

THE PROBLEM OF SOLVABILITY OF POLYLINEAR
REPRESENTATIONS OF UNIVERSAL ALGEBRAS IN SEMIROUPS

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The problem of effectivness of different kinds of embeddings of universal algebras in semigroups is treated in this paper.

1. Consider an Ω -algebra $A = (A; \Omega)$, i.e. $\Omega = \cup \{\Omega(n) \mid n \geq 1\}$ is a set of finitary operators such that $n \neq m \Rightarrow \Omega(n) \cap \Omega(m) = \emptyset$, and every n -ary operator $\omega \in \Omega(n)$ induces an n -ary operation ω_A on A . (We will use the same notation for an operator and the corresponding operation in the algebra, i.e. we will write $\omega(a_1, \dots, a_n)$ instead of $\omega_A(a_1, \dots, a_n)$.) We associate three semigroups to the algebra A as follows:

$$A_1^\Delta = \langle A \cup \Omega \mid \{a = \omega a_1 \dots a_n \mid a = \omega(a_1, \dots, a_n) \text{ in } A\} \rangle \quad (1.1)$$

$$A_2^\Delta = \langle A \cup \Omega^\wedge \mid \{a = \omega_0 a_1 \omega_1 \dots a_n \omega_n \mid a = \omega(a_1, \dots, \dots, a_n) \text{ in } A\} \rangle \quad (1.2)$$

$$A_3^\Delta = \langle A \cup \Omega \mid \{a = a_1 \omega a_2 \dots a_n \mid a = \omega(a_1, \dots, a_n) \text{ in } A\} \rangle \quad (1.3)$$

It is assumed in (1.2) that for any $\omega \in \Omega(n)$, $\omega^\wedge = \{\omega_0, \dots, \omega_n\}$ is a set with $n+1$ elements such that $\omega^\wedge \cap \tau^\wedge \neq \emptyset \Rightarrow \omega = \tau$, and $\Omega^\wedge = \cup \{\omega^\wedge \mid \omega \in \Omega\}$. It is also assumed that $\Omega(1) = \emptyset$ in (1.3).

We notice that, in all the above presentations, the right-hand sides of the defining relations have greater lengths than the ones on the left-hand sides. So, we can define reduced words to be those words which have no sub-words which are the right-hand sides of the defining relations. It is clear that for any word u there is a reduced word \bar{u} , such that \bar{u} is obtained from u by a finite application of the defining relations.

It is easy to prove the following

Theorem 1.1. The irreducible representative \bar{u} for any word u is uniquely defined in (1.1) and (1.2) for every algebra A . Every word has a unique irreducible representative in (1.3) iff the algebra A satisfies the identities

$$\omega\tau(x_1, x_2, \dots, x_{m+n-1}) = \tau(x_1, \dots, x_{m-1}, \omega(x_m, \dots, x_{m+n-1})) \quad (1.4)$$

where $\omega \in \Omega(n)$, $\tau \in \Omega(m)$. \blacksquare

An Ω -algebra A is said to be recursive iff A and Ω are recursive sets, and every operation $\omega_A: A^n \rightarrow A$ induced by $\omega \in \Omega(n)$ is recursive. As a consequence of Theorem 1.1 we have:

Corollary 1.2. If the algebra A is recursive, then the semigroups A_1^Δ and A_2^Δ are also recursive; furthermore, if the algebra A satisfies the identities (1.4), then A_3^Δ is recursive as well. \blacksquare

Since the elements of the set A are reduced in all of the presentations (1.1), (1.2) and (1.3), we have:

Corollary 1.3. For any Ω -algebra A there exists a semigroup S such that $A \cup \Omega \subseteq S$ and the equality

$$\omega(a_1, a_2, \dots, a_n) = \omega a_1 a_2 \dots a_n$$

holds for every $\omega \in \Omega(n)$, $a_1, \dots, a_n \in A$. \blacksquare

Corollary 1.4. For any Ω -algebra A there exists a semigroup S and a mapping $\omega \rightarrow \omega^\wedge = (\omega_0, \dots, \omega_n)$ of Ω into $\bigcup_{n=1}^{\infty} S^n$ such that $\omega \in \Omega(n) \Rightarrow \omega^\wedge \in S^{n+1}$, $A \subseteq S$ and the equality

$$\omega(a_1, \dots, a_n) = \omega_0 a_1 \omega_1 \dots a_n \omega_n$$

holds for every $\omega \in \Omega(n)$, $a_1, \dots, a_n \in A$. \blacksquare

Corollary 1.5. If the algebra A satisfies the identities (1.4), then there exists a semigroup S such that $A \cup \Omega \subseteq S$ and the equality

$$\omega(a_1, a_2, \dots, a_n) = a_1 \omega a_2 \dots a_n$$

holds for every $\omega \in \Omega(n)$, $a_1, a_2, \dots, a_n \in A$. \blacksquare

Namely, we can take S to be the semigroup A_1^Δ , A_2^Δ and A_3^Δ in the corresponding cases.

