THIRD ALGEBRAIC CONFERENCE
Beograd, 3-4 December 1982

A SET OF SEMIGROUP n-VARIETIES

Ǵ. Čupona

Let $\underline{S}=(S;\cdot)$ be a semigroup and $\underline{Q}=(Q;[\,])$ be an n-semigroup such that $Q\subseteq S$ and $[a_1\ldots a_n]=a_1\ldots a_n$, for any $a_\nu\in Q$. Then, $\underline{Q}$ is called an n-subsemigroup of $\underline{S}$. If $\mathcal{V}$ is a semigroup variety, then we denote by $\mathcal{V}(n)$ the class of n-semigroups that are n-subsemigroups of $\mathcal{V}$-semigroups, and it is well known that $\mathcal{V}(n)$ is a quasivariety of n-semigroups. (See, for example [6; p.274], or [3].) We say that $\mathcal{V}$ is an n-variety iff $\mathcal{V}(n)$ is a variety of n-semigroups; otherwise, i.e. if $\mathcal{V}(n)$ is a proper quasivariety, $\mathcal{V}$ is called a quasi n-variety. (Clearly, $\mathcal{V}(2)=\mathcal{V}$ for every semigroup variety). It is well known that both the set of semigroup n-varieties, and the set of semigroup quasi n-varieties are infinite for any $n\geq 3$. The same is true for the varieties of abelian semigroups. (The corresponding results can be found in [1], [7], [8] and [2]). Here we establish a sufficient condition for a semigroup variety to be an n-variety. It is shown that almost all the known n-varieties satisfy that condition, and some new examples are obtained.

0. PRELIMINARIES

0.1. Let $X=\{x_1,x_2,\ldots\}$ be an infinite countable set, elements of which are called variables and let X^+ be the free semigroup on X. Elements of X^+ are called semigroup terms, and if ξ,η are semigroup terms, then (ξ,η) is said to be a semigroup identity. A semigroup $\underline{S}=(S;\cdot)$ satisfies a semigroup identity $(x_{i_1}\ldots x_{i_p},\ x_{j_1}\ldots x_{j_q})$ if for every sequence $a_1,a_2,\ldots$ of elements of S the following equation holds in $\underline{S}$: $a_{i_1}\ldots a_{i_p}=a_{j_1}\ldots a_{j_q}$. If Λ is a set of semigroup identities, then by $\mathrm{Var}\Lambda$ we denote the variety of semigroups which satisfy all the semigroup identities belonging to Λ. The complete system $\langle\Lambda\rangle$ of semigroup identities which are consequences of Λ is the transitive extension of Λ_2, where:

$$\Lambda_0 = \Lambda\cup\Lambda^{-1}\cup\{(\xi,\xi)\mid \xi\in X^+\},$$

$$\Lambda_1 = \{(\xi_{i_1}\ldots\xi_{i_p},\xi_{j_1}\ldots\xi_{j_q})\mid(x_{i_1}\ldots x_{i_p},x_{j_1}\ldots x_{j_q})\in\Lambda_0,\ \xi_k\in X^+,$$

$$\Lambda_2 = \{(\xi_1\ldots\xi_s,\ \eta_1\ldots\eta_s)\mid(\xi_k,\eta_k)\in\Lambda_1,\ s\geq 1\}.$$

(See also [4] or [5].)

If $\xi \in X^+$ and $x_i \in X$, then we denote by $|\xi|_i$ the number of occurences of x_i in ξ, and thus $|\xi| = \Sigma|\xi|_i$ is the length of ξ.

A semigroup term ξ is said to be (n,Λ)-irreducible iff $(\xi,\eta) \in \langle\Lambda\rangle$ implies $|\xi| \equiv |\eta|$ (mod n-1). Otherwise, i.e. if there is a $\zeta \in X^+$ such that $(\xi,\zeta) \in \langle\Lambda\rangle$ and $|\xi| \not\equiv |\zeta|$ (mod n-1), then ξ is (n,Λ)-reducible.

