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SOME CONNECTIONS BETWEEN FINITE SEPARABILITY PROPERTIES OF AN n-SEMIGROUP AND ITS UNIVERSAL COVERING

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Abstract. It is known that for any n-semigroup there exists a universal covering semigroup, and there is a connection between some properties of an n-semigroup and its universal covering. In this paper such a connection for finite separability properties is studied. It is proved that:

1. If a covering semigroup $\underline{Q}'$ of an n-semigroup $\underline{Q}$ is residually finite, then $\underline{Q}$ is residually finite as well.
2. If a cancellative n-semigroup $\underline{Q}$ is residually finite, then the cancellative universal covering semigroup $\underline{Q}^{\sim}$ is residually finite as well.
3. If the universal covering group $\underline{Q}^{\wedge}$ of an n-group $\underline{Q}$ has the finite separability property , so does $\underline{Q}$.

As a consequence of these results, the results given in [3] , some known results for n-semigroups, and the fact that finite separability properties imply solvability of algorithmic problems, some n-semigroup classes with solvable algorithmic problems are obtained.

1. Preliminary definitions

An n-semigroup is an algebra $(Q,[\,])$ with an associative n-ary operation $[\,]:(x_1,x_2,\dots,x_n)\mapsto [x_1x_2\dots x_n]$. Then the semigroup $\underline{Q}^{\wedge}$ given by the following presentation (in the class of all semigroups)

$$\langle Q;\ \{a=a_1a_2\dots a_n \mid a=[a_1a_2\dots a_n] \text{ in } \underline{Q}\}\rangle \qquad (1)$$

is called the universal covering semigroup of $\underline{Q}$. It can be assumed that $Q\subseteq Q^{\wedge}$, moreover, Q is a generating subset of $\underline{Q}^{\wedge}$ and any element $u\in Q^{\wedge}$ has a form $u=a_1a_2\dots a_i$, where

$1 \leq i < n$, $a_\nu \in Q$, and $i = |u|$ is uniquely determined by u. If $\underline{P}$ is an n-subsemigroup of $\underline{Q}$ then there is a (unique) homomorphism $\lambda : \underline{P}^{\wedge} \to \underline{Q}^{\wedge}$ such that $\lambda(p) = p$, for any $p \in P$. $\underline{P}$ is said to be compatible in $\underline{Q}$ if λ is injective, and then we can assume $\underline{P}^{\wedge}$ to be a subsemigroup of $\underline{Q}^{\wedge}$ ([1]).

A cancellative n-semigroup is an n-semigroup which satisfy the cancellative laws. Then the semigroup $\underline{Q}^{\sim}$ given by the presentation (1) (in the class of cancellative semigroups) is called the universal cancellative covering semigroup of $\underline{Q}$. We note that $a_1 a_2 \ldots a_i = b_1 b_2 \ldots b_i$ in $\underline{Q}$ iff $[a^{n-i} a_1 \ldots a_i] = [a^{n-i} b_1 \ldots b_i]$ in $\underline{Q}$, for each $a \in Q$.

An n-semigroup $(Q, [\ ])$ is called an n-group if $(\forall a_1, \ldots, a_n \in Q)(\exists x, y \in Q)[x a_1 \ldots a_{n-1}] = a_n$, $[a_1 \ldots a_{n-1} y] = a_n$, or equivalently, if $\underline{Q}^{\wedge}$ is a group. An n-group $\underline{Q}$ is a cancellative n-semigroup and $\underline{Q}^{\sim} = \underline{Q}^{\wedge}$.

We note that every n-subgroup $\underline{P}$ of an n-semigroup $\underline{Q}$ is compatible in $\underline{Q}$ ([1]).

2. Some connections between finite separability properties of an n-semigroup and its universal covering

Let $\mathcal{K}$ be a class of n-semigroups and $\underline{Q} \in \mathcal{K}$.

DEFINITION 1. *$\underline{Q}$ is said to be residually finite in $\mathcal{K}$ if for each $x, y \in Q$, $x \neq y$, there is a surjective homomorphism φ from $\underline{Q}$ to a finite n-semigroup of $\mathcal{K}$ such that $\varphi(x) \neq \varphi(y)$.*

DEFINITION 2. *$\underline{Q}$ is said to have the finite separability property in $\mathcal{K}$ if for each $x \in Q$, and n-subsemigroup $\underline{P}$ of $\underline{Q}$, $x \notin P$, there is a surjective homomorphism φ from $\underline{Q}$ to a finite n-semigroup of $\mathcal{K}$, such that $\varphi(x) \notin \varphi(\underline{P})$.*

Replaceing the words "n-semigroup","n-subsemigroup" by "n-group", "n-subgroup" respectively, we obtain the corresponding classes of n-groups.

Remark In the propositions below by a residually finite n-semigroup we will always mean a residually finite n-semigroup in a class of n-semigroups. The cinsidered class of n-semigroups will be clearly understood by the context.

PROPOSITION 2.1. *If a covering semigroup $\underline{Q}'$[1] of an n-semigroup $\underline{Q}$ is residually finite, then $\underline{Q}$ is residually finite as well.*

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1) $\underline{Q}'$ is a covering semigroup of an n-semigroup $\underline{Q}$ is Q is a generating subset of Q' and $[x_1 \ldots x_n] = x_1 \ldots x_n$ for any $x_\nu \in Q$.

