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ON A QUASIIDENTITY IN n-ARY QUASIGROUPS

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Abstract. A generalization of the Reidemeister condition for n-ary quasigroups is considered. (A generalization of this condition for ternary nets is given in [1].)

First we will give some definitions.

Let Q be a nonempty set, Q^n the Cartesian n-th power of Q and A: $Q^n \to Q$ a mapping. The ordered pair (Q,A) is called an n-quasigroup if for any $(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n,b) \in Q^n$ and $i \in \{1,\dots,n\}$ the equation

$$A(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n) = b \tag{1}$$

has a unique solution.

Further on we will denote the sequence $(a_1,\dots,a_n) \in Q^n$ shortly by $\bar{a}$, and the sequence $(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n)$ by $i(\bar{a})$.

If $\bar{a}$ is any fixed element of Q^n and $i \in \{1,\dots,n\}$, then the mapping

$$L_{i(\bar{a})}: x \to A(a_1,\dots,a_{i-1},x,a_{i+1},\dots,a_n)$$

is called a translation of the n-quasigroup (Q,A) [2]. We note that any translation of (Q,A) is a permutation of the set Q. The n-ary operation B on Q, defined by:

$$B(x_1,\dots,x_n) = A(L^{-1}_{1(\bar{a})}x_1,\dots,L^{-1}_{n(\bar{a})}x_n), \tag{2}$$

where $\bar{a}$ is a given element of Q^n, is called an LP-isotop of the n-quasigroup (Q,A). So, any fixed element $(a_1,\dots,a_n) \in Q^n$ determines an LP-isotop of (Q,A) by (2).

It can be shown that (Q,B) is an n-quasigroup with an identity element $e = A(a_1,\dots,a_n)$, i.e. (Q,B) is an n-loop.

We can write the equality (2) in the following form:

$$A(x_1,\dots,x_n) = B(L_{1(\bar{a})}x_1,\dots,L_{n(\bar{a})}x_n). \tag{2'}$$

We will investigate some properties of LP-isotops when the n-quasigroup (Q,A) satisfies a special condition.

Let (Q,A) be an n-quasigroup. Consider the quasiidentity

$$A(x_1^1,\dots,x_n^1) = A(x_1^3,\dots,x_n^3)$$

$$\bigwedge_{i=1}^{n} A(x_1^1,\dots,x_{i-1}^1,x_i^2,x_{i+1}^1,\dots,x_n^1) = \tag{3}$$

$$= A(x_1^3,\dots,x_{i-1}^3,x_i^4,x_{i+1}^3,\dots,x_n^3)$$

$$\Rightarrow A(x_1^2,\dots,x_n^2) = A(x_1^4,\dots,x_n^4),$$

where $x_1,\dots,x_n$ are variables. If n=2, i.e. A is a binary operation, then (3) is the Reidemeister condition [1].

THEOREM. The condition (3) in an n-quasigroup (Q,A) holds if and only if any two LP-isotops of (Q,A) with a (given in advance) common identity element are equal.

Proof. Let (Q,C) and (Q,D) be LP-isotops of an n-quasigroup (Q,A). By (2′) we obtain

$$A(x_1,\dots,x_n) = C(L_{1(\overline{x})}x_1,\dots,L_{n(\overline{x})}x_n), \tag{4}$$

$$A(x_1,\dots,x_n) = D(L_{1(\overline{y})}x_1,\dots,L_{n(\overline{y})}x_n), \tag{5}$$

where $\overline{x} = (x_1^1,\dots,x_n^1)$ and $\overline{y} = (x_1^3,\dots,x_n^3)$ are fixed elements of Q^n, $A(\overline{x})$ adn $A(\overline{y})$ are identity elements of (Q,C) and (Q,D) respectively.

Let the following condition

$$e_C = e_D \Rightarrow C = D \tag{6}$$

be satisfied. By the hypothesis $e_C=e_D$ it follows that

$$A(x_1^1,\dots,x_n^1) = A(x_1^3,\dots,x_n^3).$$

Substituing $x_1=x_1^2,\dots,x_n=x_n^2$ in (4), $x_1=x_1^4,\dots,x_n=x_n^4$ in (5) and using the assumption that the condition on the left-hand side of the implication (3) holds and C=D, we obtain

$$A(x_1^2,\dots,x_n^2) = A(x_1^4,\dots,x_n^4),$$

which means that (3) is satisfied.

Conversely, suppose that (3) holds in an n-quasigroup (Q,A) and let $e_C=e_D$ be an identity element of the LP-isotops of (Q,A), defined by (4) and (5). It is necessary to show that C=D. By the equalities

$$A(x_1^2,x_2^1,x_3^1,\dots,x_n^1) = A(x_1^4,x_2^3,x_3^3,\dots,x_n^3) = t_1,$$
$$A(x_1^1,x_2^2,x_3^1,\dots,x_n^1) = A(x_1^3,x_2^4,x_3^3,\dots,x_n^3) = t_2, \quad (7)$$
$$\vdots$$
$$A(x_1^1,\dots,x_{n-1}^1,x_n^2) = A(x_1^3,\dots,x_{n-1}^3,x_n^4) = t_n,$$

where $t_1,\dots,t_n$ are arbitrarily chosen elements of Q, we conclude that the elements $x_1^2,\dots,x_n^2,x_1^4,\dots,x_n^4$ are uniquele determined. By (3), (4), (5) it follows that

$$C(t_1,\dots,t_n) = A(x_1^2,\dots,x_n^2) = A(x_1^4,\dots,x_n^4) = D(t_1,\dots,t_2),$$

i.e. C=D. The proof of the Theorem is completed.

COROLLARY 1. If an n-quasigroup (Q,A) satisfies condition (3), then the number of their LP-isotops is not greater than $|Q|$.

Namely, let $a \in Q$. For any solution of the equation $A(x_1,\dots,x_n) = a$ we obtain an LP-isotop of (Q,A), determined by (2). All of them have a common identity element e=a, and thus they are equal.

Let (Q,A) be an n-quasigroup. If we substitute the variables $x_{i_1},\dots,x_{i_{n-2}}$ by arbitrary fixed elements of Q, then we will obtain a binary quasigroup (a binary retract) $(Q,\cdot)$ of the n-quasigroup (Q,A).

COROLLARY 2. If an n-quasigroup (Q,A) satisfies (3) then any of its binary retracts is isotopic with a group.

Namely, if n-2 variables in (3) are fixed, then the Reidemeister condition for the binary retracts of the n-quasigroup (Q,A) is satisfied, and thus any binary retract is isotopic with a group [3].

REFERENCES

1. F.Radó, Eine Bedingung für die Regularität der Gewebe, Mathematica 2(25) (1960), 325-334
2. В.Д.Белоусов, n-арные квазигруппы, Штиинца, Кишинев (1972)
3. В.Д.Белоусов, Основы Теории квазигрупп и луп, Наука, Москва (1967)

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