0.2. To every set Λ of semigroup identities we associate an index r=ind Λ and a period m=per Λ. First, if $|\xi|_i = |\eta|_i$ for every $i \in \{1,2,\ldots\}$ and for every semigroup identity $(\xi,\eta) \in \Lambda$, then we write indΛ=1, perΛ=0. (Namely, this is satisfied iff the variety of abelian semigroups ABSEM is a subvariety of Var Λ). Assume now that there exists a semigroup identity $(\xi,\eta) \in \Lambda$ and an integer $i \in \{1,2,\ldots\}$ such that $|\xi|_i \neq |\eta|_i$. Then, per Λ and indΛ are defined by:

$$\mathrm{per}\Lambda = \text{g.c.d.}\{|\xi|_i - |\eta|_i \mid (\xi,\eta) \in \Lambda,\ i \in \{1,2,\ldots\}\},$$
$$\mathrm{ind}\Lambda = \min\ \{|\xi| \mid (\exists\eta)\ (\xi,\eta) \in \Lambda_1,\ |\xi| \neq |\eta|\}.$$

It can be easily seen that indΛ=ind⟨Λ⟩ and perΛ=per⟨Λ⟩, and thus we can say that ind Λ(perΛ) is the index (the period) of the variety VarΛ. We notice that if m=perΛ > 0 and r=indΛ, then $(x_1^r, x_1^{r+m}) \in \langle\Lambda\rangle$, and moreover if $(x_1^s, x_1^{s+k}) \in \langle\Lambda\rangle$, where $k \geq 1$, then $s \geq r$ and m is a divisor of k.

0.3. Let $\underline{Q}=(Q;[\ ])$ be an n-semigroup. Then the general associative law holds, i.e. for any $k \geq 1$ and $a_o,\ldots,a_{k(n-1)} \in Q$, the "product" $[a_o \ldots a_{k(n-1)}]$ is uniquelly determined in Q; we also write $[a] = a$, for every $a \in Q$.

If (ξ,η) is a semigroup identity such that $|\xi| \equiv |\eta| \equiv 1$ (mod n-1) then it can be also interpreted as an n-semigroup identity. And, if every semigroup identity $(\xi,\eta) \in \Lambda$ is an n-semigroup identity, then ve denote by $\mathrm{Var}_n\Lambda$ the variety of n-semigroups which satisfy all the n-semigroup identities $(\xi,\eta) \in \Lambda$.

Assume now that Λ is a set of semigroup identities, and denote by $\Lambda^{[n]}$, the set of n-semigroup identities belonging to ⟨Λ⟩. It is clear that if $\mathcal{V}$=VarΛ, $\mathcal{V}_n = \mathrm{Var}_n\Lambda^{[n]}$, then $\mathcal{V}(n) \subseteq \mathcal{V}_n$. Moreover: $\mathcal{V}$ is an n-variety iff $\mathcal{V}(n) = \mathcal{V}_n$.

1. MAIN RESULT

Theorem. Let $\mathcal{V}=\mathrm{Var}\Lambda$ be a semigroup variety with a period m, and let $n \geq 2$ be such that the following condition is satisfied:

(α) If ξ is an (n,Λ)-reducible semigroup term, then there exist $x,y \in X$ such that $(x^{km}\xi,\xi)$, $(\xi,\xi y^{km}) \in \langle\Lambda\rangle$, for every positive integer k.

Then $\mathcal{V}$ is an n-variety.

The proof will be given in three steps, and the condition (α) will be not assumed in the first two of them.

1.1. Let $\underline{Q}=(Q;[\;])$ be an n-semigroup and let Q_Λ be the free semigroup in $\mathcal{V}$ with a basis Q. Thus, Q is a generating subset of Q_Λ, and if $a_1,a_2,\ldots$ is a set of different elements of Q then $a_{i_1}\ldots a_{i_p}=a_{j_1}\ldots a_{j_q}$ iff $(x_{i_1}\ldots x_{i_p}, x_{j_1}\ldots x_{j_q}) \in \langle\Lambda\rangle$. Define a relation $\vdash$ in Q_Λ by: $\ldots a \ldots \vdash \ldots a_0 \ldots a_{k(n-1)} \ldots$, where $a=[a_0\ldots a_{k(n-1)}]$ in $\underline{Q}$. Let $\vdash\!\dashv$ be the symmetric extension of $\vdash$, and $\approx$ be the transitive extension of $\vdash\!\dashv$. The following two propositions are obvious.

1.1.1. $\approx$ is a congruence on the semigroup Q_Λ.