Remark that Corollary 1.3 is the well known Cohn-Rebane's theorem ([2], [7]) and Corollary 1.5 is proved in [3].

2. We will consider here more general representations of Ω -algebras into semigroups, and (1.1), (1.2) and (1.3) will be special cases of them.

Let Ω be a set of finitary operations, C be a given set and $e \notin \Omega \cup C$. Assume that for any $\omega \in \Omega(n)$ we have a sequence $\omega^\Delta = (\omega_0,$

$\omega_1, \dots, \omega_n$), where $\omega_i \in C \cup \{e\}$. If \mathbf{A} is a given Ω -algebra with a carrier A , then we consider the semigroup \mathbf{A}^Δ given by the following presentation:

$$\begin{aligned} \mathbf{A}^\Delta = \langle A \cup C; \{a = \omega_0 a_1 \omega_1 \dots \omega_{n-1} a_n \omega_n \mid a = \\ = \omega(a_1, \dots, a_n) \text{ in } \mathbf{A}\} \rangle \end{aligned} \quad (2.1)$$

We suppose that e is the empty word in (2.1), i.e. if $\omega_i = e$ for some i , then we do not write ω_i on the corresponding right-hand side of the defining relation. (We say that Δ is the kind of the multilinearity)

It is clear that (1.1), (1.2) and (1.3) are special cases of (2.1). Namely, if $C = \Omega$ and $\omega^\Delta = (\omega, e, \dots, e)$ ($\omega^\Delta = (e, \omega, e, \dots, e)$), we obtain (1.1) ((1.3)). If $\omega_i \neq e$ for every $\omega \in \Omega(n)$, $i \in \{0, 1, \dots, n\}$ and $\omega_i = \tau_j \Leftrightarrow \omega = \tau, i = j$, then we obtain (1.2).

The reduced words could be defined as above, and so we have

Theorem 2.1. Let the algebra \mathbf{A} be defined such that for any word u in the presentation (2.1) there exists a unique reduced representative \bar{u} .

Then, if the algebra \mathbf{A} is recursive, the semigroup \mathbf{A}^Δ is recursive as well. ■

We are looking now for conditions under which we can have a unique reduced representatives for a given word.

Define the set of Ω -words, which is a subset of $(A \cup C)^+$, in this inductive way:

- (i) every element of A is an Ω -word;
- (ii) if u_1, u_2, \dots, u_n are Ω -words and $\omega \in \Omega(n)$, then $\omega_0 u_1 \omega_1 u_2 \dots \omega_{n-1} u_n \omega_n$ is an Ω -word;
- (iii) a word $u \in (A \cup C)^+$ is an Ω -word iff it is obtained by a finite application of (i) and (ii).

Let \mathbf{A} be an Ω -algebra. For every Ω -word u let us define its value $[u] \in A$ as follows:

$$a \in A \Rightarrow [a] = a;$$

if $\omega \in \Omega(n)$ and u_1, u_2, \dots, u_n are Ω -words with values

$[u_i] = b_i$, $i = 1, 2, \dots, n$, then $b = \omega(b_1, b_2, \dots, b_n)$ is one value of the Ω -word $u = \omega_0 u_1 \omega_1 \dots \omega_{n-1} u_n \omega_n$. Thus the value of an Ω -word need not be uniquely defined.

It is clear what we mean by „an Ω -word u is a maximal Ω -subword of a given word v “. (Note that u can have both maximal and non maximal appearances in v .)

We can formulate now the wanted condition:

¹⁾ B^+ is the free semigroup on B .

Theorem 2.2. Let the algebra \mathbf{A} and Δ satisfy the conditions:

1) Every word $v \in (A \cup C)^+$ can be represented uniquely in the form

$$v = \alpha_0 u_1 \alpha_1 u_2 \dots \alpha_{p-1} u_p \alpha_p \quad (2.2)$$

where $\alpha_i \in C^{*1}$ and u_1, u_2, \dots, u_p are maximal Ω -subwords of v . (We say that (2.2) is a canonical representation of v .)

2) Every Ω -word u has a uniquely defined value $[u]$.

Define a relation \approx on $(A \cup C)^+$ as follows: $v \approx w$ iff v has a canonical representation of the form (2.2), w has a canonical representation

$$w = \alpha_0 u'_1 \alpha_1 u'_2 \dots \alpha_{p-1} u'_p \alpha_p$$

and $[u'_i] = [u_i]$ for $i = 1, 2, \dots, p$.

Then \approx is an equivalence on $(A \cup C)^+$.

