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COHN-REBANE THEOREM FOR VECTOR VALUED ALGEBRAS

Dedicated to Academician Petar Serafimov on occasion of his 70-th anniversary

A vector valued variant of the well-known Cohn-Rebane's Theorem is given in this paper.

0. Necessary preliminaries will be given first, and then the main result of the paper will be stated.

Let A be a non empty set, n, m positive integers, and f a mapping from A^n into A^m, where A^s is the s-th cartesian power* of A. Then we say that f is an (n, m)-operation on A, and write $\delta(f) = n$, $\rho(f) = m$; f will also be called a v e c t o r v a l u e d o p e r a t i o n on A. If F is a set of vector valued operations on A, then $(A; F)$ is called a v e c t o r v a l u e d a l g e b r a or, shortly, a v.v.a.

A v.v.a. $(Q; f)$ with an $(m+k, m)$-operation f, where $m, k \geqslant 1$, is said to be an $(m+k, m)$-s e m i g r o u p if the following equation

$$f(f(x_1^{m+k})x_{m+k+1}^{m+2k}) = f(x_1^j f(x_{j+1}^{j+m+k})\, x_{j+m+k+1}^{m+2k}) \tag{0.1}$$

is an identity on Q, for every $j \in N_k$, where $N_r = \{1, 2, \ldots, r\}$.

Thus: $(Q;f)$ is a (2,1)-semigroup iff it is a usual (binary) semigroup.

The general associative law holds in an $(m+k,m)$-semigroup. Namely, if $(Q;f)$ is an $(m+k, m)$-semigroup, and s is a positive integer, an $(m+sk, m)$-operation f^s can be defined by $f^1 = f$, and

$$f^{s+1}(x_1^{m+(s+1)k}) = f^s(f(x_1^{m+k})\, x_{m+k+1}^{m+(s+1)k}). \tag{0.2}$$

Then:

0.1. The following equation

$$f^s(x_1^j f^t(y_1^{m+tk})x_{j+1}^{sk}) = f^{s+t}(x_1^j\, y_1^{m+tk}\, x_{j+1}^{sk}) \tag{0.3}$$

is an identity on Q, for any $s, t \geqslant 1$ and $j \in N_{sk}$.

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* The elements of A^s will be denote by (α_1^s), and here we use the abbrevation x_α^β for $x_\alpha x_{\alpha+1} \ldots x_\beta$ if $\alpha \leqslant \beta$, and x_α^β will be "empty" if $\alpha > \beta$.

0.2. $(Q; f^s)$ is an $(m+sk,\ m)$-semigroup for every $s \geqslant 1$.

Further, if $(Q; f)$ is an $(m+k,\ m)$-semigroup, we will write $[x_1^{m+sk}]$ instead of $f^s(x_1^{m+sk})$, and, sometimes, $[x_1^m]$ instead of (x_1^m).

Now we can state the main result of this paper.

Theorem. Let $(A; F)$ be a v.v.a. and let m be a positive integer such that $\delta(f) \geqslant m$, $\rho(f) \geqslant m$, for every $f \in F$. There is an $(m+1,\ m)$-semigroup $(Q; [\ ])$ and a mapping $\alpha: f \mapsto \mathbf{f}$ from F into Q such that $A \subseteq Q$ and

$$f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Leftrightarrow [\mathbf{f}\ a_1^{\delta(f)}] = [b_1^{\rho(f)}] \tag{0.4}$$

for any $a_\nu, b_\lambda \in A$, $f \in F$.

We will prove the Theorem in the next two parts. From now on, we assume that $(A; F)$ is a given v.v.a., and m a positive integer such that $\delta(f) \geqslant m$, $\rho(f) \geqslant m$ for every $f \in F$. Also, $\alpha: f \mapsto \mathbf{f}$ will be a bijection from F onto $\mathbf{F} = \{\mathbf{f} \mid f \in F\}$, $A \cap \mathbf{F} = \emptyset$ and $B = A \cup \mathbf{F}$.

1. Here we will consider the case $m = 1$.

Let B^+ (B^*) be the free semigroup (monoid) on B, i.e.

$$B^+ = \{b_1 \dots b_t \mid b_\nu \in B,\ t \geqslant 1\}, \quad B^* = B^+ \cup \{1\}$$

where 1 is the empty sequence on B.