1.1.2. $\underline{Q} \in \mathcal{V}(n)$ iff the following statement is satisfied:

$$a,b \in Q \Longrightarrow (a \approx b \Longrightarrow a = b).$$

1.2. Assume now that $\underline{Q} \in \mathcal{V}_n=\mathrm{Var}_n\Lambda^{[n]}$, and that Q_Λ, $\vdash$, $\vdash\!\dashv$, $\approx$ are defined as in 1.1. A partial mapping $u \mapsto [u]$ from Q_Λ in Q can be defined in a usual way. Namely, $u \in Q_\Lambda$ is in the domain of $[\;]$ iff $u=a_0a_1\ldots a_{k(n-1)}$, where $a_\nu \in Q$, and then the "value" $[u]$ of u is defined by $[u]=[a_0a_1\ldots a_{k(n-1)}]$. The assumption $\underline{Q} \in \mathcal{V}_n$ implies that $[\;]$ is a well defined partial mapping.

Let $a_1,a_2,\ldots$ be different elements of Q, and let $u=a_{i_1}a_{i_2}\ldots a_{i_p}$. We say that u is irreducible (reducible) iff the semigroup term $x_{i_1}\ldots x_{i_p}$ is (n,Λ)-irreducible ((n,Λ)-reducible).

The following three proposition can be easily shown.

1.2.1. If $u \in Q_\Lambda$ is in the domain of $[\;]$, then $[u] \vdash u$.

1.2.2. Let $u,v \ \ Q_\Lambda$ be such that $u \longmapsto v$, and u is irreducible. If u is in the domain of $[\ ]$, then v is also in the domain of $[\ ]$ and moreover $[u]=[v]$.

1.2.3. V is an n-variety iff every $\underline{Q} \in V_n$ satisfies the following condition. If $u,v \in Q_\Lambda$ are in the domain of $[\ ]$ and $u \approx v$, then $[u]=[v]$.

From 1.2.2 and 1.2.3 we obtain the following proposition.

1.2.4. If every semigroup term is (n,Λ)-irreducible, then $V=\mathrm{Var}\Lambda$ is an n-variety.

1.3. The proof of Theorem will be completed here, by assuming that the condition (α) is satisfied.

If $m=0$, then all the semigroup terms are (n,Λ)-irreducible, and by 1.2.4 we obtain that V is an n-variety. Thus, we can assume that $m > 0$.

Let $\underline{Q} \in V_n$, and $u,v \in Q_\Lambda$ be such that $u \approx v$ and both u and v are in the domain of $[\ ]$. By 1.2.3 we have to show that $[u]=[v]$.

From $u \approx v$ it follows that there exists a sequence $w_1,\dots w_k \in Q_\Lambda$ such that $k \geq 0$ and $u \longmapsto w_1 \longmapsto w_2 \longmapsto \dots \longmapsto w_k \longmapsto v$. If one of u,v is irreducible, then, by 1.2.2, the sequence $u,w_1,\dots,w_k,v$ can be shortened in the case $k > 0$, and we have $[u]=[v]$ in the case $k=0$. Thus we can assume that both u and v are reducible.

Let s be such that $w=w_s$ is reducible, and w_t is irreducible for any $t < s$. (If w_1 is reducible, then $w=w_1$, and $w=v$ if all the $w_1,\dots,w_k$ are irreducible.)

The condition (α) implies that there exist $a,b \in Q$ such that $u=a^{im}u$, $w=wb^{jm}$, for any pair of positive integers i,j. The assumption u to be in the domain of $[\ ]$ implies that i can be chosen in such a way that all the members of the sequence $a^{im}w_1,\dots,a^{im}w_{s-1},a^{im}w$ are in the domain of $[\ ]$. Then we also have: $u \longmapsto a^{im}w_1 \longmapsto \dots \longmapsto a^{im}w$, and this implies that $[u]=[a^{im}w] =\dots=[a^{im}w]$. Let j be such that $j(n-1)m \geq r$, where r is the index of V. Then we have: $w=wb^{j(n-1)m}$, $b^{j(n-1)m+im}=$

$=b^{j(n-1)m}$, and this implies that: $a^{im}ub^{j(n-1)m}=ub^{j(n-1)m+im}$.
Therefore we have:

$$a^{im}w = a^{im}wb^{j(n-1)m} \vdash\!\!\dashv \ \dots \vdash\!\!\dashv \ a^{im}ub^{j(n-1)m}=ub^{j(n-1)m+im},$$

and:

$$ub^{j(n-1)m+im} \vdash\!\!\dashv \ w_1 b^{j(n-1)m+im} \vdash\!\!\dashv \ \dots \vdash\!\!\dashv \ wb^{j(n-1)}wb^{j(n-1)m+im}=w$$

Finally, we obtain:

$$[u] = [a^{im}w] = [a^{im}ub^{j(n-1)m}] = [ub^{j(n-1)m+im}] =$$
$$=\dots= \left[wb^{j(n-1)m+im}\right] = [w].$$

This completes the proof of Theorem.