If it is satisfied the condition

3) \approx is a congruence on the semigroup $(A \cup C)^+$, then \mathbf{A}^Δ is isomorphic to $(A \cup C)^+ / \approx$ and every word v with a canonical representation (2.2) has uniquely defined reduced representation

$$\bar{v} = \alpha_0 a_1 \alpha_1 a_2 \dots \alpha_{p-1} a_p \alpha_p \quad (2.2)$$

where $[u_i] = a_i$, $i = 1, 2, \dots, p$. ■

The condition 3) is independent from 1) and 2). Namely, let $\Omega = \{\tau, \omega\}$, where $\omega \in \Omega(3)$, $\tau \in \Omega(4)$, and let $\omega^\Delta = (e, e, e, e)$, $\tau^\Delta = (e, e, e, e, e)$, $C = \emptyset$. Then if $u \in A$ or $u = a_1 a_2 \dots a_p$, $p \geq 3$, u is an Ω -word, and if $u = ab$, $a, b \in A$, then a and b are maximal subwords of u . Thus, 1) is satisfied for any algebra $\mathbf{A} = (A; \omega, \tau)$. The condition 2) is satisfied iff \mathbf{A} satisfies the general associative law, i.e. iff \mathbf{A} is an associative ([1], [4]). But, there are associatives which do not satisfy 3).

For example, let $A = \{a, b, c\}$ and $\omega(x_1, x_2, x_3) = a$ for every $x_1, x_2, x_3 \in A$, $\tau(x_1, x_2, x_3, x_4) = a$ for every $x_1, x_2, x_3, x_4 \in A$ such that $x_i \neq c$ for at least one i , and $\tau(c, c, c, c) = b$. Then $(A; \omega, \tau)$ is an associative ([4]), but the relation \approx is not a congruence, since $c^3 \approx a^3$, $c^4 \approx b$, $a^3 c \approx a$, but a and b are not equivalent.

Notice that the condition 1) depends on Δ , but not on the considered algebra, and that 2) is satisfied iff the algebra \mathbf{A} satisfies some corresponding system of identities. Thus 2) is satisfied in (1.3) iff the algebra \mathbf{A} satisfies the identities (1.4).

3. Assume that the Ω -algebra \mathbf{A} has a presentation

$$\mathbf{A} = \langle B; \Lambda \rangle \quad (3.1)$$

1) B^* is the free monoid on B .

in the class of all Ω -algebras, or in some variety of Ω -algebras. We want to give a presentation of the semigroup A^Δ , for a given Δ .

Define the set Ω_A of Ω -terms without variables to be the intersection of all subsets H of $(A \cup C)^+$ with the properties

$$(i) A \subseteq H$$

$$(ii) \omega \in \Omega(n), \xi_1, \dots, \xi_n \in H \Rightarrow \omega \xi_1 \dots \xi_n \in H.$$

For every Ω -term ξ without variables we have an Ω -word ξ^Δ , the „translation“ of ξ , obtained in this inductive way:

$$a^\Delta = a, \text{ for every } a \in A,$$

$$(\omega \xi_1 \dots \xi_n)^\Delta = \omega_0 \xi_1^\Delta \omega_{11} \dots \xi_n^\Delta \omega_n \text{ for every}$$

$$|\omega \in \Omega(n), \xi_1, \xi_2, \dots, \xi_n \in \Omega_A.$$

The translation Λ^Δ of Λ is defined by

$$\Lambda^\Delta = \{\xi^\Delta = \eta^\Delta \mid \xi = \eta \in \Lambda\}$$

Now, we have the following results:

Theorem 3.1 If the algebra A has the presentation (3.1), then the semigroup A^Δ has the presentation

$$A^\Delta = \langle B \cup C \mid \Lambda^\Delta \rangle \blacksquare \quad (3.2)$$

Theorem 3.2. Suppose that the conditions of Theorem 2.2 are satisfied. Then the presentation (3.1) is solvable iff the presentation (3.2) is solvable. \blacksquare

We notice that an n -group is recursive iff its universal covering is recursive ([6], [5]), but it is not known whether the same result is true when n -semigroups are considered.

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ПРОБЛЕМОТ НА РЕШЛИВОСТ НА ПОЛИЛИНЕАРНИТЕ ПРЕТСТАВУВАЊА
НА УНИВЕРЗАЛНИ АЛГЕБРИ ВО ПОЛУГРУПИ

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Една Ω -алгебра (A, Ω) е полилинеарна подалгебра од полугрупа (S, \cdot) ако $A \subseteq S$ и за секои $\omega \in \Omega$, $a_1, a_2, \dots, a_n \in A$,

$$(*) \quad \omega(a_1, a_2, \dots, a_n) = \omega_0 a_1 \omega_1 a_2 \omega_2 \dots a_n \omega_n,$$

каде на секое $\omega \in \Omega(n)$ му одговара низа $(\omega_0, \dots, \omega_n)$ од елементи од S , при што се дозволува некои ω_i да не се јавуваат во $(*)$. Во работава се разгледуваат повеќе видови полилинеарни сместувања на универзални алгебри во полугрупи, при што главен акцент е ставен на ефикасноста на сместувањето. Основен резултат на работава е Теорема 2.2.