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ДОКЛАДЫ

EMBEDDINGS OF ASSOCIATIVES IN SEMIGROUPS

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Abstract. Here we characterize associativies and ℓ-unars as subalgebras of semigroups and we also show that the class of weak associatives as well as the classes of associatives and ℓ-unars satisfy the corresponding condition of compatibility.

1. Preliminaries and main results

A universal algebra $\underline{A} = (A;\Omega)$ (i.e. „an Ω-algebra") is called an Ω-subsemigroup of a semigroup S iff $A \subseteq S$ and

$$\omega(a_1,a_2,\dots,a_n) = a_1a_2\dots a_n \tag{1.1}$$

for every $\omega\in\Omega_n$, $n\geq 0$ and $a_1,\dots,a_n\in A$ (where Ω_n is the set of n-ary operators belonging to Ω). By a more general result ([7, p. 274] or [5]) the class of Ω-subsemigroups of semigroups is a quasivariety. The following set of identities (1.2), (1.3) and (1.4) is an axiom system for the variety generated by this quasivariety.

$$\omega\tau(x_1,\dots,x_{m+n-1}) = \tau\omega(x_1,\dots,x_{m+n-1}), \tag{1.2}$$

$$\omega'(x_1,\dots,x_{k-1},\tau'(x_k,\dots,x_{k+p-1}),\dots,x_{p+q-1}) = \omega'\tau'(x_1,\dots,x_{p+q-1}) \tag{1.3}$$

$$\omega_1\omega_2\dots\omega_r(x_1,\dots,x_t) = \tau_1\tau_2\dots\tau_s(x_1,\dots,x_t), \tag{1.4}$$

for any $\omega\in\Omega_n$, $\tau\in\Omega_m$, $\omega'\in\Omega_q$, $\tau'\in\Omega_p$, $m,n,q\geq 1$, $p\geq 0$, $1\leq k\leq q$, and for any $\omega_\nu\in\Omega_{n_\nu}$, $\tau_\lambda\in\Omega_{m_\lambda}$ such that

$$n_1+\dots+n_r - r+1 = t = m_1+\dots+m_s-s+1 \geq 0 \tag{1.5}$$
$$n_1+\dots+n_i - i+1 \geq 0,\quad m_1+\dots+m_j-j+1\geq 0$$

for any $i\leq r$, $j\leq s$.

Algebras belonging to this variety are called Ω-associatives; if an Ω-algebra satisfies all the identities (1.2) and (1.3), then it is called an Ω-weak associative.

Associatives are investigated in several papers of authors, and a survey of results can be found in [2]. We note that, in general[1)], the class of Ω-subsemigroups of semigroups is a proper subquasivariety of the variety of Ω-associatives, and if $|\Omega| \geq 2$, then the variety of Ω-associatives is a proper subvariety of the variety of Ω-weak associatives ([2]).

A unary Ω-algebra $\underline{A} = (A;\Omega)$ (i.e. an Ω-algebra with $\Omega = \Omega_1$) is called an ℓ-subunar of a unar $\underline{B} = (B;f)$ if $A \subseteq B$ and

$$\omega(a) = f^{\ell(\omega)}(a), \tag{1.6}$$

for any $\omega \in \Omega$ and $a \in A$, where $\ell: \omega \mapsto \ell(\omega)$ is a mapping from Ω into the set of positive integers.

Unary Ω-algebras belonging to the variety of unary Ω-algebras generated by the quasivariety of ℓ-subunars of unars are called ℓ-unars. This variety is defined by all the identities of the following form:

$$\omega_1 \ldots \omega_r(x) = \tau_1 \ldots \tau_s(x), \tag{1.7}$$

where

$$\ell(\omega_1)+\ldots+\ell(\omega_r) = \ell(\tau_1)+\ldots+\ell(\tau_s). \tag{1.8}$$

The quasivariety of ℓ-subunars of unars is, in general[2)], a proper subquasivariety of the variety of ℓ-unars ([3]).

By the well known Cohn-Rebane's theorem ([1], [6]), if $\underline{A} = (A;\Omega)$ is an Ω-algebra, then there is a semigroup $\underline{S} = (S,\cdot)$ and a mapping $\omega \mapsto \bar{\omega}$ of Ω in S such that $A \subseteq S$ and

$$\omega(a_1,\ldots,a_n) = \bar{\omega}a_1\ldots a_n \tag{1.9}$$

for any $\omega \in \Omega_n$, $n \geq 0$ and $a_1,\ldots,a_n \in A$. Then we say that $\underline{S}$ is Cohn-Rebane's semigroup (or CR-semigroup) for the algebra $\underline{A}$.