Define a mapping dg from B^* into the set of positive integers in the following way:

$$dg(1) = 0; \quad a \in A \Rightarrow dg(a) = 0; \quad f \in F \Rightarrow dg(\mathbf{f}) = 1$$
$$u, v \in B^* \Rightarrow dg(uv) = dg(u) + dg(v).$$

An element $u \in B^+$ is called r e d u c i b l e iff u has a form $u = u'\mathbf{f}\, a_1 \dots a_{\delta(f)} u''$, where $u', u'' \in B^*, f \in F, a_\nu \in A$. And, if $u \in B^+$ is not reducible, then we say that u is a r e d u c e d element. Denote the set of all reduced elements of B^+ by $B^\wedge$. We will define a r e d u c t i o n m a p p i n g $\varphi: B^+ \to B^\wedge$ as follows.

First, if $u \in B^\wedge$ then $\varphi(u) = u$.

Assume now that $u \in B^+$ is a reducible element, and that for every $v \in B^+$ such that $dg(v) < dg(u)$ the reduction $\varphi(v) \in B^\wedge$ is well determined, and furthermore:

$$v \neq \varphi(v) \Leftrightarrow v \in B^+ \setminus B^\wedge \Leftrightarrow dg(\varphi(v)) < dg(v).$$

From the reducibility of u it follows that there exist uniquely determined elements $u', u'' \in B^*$, $f \in F$, $a_1, \dots, a_{\delta(f)} \in A$ such that $u = u'\mathbf{f} a_1 \dots a_{\delta(f)} u''$ and ($u' = 1$ or $u' \in B^\wedge$). If $w = u' b_1 \dots b_{\delta(f)} u''$, where $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$, then $dg(w) = dg(u) - 1$, and thus $\varphi(w)$ is well defined element of $B^\wedge$. Now we define $\varphi(u)$ by

$$\varphi(u) = \varphi(w)$$

Thus $\varphi : B^+ \to B^\wedge$ is well determined and, by induction, it can easily be seen that the following statements are satisfied:

1.1. $(\forall\, u \in B^+)\ [u \notin B^\wedge \Leftrightarrow \varphi(u) \neq u \Leftrightarrow dg\,(\varphi(u)) < dg\,(u)]$.

1.2. $(\forall\, u, v \in B^+)\ \varphi(uv) = \varphi(\varphi(u)\, v) = \varphi(u\varphi(v) = \varphi(\varphi(u)\, \varphi(v))$.

Define an operation $\bullet$ on $B^\wedge$ in the following way:

$$u, v \in B^\wedge \Rightarrow u \bullet v = \varphi(uv)$$

Then, by **1.2**, we find that

1.3. $(B^\wedge ;\bullet)$ is a semigroup.

We will show that (0.4) is satisfied.

Assume first that $f\,(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ in $(A;F)$. Then we have:

$$\mathbf{f}\bullet a_1 \bullet \ldots \bullet a_{\delta(f)} = \varphi\,(\mathbf{f}\,a_1 \ldots a_{\delta(f)}) = b_1 \ldots b_{\rho(f)} = \varphi(b_1 \ldots b_{\rho(f)}) = b_1 \bullet \ldots \bullet b_{\rho(f)},$$

i.e. $[\mathbf{f}a_1^{\delta(f)}] = [b_1^{\rho(f)}]$ if we use [ ]-notation.

Conversely, if $f \in F$, a_ν, $b_\lambda \in A$ are such that

$$\mathbf{f}\bullet a_1 \bullet \ldots \bullet a_{\delta(f)} = b_1 \bullet \ldots \bullet b_{\rho(f)}$$

then we have

$$\varphi\,(\mathbf{f}\,a_1 \ldots a_{\delta(f)}) = \varphi\,(b_1 \ldots b_{\rho(f)}), \text{ i.e. } f\,(a_1^{\delta(f)}) = (b_1^{\rho(f)}).$$

This completes the proof of the Theorem in the case $m = 1$.

2. We assume that $m \geqslant 2$ in this part of the paper.

Define a *dimension* dm of the elements of a monoid X^* as follows: $dm\,(1) = 0$; $x \in X \Rightarrow dm\,(x) = 1$; $u, v \in X^* \Rightarrow dm\,(uv) = dm\,(u) + dm(v)$.

