PROCEEDINGS OF THE CONFERENCE
„ALGEBRA AND LOGIC", ZAGREB 1984

A NOTE ON INVARIANT n-SUBGROUPS OF n-GROUPS

Naum Celakoski, Snežana Ilić

Invariant n-subgroups of n-groups are considered here, and the so called "indirect method" for proving theorems on polyadic groups is used.

0. PRELIMINARIES

Invariant n-subgroups of n-groups are considered in Rusakov [3], [4] and some properties are investigated there by "direct technics" (which are used in most papers on n-groups). An "indirect method" which uses binary groups for proving theorems on polyadic groups is proposed in Čupona, Celakoski [2].

We use this method here to give an analogy of the well known result of the binary case that all normal subgroups of a group are exhausted by the kernels of homomorphisms, giving firstly some characterizations of normal n-subgroups of an n-group by the universal covering group.

We will use some definitions and notations as in [1] - [4].

An algebra $\underline{Q} = (Q,[\ ])$ with the carrier Q and an n-ary associative operation on Q, $[\ ]:(x_1,\ldots,x_n)\mapsto [x_1\ldots x_n]$ (n being fixed) is called an n-__semigroup__. $\underline{Q}$ is called an n-__group__ if, in addition, all the equations $[a_1\ldots a_{n-1}x] = b$, $[ya_1\ldots a_{n-1}] = b$ on x and y are solvable in $\underline{Q}$. The semigroup $\underline{Q}^\wedge = (Q^\wedge,\cdot)$ generated by the set Q with the set of defining relations:
$a = a_1\ldots a_n$ for every equality $a = [a_1\ldots a_n]$ in $\underline{Q}$, i.e.

$$\underline{Q}^\wedge = \langle Q;\ \{a = a_1\ldots a_n \mid a = [a_1\ldots a_n] \text{ in } Q\}\rangle$$

is called the __universal covering semigroup of__ $\underline{Q}$. The set

$Q\hat{} = \bigcup_{1}^{\infty} Q^m$, where $Q^m = \{a_1 \ldots a_m \mid a_\nu \in Q\}$ can be written in the form ([1; p.25], [2; p.136]):

$$Q\hat{} = Q \cup Q^2 \cup \ldots \cup Q^{n-1}, \quad \text{where} \quad Q^i \cap Q^j = \emptyset \quad \text{if} \quad i \neq j.$$

An n-semigroup $\underline{Q}$ can be considered as an n-subsemigroup of its universal covering semigroup $\underline{Q}\hat{}$. If $\underline{Q}$ is an n-group, then $Q\hat{}$ is a group and vice versa.

1. INVARIANT n-SUBGROUPS AND THE UNIVERSAL COVERING GROUP

Let $\underline{Q}$ be an n-group. An n-subgroup $\underline{H}$ of $\underline{Q}$[1)] is said to be invariant (or normal) in $\underline{Q}$ iff

$$(\forall x \in Q)(\forall i \in \{2,\ldots,n\}) \quad [xH^{n-1}] = [H^{i-1}xH^{n-i}]. \tag{1.1}$$

This is equivalent to the statement ([4; p.104])

$$(\forall x_1,\ldots,x_{n-1} \in Q)(\forall i \in \{2,\ldots,n-1\})\,[x_1^{n-1}H] = [x_1^{i-1}Hx_i^{n-1}]. \tag{1.2}$$

(Here, for example, $[H^{i-1}xH^{n-i}]$ is the set $\{[h_1^{i-1}xh_i^{n-1}] \mid h_\nu \in H\}$, where h_k^m stands for $h_k h_{k+1} \ldots h_m$ if $k \le m$, or for the empty symbol if $k > m$.)

The following Lemma gives a characterization of invariant n-subgroups in terms of the universal covering group.

1.1. LEMMA. *An n-subgroup $\underline{H}$ of an n-group $\underline{Q}$ is invariant in $\underline{Q}$ iff*

$$(\forall x \in Q) \quad xH = Hx \quad \text{in} \quad Q\hat{}.$$

Proof. If $\underline{H}$ is invariant in $\underline{Q}$ and $x \in Q$, then by (1.2) $[x^{n-1}H] = [x^{n-2}Hx]$, which becomes $x^{n-1}H = x^{n-2}Hx$ in $\underline{Q}\hat{}$ and thus (by cancelling x^{n-2} in the group $\underline{Q}\hat{}$) $xH = Hx$.