2. COROLLARIES

Cor. 1. If ABSEM is a subvariety of a variety $\mathcal{V}$, then $\mathcal{V}$ is an n-variety for any $n \geq 2$.

Proof. The assumption is equivalent to the statement that $\mathrm{per}\mathcal{V}=0$, and then all the semigroup terms are (n,Λ)-irreducible for every $n \geq 2$.

Cor. 2. Let m be a non-negative integer and $n \geq 2$ be such that n-1 is a divisor of m. If $\mathcal{V}$ is a semigroup variety with a period m, then $\mathcal{V}$ is an n-variety.

Proof. If $\mathcal{V}=\mathrm{Var}\Lambda$, then every semigroup term is (n,Λ)-irreducible. (Clearly Cor. 1 is a special case of Cor. 2.)

Cor. 3. If $\mathcal{V}$ is a semigroup variety with an index $r=1$, then $\mathcal{V}$ is an n-variety for every $n \geq 2$.

Proof. Let $m=\mathrm{per}\mathcal{V}$. If $m=0$, then we can apply Cor. 1, and thus we can assume that $m > 0$. If $\xi=x_i\dots x_j$, and $k > 0$, then we have: $(x_i^{km}\xi,\xi)$, $(\xi,\xi x_j^{km}) \in \langle\Lambda\rangle$, where $\mathcal{V}=\mathrm{Var}\Lambda$. Thus, the condition (α) is satisfied.

A semigroup variety $\mathcal{V}=\mathrm{Var}\Lambda$ is a variety of periodic groups iff $\mathrm{ind}\mathcal{V}=1$, $\mathrm{per}\Lambda=m \geq 1$ and $(x_1x_2^m,x_1)$, $(x_1^m x_2,x_2) \in \langle\Lambda\rangle$. From Cor. 3 we obtain the following one:

Cor. 4. A variety of periodic groups is an n-variety for every $n \geq 2$.

Cor. 5. Let $\mathbb{V}$=VarΛ be a variety of abelian semigroups with an index r, and let the following condition be satisfied:

(β) If (ξ,η) is a nontrivial semigroup-identity belonging to Λ, i.e. $(\xi,\eta) \in \Lambda$ is such that $|\xi|_i \neq |\eta|_i$ for some $i \geq 1$, then there exist $j,k \geq 1$ such that $|\xi|_j \geq r$ and $|\eta|_k \geq r$.

Then $\mathbb{V}$ is an n-variety for every $n \geq 2$.

Proof. We notice first that $\langle\Lambda\rangle$ also satisfies the condition (β). If r=1, then the conclusion follows from Cor. 3. Thus we can assume that $r > 1$ and $m > 0$. Let ξ be an (n,Λ)-reducible semigroup-term. Then there is a semigroup term η such that $(\xi,\eta) \in \langle\Lambda\rangle$, and $|\xi| \equiv |\eta|$ (mod n-1). Therefore, $|\xi|_i \neq |\eta|_i$ for some $i \in \{1,2,...\}$ and this implies that there is an $x_j \in X$ such that $|\xi|_j \geq r$. Thus, we have $(x_j^{km}\xi,\xi)$, $(\xi,\xi x_j^{km}) \in \langle\Lambda\rangle$, for any $k > 0$, and we can apply Theorem.

Cor. 5'. $\mathbb{A}_{r,m}$=Var$\{x_1x_2=x_2x_1,\ x_1^r=x_1^{r+m}\}$ is an n-variety for every $n \geq 2$, $r \geq 1$, $m \geq 0$. (This is in fact Theorem 2 of [1].)

Cor. 6. Denote by $\Delta_{(k)}$ the following set of semigroup identities:

$$\Delta_{(k)} = \{(x_1...x_k, x_1...x_ix_jx_{i+1}...x_k) \mid 2 \leq i \leq k-1,\ j \in \{1,k\}\},$$

where $k \geq 3$. Then $\mathbb{D}_k$=Var $\Delta_{(k)}$ is an n-variety for every $n \geq 2$.