Let $B_o = B$, $C_p = \{u \in B_p^* \mid dm\,(u) \geqslant m+1\}$, $B_{p+1} = B_p \cup C_p \times N_m$ and $\mathbf{B} = \bigcup_{p \geqslant o} B_p$.

Thus we have:

$$u \in \mathbf{B} \text{ iff } (u \in B \text{ or } u = (v, i),\ v \in \mathbf{B}^+,\ dm\,(v) \geqslant m+1,\ i \in N_m).$$

In what follows we will use the following notations:

(i) a, b, c (with or without indices) $\in A$.

(ii) f, g, h (with or without indices) $\in F$.

(iii) x, y, z, u, v, w (with or without indices) $\in \mathbf{B}^*$.

(iv) (x, i) will always mean that $i \in N_m$, and $x \in \mathbf{B}^*$ is such that $dm\,(x) > m$, i.e. $(x, i) \in \mathbf{B}$.

Define a degree dg and a length $|\,|$ of the elements of $\mathbf{B}^*$ as follows:

$$dg\,(a) = 0;\ dg\,(\mathbf{f}) = 1;\ dg\,(uv) = dg\,(u) + dg\,(v);\ dg\,(u, i) = dg(u).$$

$$|a| = |\mathbf{f}| = 1;\quad |uv| = |u| + |v|;\quad |(u, i)| = |u|.$$

An element $u \in \mathbf{B}$ is said to be reducible iff

1) there is an appearance of $\mathbf{f}a_1 \ldots a_{\delta(f)}$ in u, for some $f, a_1, \ldots, a_{\delta(f)}$, or

2) there is an appearance of $(y, 1)\,(y, 2) \ldots (y, m)$ in u, for some y.

And, if $u \in \mathbf{B}$ is not reducible, then it is called reduced. The set of reduced elements of $\mathbf{B}$ will be denoted by $B^\wedge$: We note that $B \subseteq B^\wedge$.

Define a reduction mapping $\varphi : \mathbf{B} \to B^\wedge$ as follows:

First, if $u \in B^\wedge$, then we put $\varphi\,(u) = u$.

Let $u = (x, i) \in \mathbf{B} \setminus B^\wedge$ and assume that for every $v \in \mathbf{B}$ which satisfies the condition

$$dg\,(v) < dg\,(u) \quad \text{or} \quad (dg\,(v) = dg\,(u)\ \&\ |v| < |u|) \tag{2.1}$$

$\varphi(v)$ is a well determined element of $B^\wedge$, and that the following statement holds:

$$\varphi(v) \neq v \Leftrightarrow v \in \mathbf{B} \setminus B^\wedge \Leftrightarrow$$
$$\Leftrightarrow [dg\,(\varphi(v)) < dg\,(v) \quad \text{or} \quad (dg\,(\varphi(v)) = dg\,(v)\ \&\ |\varphi(v)| < |v|)] \tag{2.2}$$

Let $x = x_1 x_2 \ldots x_r$, $x_\nu \in \mathbf{B}$. Then $\varphi(x_j)$ is well defined for any $j \in N_r$. Denote $\varphi(x_1)\,\varphi(x_2) \ldots \varphi(x_r)$ by $\varphi(x)$. If $\varphi(x) \neq x$, and if we put $v = (\varphi(x), i)$, then (2.1) is satisfied, and thus $\varphi(v)$ is defined. Then we put $\varphi(u) = \varphi(v) = \varphi(\varphi(x), i)$.

Assume now that $\varphi(x) = x$, but x has the following form: $x = x'\,\mathbf{f}a_1 \ldots a_{\delta(f)}x''$, where x' is chosen in such a way that it has the minimal possible dimension. Let $f\,(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ in $(A;F)$. If $\rho(f) > m$ or $x'x'' \neq 1$ then $v = (x'\,b_1, \ldots\ b_{\rho(f)}x'', i) \in \mathbf{B}$ and $dg\,(v) < dg\,(u)$. Thus we can put $\varphi(u) = \varphi(v)$. If $\rho\,(f) = m$ and $x'x'' = 1$, then we put $\varphi(u) = \varphi(\mathbf{f}a_1 \ldots a_{\delta(f)}, i) = b_i$.