Conversely, let $xH = Hx$ in $\underline{Q}\hat{}$ for every $x \in Q$. Then

$$[xH^{n-1}] = xH^{n-1} = HxH^{n-2} = \ldots = H^{i-1}xH^{n-i} = [H^{i-1}xH^{n-i}]$$

for every $i \in \{2,\ldots,n\}$. Thus, $\underline{H}$ is invariant in $\underline{Q}$. ∎

If $\underline{H}$ is an n-subgroup of an n-group $\underline{Q}$, then $\underline{H}\hat{}$ is a subgroup of $\underline{Q}\hat{}$ ([1; 3.2, 3.9]) and $H\hat{} = H \cup H^2 \cup \ldots \cup H^{n-1}$. Therefore, by using Lemma 1.1, we have the following

---

1) *Throughout the paper $\underline{Q}$ will denote an n-group and $\underline{H}$ an n-subgroup of $\underline{Q}$.*

1.2. THEOREM. *An n-subgroup H of an n-group Q is invariant in Q iff the subgroup $H^\wedge$ is invariant in $Q^\wedge$.*

Proof. Let *H* be invariant in *Q*. Then, for every $x \in Q$, $xH = Hx$ in $Q^\wedge$ and

$$xH^\wedge = x(H \cup H^2 \cup \ldots \cup H^{n-1}) = xH \cup xH^2 \cup \ldots \cup xH^{n-1} =$$
$$= Hx \cup H^2x \cup \ldots \cup H^{n-1}x = (H \cup H^2 \cup \ldots \cup H^{n-1})x = H^\wedge x.$$

If $a \in Q^\wedge$, i.e. $a = a_1 \ldots a_i$, $a_\nu \in Q$, then

$$aH^\wedge = a_1 \ldots a_i (H \cup H^2 \cup \ldots \cup H^{n-1}) = a_1 \ldots a_{i-1}(a_i H \cup \ldots \cup a_i H^{n-1}) =$$
$$= a_1 \ldots a_{i-1}(Ha_i \cup \ldots \cup H^{n-1}a_i) = a_1 \ldots a_{i-1}(H \cup \ldots \cup H^{n-1})a_i =$$
$$= \ldots = (H \cup \ldots \cup H^{n-1})a_1 \ldots a_i = H^\wedge a.$$

Thus, $H^\wedge$ is invariant in $Q^\wedge$.

Conversely, let $H^\wedge$ be invariant in $Q^\wedge$. Then $(\forall x \in Q)\ xH^\wedge = H^\wedge x$, i.e.

$$xH \cup xH^2 \cup \ldots \cup xH^{n-1} = Hx \cup H^2x \cup \ldots \cup H^{n-1}x;$$

this is equivalent to the following sequence of equalities in $Q^\wedge$:

$$xH = Hx, \quad xH^2 = H^2x, \ldots, xH^{n-1} = H^{n-1}x;$$

by Lemma 1.1, *H* is invariant in *Q*. ▯

An n-group *Q* is called a *Dedekind n-group* ([3; p.89]) iff every n-subgroup of *Q* is invariant in *Q*.

1.3. PROPOSITION. *If Q is an n-group and $Q^\wedge$ is a Dedekind group, then Q is a Dedekind n-group.*

Proof. Let *H* be any n-subgroup of *Q*. Since $Q^\wedge$ is a Dedekind group, it follows that $H^\wedge$ is invariant in $Q^\wedge$ and by Th. 1.2, *H* is invariant in *Q*. Thus *Q* is a Dedekind n-group. ▯

The question for the converse of Prop. 1.3:

P.1. Is $Q^\wedge$ a Dedekind group when *Q* is a Dedekind n-group?

remains here without an answer.

The set of all elements x of Q such that

$$[xH^{n-1}] = [H^{i-1}xH^{n-i}] \quad \text{all} \quad i \in \{2, \ldots, n\} \tag{1.3}$$

is called the *normalizer* of the n-subgroup *H* in the n-group *Q* ([3; p.111]) and it is denoted by $N_Q(H)$ or shortly $N(H)$.

Clearly, N(H) $\neq \emptyset$ since $H \subseteq N(H)$. If $x_1, \ldots, x_n \in N(H)$, then

$$[x_1 \ldots x_n]H = x_1 \ldots x_n H = x_1 \ldots x_{n-1} H x_n = \ldots = H x_1 \ldots x_n = H[x_1 \ldots x_n]$$

in $Q^\wedge$, by which follows that $[x_1 \ldots x_n] \in N(H)$. It is easy to verify that any equation $[a_1 \ldots a_{n-1}x] = a_n$ on x and $[ya_1 \ldots a_{n-1}] = a_n$ on y in N(H) is solvable in N(H) and thus N(H) is an n-subgroup of Q. By the definition of N(H), H is invariant in N(H) and there is no element $x \in Q \setminus N(H)$ which satisfies the condition (1.3). Thus:

1.4. PROPOSITION. *The normalizer N(H) of an n-subgroup H of Q is the largest n-subgroup of Q such that H is invariant in N(H).* ∎

We note that the universal covering group $(N(H))^\wedge$ of N(H) is contained in

$$N(H^\wedge) = \{x_1 \ldots x_i \in Q^\wedge \mid x_1 \ldots x_i H^\wedge = H^\wedge x_1 \ldots x_i\},$$

i.e.

$$(N(H))^\wedge \subseteq N(H^\wedge). \tag{1.4}$$

Namely, if $x_1 \ldots x_i \in (N(H))^\wedge$, where $x_\nu \in N(H)$, then by 1.4 and 1.1

$$\begin{aligned} x_1 \ldots x_i H^\wedge &= x_1 \ldots x_i (H \cup H^2 \cup \ldots \cup H^{n-1}) = x_1 \ldots x_{i-1}(x_i H \cup \ldots \cup x_i H^{n-1}) = \\ &= x_1 \ldots x_{i-1}(Hx_i \cup \ldots \cup H^{n-1}x_i) = \ldots = \\ &= Hx_1 \ldots x_i \cup \ldots \cup H^{n-1}x_1 \ldots x_i = \\ &= (H \cup \ldots \cup H^{n-1})x_1 \ldots x_i = H^\wedge x_1 \ldots x_i, \end{aligned}$$

that is $x_1 \ldots x_i \in N(H^\wedge)$. Thus (1.5).

P.2. Does (or under what conditions) equality hold in (1.4)?

The indirect method can be used in obtaining shorter proofs of other results as well as of the following three:

1) If H and K are n-subgroups of an n-group Q such that $M = H \cap K \neq \emptyset$, and H is invariant in Q, then M is invariant in K [4; p.107] and $M^\wedge = H^\wedge \cap K^\wedge$.

2) If X and H are invariant n-subgroups of an n-group Q such that $[XH^{n-1}] = [H^{n-1}X]$, then the n-subgroup $B = [XH^{n-1}]$ is invariant in Q ([4; p.107]) and $B^\wedge = X^\wedge H^\wedge$.

3) The center of Q, i.e. the set

$$Z(Q) = \{z \in Q \mid (\forall x \in Q)\ [xz^{n-1}] = [z^{i-1}xz^{n-i}],\ i=2,\dots,n\}$$

is a commutative invariant n-subgroup of Q if it is not empty; in that case $(Z(Q))^{\wedge} = Z(Q^{\wedge})$.

(We note that the condition of "non-emptiness" above is omited in [4; p.106], which is a mistake. For example, Z(Q) of the 3-group $Q = \{\sigma \mid \sigma$ is an odd permutation of $\{1,2,3\}\}$ with $[xyz] = x\circ y\circ z$, is empty and thus it is not an n-subgroup of Q.)

2. HOMOMORPHISMS AND INVARIANT n-SUBGROUPS

The notion of homomorphisms of n-groups one defines in a usual way. The well known properties of the surjective homomorphisms (i.e. epimorphisms) of groups that the homomorphic image of a normal subgroup is a normal subgroup one proves easily for the n-ary case directly or indirectly. But the fact that an n-group might have more than one identities or no identity element at all brings the situation that the notion of a kernel of such a homomorphism one can not translate in a usual way.

Therefore we will consider the case when $\phi: Q \to Q'$ is a surjective homomorphism of n-groups, where Q' is an n-group with at least one identity. In this case, for every identity $e' \in Q'$ there exists a kernel

$$\mathrm{Ker}_{e'}\phi = \{x \in Q \mid \phi(x) = e'\}. \tag{2.1}$$

An analogous relation between the invariant n-subgroups of an n-group and kernels of homomorphisms (of n-groups) can be stated as in the binary case. We note that every homomorphism $\phi: Q \to Q'$ of n-groups induces a homomorphism $\phi^{\wedge}: Q^{\wedge} \to Q'^{\wedge}$ between their universal covering groups, defined by ([1; p.26])

$$\phi^{\wedge}(x_1 \dots x_i) = \phi(x_1)\dots\phi(x_i), \quad 1 \le i \le n-1, \quad x_\nu \in Q. \tag{2.2}$$

If ϕ is an epimorphism (monomorphism) of n-groups, then $\phi^{\wedge}$ is an epimorphism (a monomorphism) too ([1; 2.2,2.3]). We will prove first the following

2.1. TEHOREM. *If $\phi: Q \to Q'$ is an epimorphism of n-groups and H' is an invariant n-subgroup of Q', then the complete inverse image of H',*

$$H = \phi^{-1}(H') = \{h \in Q \mid \phi(h) \in H'\}$$

is an invariant n-subgroup of Q.