If we allow a_ν in (1.9) to be arbitrary elements of S, then we obtain an Ω-algebra $\underline{S}(\Omega) = (S;\Omega)$, containing the given Ω-algebra $\underline{A}$ as a subalgebra.

[1)] *The class of Ω-subsemigroups of semigroups is a variety iff $d \in J$, where d is the greatest common divisor of the elements of the set $J = \{n-1 \mid \Omega_n \neq \emptyset\}$.*

[2)] *The class of ℓ-subunars of unars is a variety (i.e. the class of ℓ-unars coincides with the class of ℓ-subunars of unars) iff there exists a $\tau \in \Omega$ such that $\ell(\tau)$ is a divisor of $\ell(\omega)$ for all $\omega \in \Omega$.*

A class $\mathcal{D}$ of Ω-algebras is said to be compatible with SEM (the class of semigroups) if for any $\underline{A}\in\mathcal{D}$ there is a CR-semigroup $\underline{S}$ such that $\underline{S}(\Omega)\in\mathcal{D}$. The problem of compatibility is investigated in [9], and there can be found classes which satisfy the condition of compatibility and classes which do not satisfy that condition.

Now we can state the main results of the paper.

THEOREM 1. *If $\underline{A}$ is an Ω-weak associative, then there is a CR-semigroup $\underline{S}$ for $\underline{A}$ such that $\overline{\omega}$ is in the center of $\underline{S}$, for any $\omega\in\Omega$.*

If, moreover, $\underline{A}$ is an Ω-associative, then

$$\overline{\omega}_1\ldots\overline{\omega}_r = \overline{\tau}_1\ldots\overline{\tau}_s \tag{1.10}$$

for any $\omega_\nu\in\Omega_{n_\nu}$, $\tau_\lambda\in\Omega_{m_\lambda}$ such that (1.5) hold.

THEOREM 2. *If $\underline{A} = (A;\Omega)$ is an ℓ-unar, then there is a CR-semigroup $\underline{S}$ for $\underline{A}$ such that $\overline{\omega}$ is in the center of $\underline{S}$ for any $\omega\in\Omega$, and (1.10) hold for any ω_ν, $\tau_\lambda\in\Omega$ such that (1.8) is satisfied.*

THEOREM 3. *Any of the varieties: a) Ω-weak associatives, b) Ω-associatives, c) ℓ-unars, is compatible with SEM.*

2. Commutative semigroups of operations

Let A be a nonempty set, let $\mathcal{O}_n(A) = A^{A^n}$ be the set of all n-ary operations on A $(n\geq 1)$ and let $\mathcal{O}(A) = \cup\{\mathcal{O}_n(A)\mid n\geq 1\}$ be the set of all nonnullary finitary operations on A. $\mathcal{O}(A)$ is a semigroup with respect to the usual superposition "$\circ$" of mappings, i.e.

$$\omega\circ\tau(x_1,\ldots,x_{m+n-1}) = \omega(\tau(x_1,\ldots,x_m),x_{m+1},\ldots,x_{m+n-1}), \tag{2.1}$$

where $\omega\in\mathcal{O}_n(A)$ and $\tau\in\mathcal{O}_m(A)$, with an identity element 1_A, $1_A: x\mapsto x$ (see also [4]).

A subsemigroup Γ of $\mathcal{O}(A)$ is called a *commutative semigroup of operations* on A iff Γ is commutative as a semigroup and if it satisfies the following identity equality

$$\omega\tau(x_1,\dots,x_{m+n-1}) = \omega(x_1,\dots,x_{i-1},\tau(x_1,\dots,x_{i+m-1}),\dots,x_{m+n-1}) \quad (2.2)$$

for all $\omega \in \Gamma_n = \mathcal{O}_n(A) \cap \Gamma$, $\tau \in \Gamma_m$ and $i=1,2,\dots,n$. An element $e \in A$ is said to be _in the center of_ Γ iff the following identity is satisfied

$$\omega(x_1,\dots,x_{i-1},e,x_i,\dots,x_{n-1}) = \omega(e,x_1,\dots,x_{n-1}), \quad (2.3)$$

for any $\omega \in \Gamma_n$ and $i \in \{1,\dots,n\}$.