Proof: Let a,b be two distinct elements of Q. Then $a\neq b$ in Q', and, by the assumtion, there is a surjective homomorphism $\varphi: Q' \to S$, such that S is a finite semigroup and $\varphi(a)\neq\varphi(b)$. If we put $\psi=\varphi_Q$ and $T=\psi(Q)$, then (T, [ ]) is a finite n-semigroup where $[x_1\dots x_n] = x_1\dots x_n$, and, thus, $\psi: Q \to T$ is a surjective homomorphism such that $\psi(a)\neq\psi(b)$. □

It is not known whether the residual finitness of an n-semigroup Q induces the corresponding property for its universal covering. We will show, now, that we have the positive answer if we consider the class of cancellative n-semigroups and its cancellative universal covering semigroup.

PROPOSITION 2.2. If a cancellative n-semigroup Q is residually finite, then the cancellative universal covering semigroup Q is residually finite as well.

Proof: Let $a\neq b$, $a=a_1\dots a_i$, $b=b_1\dots b_j \in Q^{\sim}$, $a_\nu, b_\lambda \in Q$, $1\leq i\leq j< n$. If $i\neq j$ then $|\,|: c\mapsto |c|$ is a surjective homomorphism from $Q^{\sim}$ to $(Z_n,+)$ such that $|a|\neq|b|$. Assume, now, that i=j. Then $a' = [a_1^{n-i}a_1\dots a_i] \neq [a_1^{n-i}b_1\dots b_i] = b'$, and, thus, there is a surjective homomorphism ψ from Q into a finite cancellative n-semigroup S, such that $\psi(a')\neq\psi(b')$. Then ψ induces a surjective homomorphism $\psi^{\sim}: Q^{\sim}\to S^{\sim}$, where $S^{\sim}$ is a finite cancellative semigroup. Moreover, we have $\psi^{\sim}(a)\neq\psi^{\sim}(b)$, for if $\psi^{\sim}(a)=\psi^{\sim}(b)$, then $\psi(a')=\psi(a_1^{n-i}a_1\dots a_n)= = \psi(a_1)^{n-i}\psi^{\sim}(a_1\dots a_i)=\psi(a_1)^{n-i}\psi^{\sim}(b_1\dots b_i)=\psi(b')$. □

As a consequence of these two properties we obtain:

COROLLARY 2.3. The universal covering group $Q^\wedge$ of an n-group Q is residually finite iff Q is residually finite. □

As for the finite separability properties we have the following results.

PROPOSITION 2.4. If the universal covering group $Q^\wedge$ of an n-group Q has the finite separability property, then Q also has the finite separability property.

Proof: Let P be an n-subgroup of Q and $x\in Q\setminus P$. Then $P^\wedge$ is a subgroup of $Q^\wedge$ and $x\notin P^\wedge$. Therefore, if $Q^\wedge$ has the finite separability property then there is a finite group G and a surjective homomorphism $\varphi: Q^\wedge \to G$ such that $\varphi(x)\notin\varphi(P^\wedge)$.

The restriction $\varphi_Q = \psi$ of φ on Q is a surjective homomorphism from $\underline{Q}$ into a finite n-group $\psi(\underline{Q})=\underline{G}'$ and $\psi(x) \notin \psi(\underline{P}) \subseteq \varphi(\underline{P}^\wedge)$. □

PROPOSITION 2.5. If each n-subsemigroup of an n-semigroup $\underline{Q}$ is compatible in $\underline{Q}$, and the universal covering semigroup $\underline{Q}^\wedge$ has the finite separability property, then $\underline{Q}$ also has the finite separability property.

Proof: The proof is the same as the proof of 2.4. □

3. Some n-semigroup classes with solvable algorithmic problems

Certain connections between the finite separability properties and solvability of algorithmic problems are given in [3]. To be able to state them for n-semigroup classes, let me note that if $\mathcal{P}$ is a property for n-semigroups, then a class $\mathcal{K}$ of n-semigroups is a $\mathcal{P}$-class if each finitely presented member of $\mathcal{K}$ has the property $\mathcal{P}$. Now, if a class $\mathcal{K}$ of n-semigroups is residually finite (has the finite separability property), then $\mathcal{K}$ has a solvable word problem (has a solvable generalized word problem).

Also, a table of some varieties and classes with solvable algorithmic problems and with some finite separability properties is given in [3]. Among others, the following results are given:

(i) The variety of commutative groups (commutative semigroups) is residually finite.

(ii) The class of free groups (free semigroups, free commutative semigroups) has the finite separability property.

Using these results, the results given in 2., as well as known results for n-semigroups and n-groups, some corollaries are obtained.

COROLLARY 3.1. The variety of commutative n-groups is residually finite. □

COROLLARY 3.2. The variety of commutative n-semigroups is residually finite.

Proof: Let $\underline{Q}$ be a finitely presented n-semigroup. The semigroup $\underline{Q}'$ given by the presentation (1)(in the class of commutative semigroups) is the universal commutative covering semigroup of $\underline{Q}$ ([7]). $\underline{Q}'$ is finitely generated commutative semigroup, so ([5],Th 9.28, pg.172, II) it is finitely presented and is residually finite. Now, by 2.1, $\underline{Q}$ is residually finite as well.$\square$

COROLLARY 3.3. The class of free n-groups has the finite separability property.$\square$

Using the connections between finite separability properties and solvability of algorithmic problems, it follows immediately that:

1) The variety of commutative n-groups (commutative n-semigroups) has a solvable word problem.

2) The class of free n-groups has a solvable generalized word problem.

Remark: The result 1) could be obtained as a direct consequence of the results in [2] for connections between solvability of the word problem in n-semigroups (n-groups) and their universal covering. It could be proved that: if $\underline{Q}^\wedge$ is the universal covering group of an n-group $\underline{Q}$ with solvable generalized word problem, then $\underline{Q}$ has a solvable generalized word problem as well.The proof of this last property essentially uses the fact that each n-subgroup of $\underline{Q}$ is compatible in $\underline{Q}$, so this result could be proved for n-semigroups in which each n-subsemigroup is compatible.

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