The case remains when $\varphi(x) = x$, but x has not the form $x = x'\,\mathbf{f}a_1 \ldots a_{\delta(f)}x''$. By assumption $u = (x, i)$ is reducible. The reducibility of $u = (x, i)$ implies that $x = x'\,(y, 1)\,(y, 2) \ldots (y, m)\,x''$, and x', y, x'' are uniquely determined if we assume that x' has the corresponding property of minimality. Then the length of $v = (x'\,yx'', i)$ is less than $|u|$, and $dg\,(v) \leqslant dg\,(u)$. Therefore, $\varphi(v) \in B^\wedge$ is well defined, and in this case we define $\varphi\,(u)$ by $\varphi\,(u) = \varphi\,(v)$.

In such a way all possible cases are exhausted, and thus we have obtained a well defined mapping $\varphi : \mathbf{B} \to B^\wedge$.

By using an induction on the degrees and the lengths it can be proved that φ admits the following properties:

2.1. $u \in \mathbf{B} \setminus B^{\wedge} \Leftrightarrow \varphi(u) \neq u \Leftrightarrow$
$\Leftrightarrow [dg(\varphi(u)) < dg(u) \text{ or } dg(\varphi(u)) = dg(u) \;\&\; |\varphi(u)| < |u|]$.

2.2. $\varphi(x, i) = \varphi(\varphi(x), i)$; $\varphi(xyz, i) = \varphi(x\varphi(y)z, i)$.

2.3. $f(a_1^{\delta(f)}) = (b_1^{\rho(f)}) \Rightarrow \varphi(x\mathbf{f}a_1 \dots a_{\delta(f)}y, i) = \varphi(xb_1 \dots b_{\rho(f)}y, i)$.

2.4. $\varphi(x(y, 1)(y, 2) \dots (y, m)z, i) = \varphi(xyz, i)$.

Define now an $(m+1, m)$-operation [ ] on $B^{\wedge}$ by

$$[u_1^{m+1}] = (v_1^m) \Leftrightarrow (\forall\, i \in N_m) v_i = \varphi(u_1^{m+1}, i), \text{ where } u_r, v_s \in B^{\wedge}.$$

2.5. $(B^{\wedge};\, [\,])$ is an $(m+1, m)$-semigroup.

Proof: Let $u_r, v_s \in B^{\wedge}$. Then we have:

$$\begin{aligned}
[\,[u_1^{m+1}]u_{m+2}] = (v_1^m) \Leftrightarrow & \\
v_i &= \varphi(\varphi(u_1^{m+1}, 1) \dots \varphi(u_1^{m+1}, m)\, u_{m+2}, i) \\
&= \varphi((u_1^{m+1}, 1) \dots (u_1^{m+1}, m)\, u_{m+2}, i) \\
&= \varphi(u_1^{m+2}, i) \\
&= \varphi(u_1(u_2^{m+2}, 1) \dots (u_2^{m+2}, m), i) \\
&= \varphi(u_1\varphi(u_2^{m+2}, 1) \dots \varphi(u_2^{m+2}, m), i) \text{ for all } i \in N_m \\
&\Leftrightarrow [u_1[u_2^{m+2}]\,] = (v_1^m),
\end{aligned}$$

which implies that $(B^{\wedge};\, [\,])$ is an $(m+1, m)$-semigroup.

We have to show that (0.4) is satisfied.

Let f, a_r, b_s be such that $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$ and $\rho(f) > m$.

Then

$$[\mathbf{f}a_1^{\delta(f)}] = (u_1^m) \Leftrightarrow (\forall\, i \in N_m)\, u_i = \varphi(fa_1^{\delta(f)}, i) = (b_1^{\rho(f)}, i) \Leftrightarrow [b_1^{\rho(f)}] = (u_1^m).$$

Conversely, $[\mathbf{f}a_1^{\delta(f)}] = [b_1^{\rho(f)}] = (u_1^m)$ implies

$$u_i = \varphi(\mathbf{f}a_1^{\delta(f)}, i) = (f(a_1^{\delta(f)}), i),$$
$$u_i = \varphi(b_1^{\rho(f)}, i) = (b_1^{\rho(f)}, i)$$

for all $i \in N_m$, i.e. $f(a_1^{\delta(f)}) = (b_1^{\rho(f)})$.