The first two theorems are easy consequences of the following

LEMMA. _Let_ Γ _be a commutative semigroup of operations on_ A, _and let_ $E \subseteq A$ _be in the center of_ Γ. _Then there exists a semigroup_ S _with the following properties:_

(i) $\Gamma \cup A \subseteq S$;

(ii) Γ _is a subsemigroup of the center of_ S;

(iii) E _is in the center of_ S;

(iv) $f(a_1,\dots,a_n) = fa_1\dots a_n$ _for any_ $f \in \Gamma_n$ _and_ $a_\nu \in A$.

PROOF. First, it is clear that nothing will be changed if one suppose that $1_A \in \Gamma_1 \subseteq \Gamma$.

Let $B = A\setminus E$ and let E^c be the commutative monoid freely generated by E, and B^* the monoid freely generated by B. Consider the monoid

$$T = \Gamma \times E^c \times B^*.$$

Any element $u = (f,e,b)$ of T ($f \in \Gamma$, $e \in E^c$, $b \in B^*$) can be represented in the form $fa_1\dots a_n$, i.e. $f\underline{a}$, where $f \in \Gamma$, $\underline{a} \in A^*$ (A^* being the free monoid generated by A); here, $f\underline{a} = f'\underline{a}'$ iff $f = f'$, and $\underline{a}'$ can be obtained from $\underline{a}$ by displacing the elements of E.

Note that $\Gamma \times E^c$ is in the center of T. (Namely, $\Gamma \times E^c$ is the center of T whenever $|B| \geq 2$ or $B = \emptyset$.) Also we can assume that $B^* \subseteq T$.

Some elements of T will be called Γ-words. Namely, $u = fa_1a_2\dots a_n$ is a _Γ-word_ if $f \in \Gamma_n$ and $a_\nu \in A$. Then $[u] = f(a_1,\dots,a_n)$ is called the _value_ of u. (If $a \in A$, then $a = 1_A(a)$, and thus a is a Γ-word and $[a] = a$.)

Clearly, if $f\underline{a} = f\underline{a}'$, then $f\underline{a}$ is a Γ-word iff $f\underline{a}'$ is a Γ-word, and then $[f\underline{a}] = [f\underline{a}']$.

Define a relation $\sim$ in T by:

$$u = g\underline{a}'a\underline{a}'', \quad a = f(a_1,\dots,a_n) \Longrightarrow u \sim gf\underline{a}'a_1\dots a_n\underline{a}''.$$

Let $\approx$ be the equivalence in T generated by $\sim$, i.e. $u \approx v$ iff there exist $u_0,u_1,\dots,u_p \in T$ such that

$$u_0 = u, \quad u_p = v \quad \text{and} \quad u_{i-1} \sim u_i \quad \text{or} \quad u_i \sim u_{i-1}$$

for each $i \in \{1,\dots,p\}$.

It is clear that $\approx$ is a congruence on T satisfying the following propositions:

$\underline{1}^{o}$. $a = f(a_1,\dots,a_n) \Longrightarrow a \sim fa_1\dots a_n$.

$\underline{2}^{o}$. $f \in \Gamma, u \in T \Longrightarrow (f \approx u \Longleftrightarrow u = f)$.

$\underline{3}^{o}$. If $u \sim v$ and one of u,v is a Γ-word, then the other is a Γ-word too and $[u] = [v]$.

As a corollary of $\underline{3}^{o}$ we obtain:

$\underline{4}^{o}$. If $a \in A$, $u \in T$, then

$$a \approx u \quad \text{iff} \quad u \text{ is a } \Gamma\text{-word and } [u] = a.$$

Therefore:

$\underline{5}^{o}$. $a,b \in A \Rightarrow (a \approx b \Longrightarrow a = b)$.

The conclusion of Lemma is obtained by $1^{o},2^{o},5^{o}$ and the fact that $\Gamma \cup E$ is in the center of T, since the semigroup $S = T/\approx$ has the desired property.

3. Proofs of the theorems

Let $\underline{A} = (A;\Omega)$ be an Ω-weak associative. Clearly, we can assume that distinct elements of Ω define distinct operations on A, and so we may assume that the elements of Ω are operations on A.

