega)$, i.e. that (5) is not a consequence of the identities in $K(\Omega)$.

Let $\mathbf{B} = < a : \omega \, \eta'_1 (a) = \tau \, \eta'_1 (a) > VK(\Omega)$, i.e. $\mathbf{B}$ is an Ω unioid generated by the set $\{a\}$ in $VK(\Omega)$ and with one defining relation as in the presentation. The relation $\omega \, \xi' (a) = \tau \, \xi' (a)$ is false in $\mathbf{B}$. Namely, $\omega \, \xi' (a) = = \tau \, \xi' (a)$ if we can reach $\tau \, \xi' (a)$ starting from $\omega \; \xi' (a)$ and using either identities in $VK(\Omega)$ or the defining relation in $\mathbf{B}$. Because of (2), identities in $VK(\Omega)$ do not change the first symbol. Thus the defining relation in $\mathbf{B}$ should be applied (at least once) from the first symbol of the corresponding element in $\mathbf{B}$ (and, when first time applied, that is the operator symbol ω). But that is impossible because $\sigma \notin c(\eta'_2)$ and on the other hand σ appears in every element equal to $\omega \, \xi' (a)$ in $\mathbf{B}$. : :

REFERENCES

[1] Cohn P. M. *Universal Algebra*, D. Reidel, Publ. Comp. Dordrecht Holand (1981).

[2] Čupona G'. *Subalgebras of semigroups*, Bull. de la Soc. des math. et des phys. de la S. R. Macedoine, 19 (1968) 9—16.

[3] Kalajdžievski S. *Embedding of algebras in distributive semigroups*, III Algebraic Conference, Belgrade (1982) 77—83.

[4] Kalajdžievski S. *Some proper quasivarieties of subalgebras of semigroups*, Second symposium n-ARY STRUCTURES, Varna (1983), (in print)

[5] Калајџиевски С. *Подалгебри од полугрупи од некои многукратности полугрупи,-* Магистерска работа, Скопје (1983).

[6] Мальцев А. И. *Алгебраические системы* , Наука, Москва (1970).

[7] Соколовская Т. В. О *представлениях конечных универсальных алгебр в конечних полугруппах*, Матем. Заметки, Том 9, выпуск 3, (1971) 185—190.

СМЕСТУВАЊЕ НА УНОИДИ ВО ПОЛУГРУПИ КОИ ПРИПАЃААТ НА НЕКОИ РЕГУЛАРНИ МНОГУКРАТНОСТИ ПОЛУГРУПИ

Резиме

Секој (конечен) уноид $\mathbf{A} = (A: \Omega)$ може да се смести во (конечна) полугрупа така што операциите во уноидот да се реализираат како леви трансалации во полугрупата. Значи класата од сите (конечни) подуноиди од (конечни) полугрупи се совпаѓа со класата од сите (конечни) уноиди. Во оваа работа ги испитуваме класите подуноиди од полугрупи кои припаѓаат на некои многукратности полугрупи дефинирани со регуларни идентитети. Специјално, работиме на проблемот дали тие класи се многукратности.