Suppose now that $\rho(f) = m$, $f(a_1^{\delta(f)}) = (b_1^m)$. Then

$$[\mathbf{f}a_1^{\delta(f)}] = (u_1^m) \Leftrightarrow u_i = \varphi(\mathbf{f}a_1^{\delta(f)}, i) = b_i, \text{ for all } i \in N_m$$
$$\Leftrightarrow [\mathbf{f}a_1^{\delta(f)}] = (b_1^m).$$

This completes the proof of the Theorem, since $(B^\wedge;[\,])$ is the desired $(m+1, m)$-semigroup.

3. Here we make a few remarks.

3.1. The fact that an $(m+1, m)$-semigroup $(Q;[\,])$ induces an $(m+k, m)$-semigroup $(Q;[\,]^k)$ implies that the following generalization is a corollary of the Theorem.

Theorem 1. Let $(A;F)$ be a v.v.a. and let m, k be positive integers. Suppose that for every $f\in F$ there exist integers s_f, r_f such that.

$$1+\delta(f)=m+ks_f,\ \rho(f)=m+kr_f,\quad r_f\geqslant 0,\ s_f\geqslant 1.$$

Then, there is an $(m+k, m)$-semigroup $(Q;[\,])$ and a mapping $\alpha: f\mapsto \mathbf{f}$ from F into Q such that $A\subseteq Q$ and (0.4) is satisfied for any a_i, $b_j\in A$, $f\in F$.

3.2. If $(A;F)$ is a usual universal algebra, i.e. if $\rho(f)=1$ for any $f\in F$, then $m=1$ is the unique positive integer such that $\rho(f)\geqslant m$, and in this case the result of the Theorem is the well known Cohn-Rebane's theorem ([1], [4]). That is why we call our Theorem "Cohn-Rebane theorem for v.v.a.". The original Cohn-Rebane's Theorem can be "translated" for v.v.a. in the following way.

Theorem 2. If $(A;F)$ is a v.v.a. then there is a semigroup $(Q;\cdot)$ and a mapping $\alpha: f\mapsto f_1 f_2 \ldots f_{\rho(f)}$ from F into Q^+ such that $A\subseteq Q$, $f_i\in Q$ and

$$f(a_1^{\delta(f)})=(b_1^{\rho(f)})\Leftrightarrow b_i=f_i\, a_1\ldots a_{\delta(f)}$$

for any $f\in F$, a_i, $b_j\in A$.

We note that if $m\geqslant 2$ or $(\exists f\in F)\rho(f)\geqslant 2$, then the assertion of our Theorem is not a consequence of Theorem 2.

3.3. The $(m+1, m)$-semigroup $(B^\wedge;[\,])$ obtained in the proof of the Theorem has the following universal property. If $(P;[\,]')$ is an $(m+1, m)$-semigroup and $\alpha': f\mapsto \mathbf{f}'$ a mapping from F into P such that $A\subseteq P$ and

$$f(a_1^{\delta(f)})=(b_1^{\rho(f)})\Leftrightarrow [f' a_1^{\delta(f)}]'=[b_1^{\rho(f)}]',\tag{0.4'}$$

then there exists a unique homomorphism $\xi:(B^\wedge;[\,])\to(P;[\,]')$ such that $\xi(\mathbf{f})=(f')$, $\xi(a)=a$, for any $f\in F$, $a\in A$.

3.4. Throughout the paper, it was implicitly assumed that $F\neq\emptyset$, and if we allow the case $F=\emptyset$, then the $(m+1, m)$-semigroup $(B^\wedge;[\,])$ obtained in the proof of the Theorem would be the free $(m+1, m)$-semigroup with a basis A. A convenient description of free vector valued semigroups is given in paper [2], and we would like to note that in the above proof of the Theorem some ideas from that paper are used.

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ТЕОРЕМАТА НА КОН-РЕБАНЕ ЗА ВЕКТОРСКО ВРЕДНОСНИ АЛГЕБРИ

(Резиме)

Во работава се докажува следнава

Теорема: Нека (A, F) е векторско вредносна алгебра и m позитивен цел број. Постои $(m+1, m)$-полугрупа $(Q, [\ \ ])$ и пресликување $\alpha : f \mapsto \mathbf{f}$, така што $A \subseteq Q$ и

$$f(a_1^{(\delta f)}) = (b_1^{\rho(f)}) \Leftrightarrow [\mathbf{f} a_1^{\delta(f)}] = [b_1^{\rho(f)}]$$

за секои $a_\nu, b_\lambda \in A, f \in F$.

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