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n-SUBSEMIGROUPS OF SEMIGROUPS WITH NEUTRAL PROPERTIES

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In the paper [1] (this volume), G.Čupona give a sufficient condition the class of n-subsemigroups of semigroups belonging to a semigroup variety to be also a variety of n-semigroups. Here we consider some varieties of semigroups which do not satisfy the mentioned condition, but the class of their n-subsemigroups are varieties of n-semigroups as well.

1. THE VARIETY OF SEMIGROUPS $\mathcal{O}_{k,i}$. The variety of semigroups $\mathcal{O}_{k,i}$ is defined by the semigroup identity

(1.1) $$x_0 \dots x_k = x_0 \dots x_{i-1} y x_i \dots x_k,$$

where x_ν and y are variables, k and i are integers such that $k \geq 0$, $0 \leq i \leq k+1$.

1.1. The semigroup equality

(1.2) $$x_0 \dots x_s = y_0 \dots y_r$$

is a nontrivial identity in $\mathcal{O}_{k,i}$ iff $s,r \geq k$, $x_0 = y_0, \dots, x_{i-1} = y_{i-1}$, $x_{s-k+i} = y_{r-k+i}, \dots, x_s = y_r$.

It follows an easy description of the free semigroup $\underline{F}_A = (F_A, \cdot)$ in $\mathcal{O}_{k,i}$ generated by the set A. Namely, F_A consists of all nonempty sequences of elements of the set A with lengths not greater than k+1, and with an operation defined by

$$a_0 \dots a_r \cdot a_{r+1} \dots a_s = \begin{cases} a_0 \dots a_s, & \text{if } s \leq k \\ a_0 \dots a_{i-1} a_{s-k+i} \dots a_s, & \text{if } s > k. \end{cases}$$

If $\mathcal{C}$ is a class of semigroups, then by $\mathcal{C}(n)$ we denote the class of n-semigroups which are n-subsemigroups of semigroups in $\mathcal{C}$. (See [1].) Here we show that the class of n-semigroups $\mathcal{O}_{k,i}(n)$ is a variety, which is finitely axiomatizable. We denote by [...] the n-ary operation of the n-semigroups, and x's and y's are variables.

1.2. The class of n+1-semigroups $\mathcal{O}_{k,i}(n+1)$ is a variety defined by the identity

(1.3) $[x_o \ldots x_{i-1} y_1 \ldots y_{np-k} x_i \ldots x_k] = [x_o \ldots x_{i-1} z_1 \ldots$
$\ldots z_{nq-k} x_i \ldots x_k]$

<u>where</u> p,q <u>are the least positive integers such that</u> $np-k \geq 0$, $nq-k > 0$.

<u>Proof</u>: As a consequence of <u>1.1</u> we have that (1.3) is satisfied in any n+1-subsemigroup of a semigroup in $\mathcal{O}_{k,i}$ and, furthermore,

(1.4) $$[x_o \ldots x_{sn}] = [y_o \ldots y_{rn}]$$

is a nontrivial identity in the variety of n-semigroups defined by (1.3) iff $x_o=y_o, \ldots, x_{i-1}=y_{i-1}, x_{ns-k+i}=y_{nr-k+i}, \ldots, x_{ns}=y_{nr}$.

Now, let $\underline{A} = (A,[\ldots])$ be a given n+1-semigroup which satisfy the identity (1.3). We will construct a semigroup $\underline{A}\hat{} \in \mathcal{O}_{k,i}$ such that $\underline{A}$ will be an n+1-subsemigroup of $\underline{A}\hat{}$.

Let $\underline{F}_A$ be the free semigroup in $\mathcal{O}_{k,i}$ generated by the set A. Define a relation $\vdash$ in F_A by $u=---a--- \vdash ---a_o \ldots a_{mn}---=v$ $(u,v \in F_A)$, where $a = [a_o \ldots a_{mn}]$ in $\underline{A}$, and let $\vdash\dashv = \vdash \cup \vdash^{-1}$. Then, the transitive extension $\approx$ of $\vdash\dashv$ is a congruence on F_A (see [1]). It is enough to show that $\approx$ separates the elements of the set A, i.e. $a,b \in A \Rightarrow (a \approx b \Rightarrow a=b)$, because in that case we can take $\underline{A}\hat{} = \underline{F}_A/\approx$.

An element $u \in F_A$ is said to be irreducible (reducible) if its length is less than k+1 (bigger than k). Using (1.4) we define a partial mapping [ ] of F_A into A as follows: $[u] = a$ if $u = a_o \ldots a_{mn}$ in $\underline{F}_A$ and $[a_o \ldots a_{mn}] = a$ in $\underline{A}$. Note that all reducible elements of F_A are in the domain of [ ].

Let $u,v,w_\nu \in F_A$. It is easy to check this properties:

(i) $u \vdash v$, u is in the domain of [ ] $\Rightarrow [u] = [v]$.