Note that if f is an n-ary operation on A for $n \geq 1$, and $e \in A$ is a fixed element of A, then fe is an $(n-1)$-ary operation on A defined by

$$fe(x_1,\dots,x_{n-1}) = f(e,x_1,\dots,x_{n-1}).$$

If we assume the superposition of operations in this more general sence, then

$$\mathcal{O}^{o}(A) = \bigcup\{\mathcal{O}_n(A) \mid n \geq 0\} \qquad (\text{where } \mathcal{O}_o(A) = A)$$

is a partial semigroup in the following sence: if f(gh) exists, then (fg)h exists too and f(gh) = (fg)h. (Note that fg does not exists only if $f \in \mathcal{O}_o(A)$.)

Denote by Γ^o the partial subsemigroup of $\mathcal{O}^o(A)$ generated by Ω. It is easy to show that $\Gamma = \mathcal{O}(A) \cap \Gamma^o$ is a commutative semigroup of operation on A as well as $E = \mathcal{O}_o(A) \cap \Gamma^o$ is in the center of Γ.

This and Lemma of the previous section give the conclusion of the first part of Theorem 1.

By this discussion and the Lemma again, we proved in fact the second part as well.

Namely, if $\underline{A} = (A;\Omega)$ is an Ω-associative and if Γ and E are obtained as above, then the equality

$$\omega_1\omega_2\dots\omega_r = \tau_1\tau_2\dots\tau_s$$

holds in the semigroup Γ whenever (1.5) holds, where $\omega_\lambda, \tau_\nu \in \Omega$.

Suppose now that $\underline{A} = (A;\Omega)$ is an ℓ-unar. Let Ω^+ be the semigroup freely generated by Ω. If we put

$$\omega(x) = \omega_1(\omega_2\dots\omega_r(x)) \quad \text{and} \quad \ell^+(\omega) = \ell(\omega_1)+\dots+\ell(\omega_r)$$

for any $\omega = \omega_1\dots\omega_r \in \Omega^+$, then we obtain an ℓ^+-unar $\underline{A}' = (A;\Omega^+)$.

Let Γ be the semigroup of operations (i.e. transformations) on A generated by the operations which are induced by Ω^+. The fact that $\underline{A}'$ is an ℓ^+-unar implies that Γ is a commutative semigroup of transformations and we can apply again the Lemma to obtain the validity of Theorem 2.

Clearly, Theorem 3 is a direct consequence of the previous two theorems.

4. Remarks

4.1. As a corollary of the proved Lemma one obtains a description of the class of subalgebras of commutative semigroups, and namely this is the class of Ω-weak associatives $(A;\Omega)$ such that every element $a \in A$ is the „value" of some nullary operator $\omega \in \Omega_0$. (In fact, this is the main result of the paper [8]; see also [5].)

4.2. In the definition of an Ω-associative given in [2] and [10] it is assumed that $\Omega_0 \cup \Omega_1 = \emptyset$.

We point out that the assumption $\Omega(1) = \emptyset$ is not essential, because it is easy to see that if $\Omega(1) \neq \emptyset$, then all the elements of $\Omega(1)$ induce the identity unary operation 1_A. Also, if $\Omega(0) \neq \emptyset$, then all the elements of $\Omega(0)$ determine exactly one element $e \in A$ and, if $\Omega \neq \Omega(0) \cup \Omega(1)$ and if $\tau \in \Omega(k)$, $k \geq 2$, then putting

$$x \cdot y = \tau(x, y, e^{k-2})$$

we obtain a semigroup $(A, \cdot)$ such that

$$\omega(x_1, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$$

for any $\omega \in \Omega(n)$, $n \geq 2$, $x_\nu \in A$.

4.3. The results of the first two theorems suggest the following problem.

Let $\underline{A}$ be an Ω-algebra and $\Omega' \subseteq \Omega$. Assume that $\underline{A}$ satisfies any identity $\xi = \eta$, where ξ, η are Ω-terms such that the sequences of symbols from $(\Omega \setminus \Omega') \cup X$ (X is the set of variables) that occur in both ξ and η are equal, and the sequence of symbols from Ω' that occur in ξ and η are permutations of each other.

Is it true that there exists a CR-semigroup for $\underline{A}$ such that $\bar{\omega}'$ is in the center of S for any $\omega' \in \Omega'$?

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