Proof. Clearly, $H=\phi^{-1}(H')$ is an n-subgroup of $\underline{Q}$ (as a complete inverse image of the n-subgroup $\underline{H}'$ of $\underline{Q}'$).

Since $\underline{H}'$ is invariant n-subgroup of $\underline{Q}'$, it follows by Th. 1.2 that the group $\underline{H}'^{\wedge}$ is invariant in $\underline{Q}'^{\wedge}$; thus $H^{\wedge} = \phi^{\wedge -1}(H'^{\wedge})$ is invariant in $\underline{Q}^{\wedge}$ which again by Th. 1.2 implies that $\underline{H}$ is invariant in $\underline{Q}$. ∎

Now we consider the epimorphisms and invariant n-subgroups of an n-group.

Let $\phi:Q \to Q'$ be an epimorphism from an n-group $\underline{Q}$ onto an n-group $\underline{Q}'$ with at least one identity e' and let

$$\mathrm{Ker}_{e'}\phi = \{a \in Q \mid \phi(a) = e'\} = K.$$

Clearly, K is an n-subgroup of $\underline{Q}$. Since $\{e'\}$ is an invariant n-subgroup of $\underline{Q}'$, it follows by Th. 2.1 that $K = \phi^{-1}(\{e'\})$ is an invariant n-subgroup of $\underline{Q}$.

Now let $\underline{H}$ be an invariant n-subgroup of an n-group $\underline{Q}$. Define an n-ary operation / / on the set

$$Q/H = \{\left[xH^{n-1}\right] \mid x \in Q\}$$

by

$$/[x_1H^{n-1}]\ldots\left[x_nH^{n-1}\right]/ = \left[[x_1\ldots x_n]H^{n-1}\right]. \qquad (2.3)$$

Then $\underline{Q}/H = (Q/H;/\ /)$ is an n-group (called the factor group of $\underline{Q}$ by $\underline{H}$) with an identity H. The n-subgroup $\{H\}$ of $\underline{Q}$ is the kernel of the natural homomorphism $\phi:Q \to Q/H$, $\phi(x) = [xH^{n-1}]$, since $H = \phi^{-1}\{H\}$.

So, we have the following theorem:

2.2. THEOREM. *An n-subgroup $\underline{H}$ of an n-group $\underline{Q}$ is invariant in $\underline{Q}$ iff $\underline{H}$ is a kernel of a surjective homomorphism $\phi:Q \to Q'$, where $\underline{Q}'$ is an n-group with at least one identity element.* ∎

Invariant n-subgroups of an n-group can be characterized also as kernels of homomorphisms of the n-group into (binary) groups. Namely, if $\underline{Q}$ is an n-group and $\underline{G}$ a group, then a mapping $\phi:Q \to G$ is a homomorphism iff

$$(\forall x_1,\ldots,x_n)\ \phi([x_1\ldots x_n]) = \phi(x_1)\ldots\phi(x_n). \qquad (2.4)$$

Suppose that $\underline{Q}' = (Q',[\ ])$ is an n-group with an identity e' and $\phi: Q \to Q'$ a surjective homomorphism. Putting

$$(\forall x',y' \in Q') \quad x' \cdot y' = [x'y'e'^{n-2}] \tag{2.5}$$

we obtain a group $(Q',\cdot)$ with the identity e'. Moreover, if $x_1,\ldots,x_n \in Q$ and $x'_\nu = \phi(x_\nu)$, then

$$\phi([x_1\ldots x_n]) = x'_1\cdot\ldots\cdot x'_n$$

and thus ϕ is a homomorphism of the n-group $(Q,[\ ])$ onto the group $(Q',\cdot)$. Also $Ker_{e'}\phi = \{x \in Q \mid \phi(x) = e'\}$ is an invariant n-subgroup in $\underline{Q}$. Therefore the following property is true:

2.3. THEOREM. *An n-subgroup H of an n-group Q is invariant in Q iff H is a kernel of a homomorphism from Q onto a (binary) group.* ∎

REFERENCES

[1] Čupona G., Celakoski N., *On representation of n-associatives into semigroups*, Maced. Acad. Sc. Arts, Contributions VI-2 (1974), 23-34

[2] Čupona G., Celakoski N., *Polyadic subsemigroups of semigroups*, Algebraic conference Skopje 1980, 131-152

[3] Русаков С.А., *Подгрупповое строение дедекиндовых n-арных групп*, Труды Гомель. сем., Минск 1978, 81-104

[4] Русаков С.А., *К теории нильпотентных n-арных групп*, Труды Гомель. сем., Минск 1978, 104-130.

Matematički fakultet, 91000 Skopje,

Filozofski fakultet, 18000 Niš