Proof. First, it can be easily shown that if $n \geq 3$, and a semigroup term ξ is $(n,\Delta_{(k)})$-reducible, then $|\xi| \geq k$. In this case, if $\xi=x\eta y$, then $(x^i\xi,\xi)$, $(\xi,\xi y^i) \in \langle\Delta_{(k)}\rangle$ for any $i > 0$, and thus the condition (α) is satisfied.

(We notice that it is shown in the paper [8] that $\mathbb{D}=\mathbb{D}_3$ is an n-variety for any $n \geq 2$, and that the same proof can be applied for the general case.)

Cor. 7. $\mathbb{D}_k \cap$ ABSEM is an n-variety for every $k \geq 3$, $n \geq 2$.

Proof. It is easy to show that (α) is satisfied.

(Cor. 7 is also proved in [2]).

The following proposition is the main result of the paper [7].

Prop. 8. If $\mathbb{L}_k=\mathrm{Var}(x_1\ldots x_k, x_1\ldots x_k x_{k+1})$, $\mathbb{R}_k=\mathrm{Var}(x_1\ldots x_k, x_{k+1}x_1\ldots x_k)$, $\mathbb{O}_k=\mathbb{L}_k\cap\mathbb{R}_k$, then $\mathbb{L}_k$, $\mathbb{R}_k$, $\mathbb{O}_k$ are n-varieties for any $k \geq 1$, $n \geq 2$.

We note that $\mathbb{O}_k$ satisfies the condition (α), but neither of the varieties $\mathbb{L}_k$, $\mathbb{R}_k$ satisfies (α).

The above examples exhaust all the known semigroup n-varieties. A list of the known semigroup quasi n-varieties will be given below. (see [1], [7], [2]).

Prop. 9. If $r > 1$, and $n-1$ is not a divisor of m, then $\mathbb{P}_{r,m}=\mathrm{Var}(x_1^r, x_1^{k+m})$ is a quasi n-variety.

Prop. 10. If $n \geq 3$ and $\mathbb{D}^{\ell}=\mathrm{Var}(x_1x_2x_3, x_1x_2x_1x_3)$, $\mathbb{D}^r=\mathrm{Var}(x_1x_2x_3, x_1x_3x_2x_3)$, then both $\mathbb{D}^{\ell}$ and $\mathbb{D}^r$ are quasi n-varieties.

Prop. 11. Let s,m,n and k be positive integers such that:

$n\geq 3$, $m \equiv 0 \pmod{n-1}$, $m\neq 2s+1$, $m\neq 2s+2$, $s+2 \leq m$, $k \geq m+2$,

and let $\Delta_{(k)}$ be as in Cor. 6, and

$$\Delta_{(k,s,m)} = \Delta_{(k)} \cup \{(x_1^s x_2^{m-s}, x_1^{s+2} x_2^{m-s-1})\}.$$

Then both the varieties

$\mathrm{Var}\ \Delta_{(k,s,m)}$ and $\mathrm{ABSEM} \cap \mathrm{Var}\ \Delta_{(k,s,m)}$

are quasi n-varieties.

R E F E R E N C E S

[1] Ǵ.Čupona: n-subsemigroups of semigroups satisfying the law $x^r=x^{r+m}$, Annuare de la Faculté des Mathematiques, Skopje, T.30 (1979), 5-14

[2] Ǵ.Čupona: n-subsemigroups of some commutative semigroups, Annuare T.32 (1981)

[3] S.Markovski: On quasivarieties of generalized subalgebras, Algebraic Conference Skopje 1980, 125-129

[4] E.Nelson: The lattice of equational classes of commutative semigroups, Can.J. Math Vol. XXIII, No 5 (1971), 875-895

[5] А.Я.Айзенштат, Б.К.Богута: О решетке многообразий полугрупп, Ленингр. Госуд. педаг. институт, Сборник научных трудов, Ленинград (1979)

[6] А.И.Мальцев: Алгебрические системы, Москва 1970

[7] С.Марковски: За една класа полугрупи, Математ.Билтен 1(XXVII) (1978), 29-36

[8] С.Марковски: За дистрибутивните полугрупи, Годишен зборник на Матем.фак., Скопје 30(1979), 15-27

Ǵ.Čupona, Matematički fakultet,
P.F. 504,
91000 Skopje
Jugoslavija