(ii) $u \vdash\dashv w_1 \vdash\dashv w_2 \vdash\dashv \ldots \vdash\dashv w_s \vdash\dashv v$, u and v are reducible, $w_1, \ldots, w_s$ are irreducible $\Rightarrow [u] = [v]$.

We will prove only the last implication. Namely, as $w_1, \ldots, w_s$ are irreducible, we have that $|w_1| \equiv \ldots \equiv |w_s| \pmod n$, and so there exist $w \in F_A$ such that $2|w_i| + |w| \equiv 1 \pmod n$, i.e. $w_i w w_i$ is in the domain of [ ], for i=1,2,...,s. Thus we have:

$$u \vdash\dashv w_1 \vdash\dashv \ldots \vdash\dashv w_s \vdash\dashv v \Rightarrow u=uwu \vdash\dashv w_1 wu \vdash\dashv w_1 w w_1 \vdash\dashv \ldots \vdash\dashv w_s w w_s \vdash\dashv vww_s \vdash\dashv vwv = v \Rightarrow [u]=[uwu]=[w_1 w w_1]=\ldots=[w_s w w_s]=[vwv]=[v].$$

Now, let $a,b \in A$ and $a \approx b$. Then there exist $u_1,\dots,u_r \in F_A$ such that $a \vdash\!\dashv u_1 \vdash\!\dashv u_2 \dots \vdash\!\dashv u_r \vdash\!\dashv b$, and (i) and (ii) implies that $a=b$.

2. VARIETY OF SEMIGROUPS $\mathcal{O}_{k,i,j}$. The variety of semigroups $\mathcal{O}_{k,i,j}$ is defined by the identity

$$x_0 \dots x_k = x_0 \dots x_{i-1} y_i \dots y_{j-1} x_j \dots x_k,$$

where $k \geq 0$, $0 \leq i < j \leq k+1$.

2.1. The semigroup equality

$$x_0 \dots x_s = y_0 \dots y_r$$

is an identity in the variety $\mathcal{O}_{k,i,j}$ iff it is trivial or $s \geq k$, $r \geq k$ and $x_0=y_0,\dots,x_{i-1}=y_{i-1}, x_{s-k+j}=y_{r-k+j},\dots,x_s=y_r$.

As a consequence of 1.1 and 2.1 we obtain:

2.2. $i<j \Rightarrow \mathcal{O}_{k,i} \cap \mathcal{O}_{k,j} = \mathcal{O}_{k,i,j}$.

In the same manner as in 1.2 one can prove that $\mathcal{O}_{k,i,j}(n)$ is a variety, i.e. we have:

2.3. The class of $n+1$-semigroups $\mathcal{O}_{k,i,j}(n+1)$ is a variety defined by the identity

$$[x_0 \dots x_{i-1} y_i \dots y_{pn-k+j-1} x_j \dots x_k] = [x_0 \dots x_{i-1} z_i \dots z_{qn-k+j-1} x_j \dots x_k]$$

where p,q are the least integers $(p,q \geq 0)$ such that $pn-k+j-1 \geq 0$, $qn-k+j-1 > 0$.

3. REMARKS.

1) We note that the condition (α) of [1] is not satisfied in either of the varieties $\mathcal{O}_{k,i}$ and $\mathcal{O}_{k,i,j}$. In fact, the condition (α) can be made a little more complicated such that the above varieties are in its scope, but that will not give the best posible generalization.

2) One can investigate varieties of semigroups similar to $\mathcal{O}_{k,i}$ and $\mathcal{O}_{k,i,j}$. Namely, let p be a permutation of the set $\{0,1,2,\dots,k\}$, x's and y's are variables, and consider the semigroup identities

(3.1) $\quad x_0 \dots x_k = x_{p(0)} \dots x_{p(i-1)} y x_{p(i)} \dots x_{p(k)},$

(3.2) $$x_o \ldots x_k = x_{p(o)} \ldots x_{p(i_1-1)} y_{i_1} x_{p(i_1+1)} \ldots$$

$$\ldots x_{p(i_r-1)} y_{i_r} x_{p(i_r+1)} \ldots x_{p(k)}.$$

Then one can prove that either of the varieties of semigroups defined by (3.1) or (3.2) is equal to the variety $0_{k,m,M}$ for some m and M. (We assume that $p(s) \neq s$ for some s in (3.1).) Namely, let q be the least integer such that $p(q) \neq q$, and t be the biggest integer with the property $p(t) \neq t$. Then we have $m=\min\{i,s\}$ and $M=\max\{i,t+1\}$ for (3.1), and $m=\min\{q,i_1\}$, $M=\max\{i_r+1,t+1\}$ for (3.2).

3) The results of this paper are generalizations of the results obtained in [2].

REFERENCES

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