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FREE VECTOR VALUED SEMIGROUPS

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Abstract. The aim of this paper is to give a combinatorial description of free vector valued semigroups.

0. Vector valued semigroups are defined in [1], where the question about a suitable description of free vector valued semigroups is stated. In this paper we answer this question, i.e. we think the answer is satisfactory. I thank Professor Čupona for the helpfull conversations during the course of this work.

1. Here we recall the necessary definitions and known results. From now on, let n,m be integers, such that $m \geq 2$ and $n - m = k \geq 1$.

Let Q be a nonempty set and $[\ ]:Q^n \to Q^m$ a map. (Here, Q^i is the i^{th} product of Q.) Then we say that $(Q;[\ ])$ is an (n,m)-groupoid. If $[\ ]((a_1,\ldots,a_n))=(b_1,\ldots,b_m)$ then we set $[a_1^n] = (b_1^m)$, where c_i^j stands for $c_i c_{i+1}\ldots c_j$, if $i \leq j$, and for the "empty sequence" if $i > j$.

We say that an (n,m)-groupoid $(Q;[\ ])$ is an (n,m)-semigroup if for each $1 \leq j \leq k$, the identity

$$(1.1) \qquad [[x_1^n]x_{n+1}^{n+k}] = [x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}]$$

holds in $(Q;[\ ])$.

For given (n,m)-groupoid $(Q;[\ ])$ and integer $s \geq 1$, an $(sk+m,m)$-groupoid $(Q;[\ ]^s)$ is defined by:

$$[\ ]^1 = [\ ]$$

$$(1.2) \qquad [x_1^{(s+1)k+m}]^{s+1} \stackrel{def}{=} [[x_1^n]x_{n+1}^{(s+1)k+m}]^s .$$

By taking Q with all the $[\,]^s$, $s \geq 1$, we get an m-dimensional vector valued algebra $(Q; \{[\,]^s \mid s \geq 1\})$. The proof of the following fact is by induction.

Proposition 1. An (n,m)-groupoid $(Q; [\,])$ is an (n,m)-semigroup if and only if for each $r,s \geq 1$, and each $0 \leq j \leq sk$, the identity

$$(1.3) \qquad [x_1^j [y_1^{rk+m}]^r x_{j+1}^{sk}]^s = [x_1^j\, y_1^{rk+m}\, x_{j+1}^{sk}]^{s+r}$$

holds in the vector valued algebra $(Q; \{[\,]^s \mid s \geq 1\})$. ∎

To each (n,m)-groupoid $(Q; [\,])$ we can associate two universal algebras $(Q; [\,]_1, \ldots, [\,]_m)$ and $(Q; \{[\,]_i^s \mid s \geq 1,\ 1 \leq i \leq m\})$, defined by

$$(1.4) \qquad \begin{aligned} &[\,]_i = [\,]_i^1 \\ &[x_1^{sk+m}]^s = (y_1^m) \iff [x_1^{sk+m}]_i^s = y_i \end{aligned} .$$

These universal algebras are called <u>component algebras</u> for $(Q; [\,])$. The definition of (n,m)-semigroups and Proposition 1. imply:

Proposition 2. (A) An (n,m)-groupoid $(Q;[\,])$ is an (n,m)-semigroup if and only if for each $1 \leq i \leq m$ and each $1 \leq j \leq k$, the identity

$$(1.1') \qquad [[x_1^n]_1 \ldots [x_1^n]_m x_{n+1}^{n+k}]_i = [x_1^j [x_{j+1}^{j+n}]_1 \ldots [x_{j+1}^{j+n}]_m x_{j+n+1}^{n+k}]_i$$

holds in the algebra $(Q; [\,]_1, \ldots, [\,]_m)$.

(B) An (n,m)-groupoid $(Q;[\,])$ is an (n,m)-semigroup if and only if for each $1 \leq i \leq m$, $r,s \geq 1$, $0 \leq j \leq k$, the identity

$$(1.3') \qquad [x_1^j [y_1^{rk+m}]_1^r \ldots [y_1^{rk+m}]_m^r x_{j+1}^{sk}]_i^s = [x_1^j y_1^{rk+m} x_{j+1}^{sk}]_i^{s+r}$$

holds in the algebra $(Q; \{[\,]_i^s \mid s \geq 1,\ 1 \leq i \leq m\})$. ∎

2. The fact that the (n,m)-groupoids can be characterized by the associated component algebras, allows us to translate all the notions from the universal algebras to the class of (n,m)-groupoids. It is clear that each of these notions can be defined directly for the (n,m)-groupoids. The same is true for the (n,m)-semigroups. Here we do not give explicit formulations of the corresponding definitions.

Proposition 2. implies the following:

Proposition 3. An arbitrary nonempty set B is a basis of a free (n,m)-semigroup, and moreover, B can be thought as a subset of the (n,m)-semigroup. ∎

The Proposition 3. is stated in [2], but a suitable description for free (n,m)-semigroups was not given. The aim of this paper is to give a combinatorial description of free (n,m)-semigroups.

In the following, for a nonempty set X, the set of all finite sequences with elements from X will be denoted by X^*.

Let B be a nonempty set. We define a sequence of sets $B_o, B_1, \dots, B_p, B_{p+1}, \dots$ by induction as follows:

$B_o = B$;

Let B_p be defined, and let C_p be the subset of B_p^* which consists of all elements u_1^{sk+m}, $u_\alpha \in B_p$, $s \geq 1$. Define B_{p+1} to be $B_p \cup C_p \times \mathbb{N}_m$, where $\mathbb{N}_m = \{1,2,\dots,m\}$.

Let $\overline{B} = \bigcup_{p \geq 0} B_p$.

Then $u \in \overline{B}$ if and only if $u \in B$ or $u = (u_1^{sk+m}, i)$ for some $u_\alpha \in \overline{B}$, $s \geq 1$, $i \in \mathbb{N}_m$.

Remark. By giving different "names" to the elements in B, we may assume that for each p, $C_p \times \mathbb{N}_m \cap B = \emptyset$, and B_p^* does not contain elements of the form u_1^r, $r \geq 2$, $u_\alpha \in B_p$.

Define a length for elements of $\overline{B}$, i.e. a map $|\ |: \overline{B} \longrightarrow \mathbb{N}$, ($\mathbb{N}$- the set of positive integers) as follows:

If $u \in B$ then $|u| = 1$;

If $u = (u_1^{sk+m}, i)$ then $|u| = |u_1| + |u_2| + \dots + |u_{sk+m}|$.

By induction on the length we are going to define a map $\varphi: \overline{B} \longrightarrow \overline{B}$. For $b \in B$, let $\varphi(b) = b$. Let $u \in \overline{B}$ and suppose that for each $v \in \overline{B}$ with $|v| < |u|$, $\varphi(v) \in \overline{B}$, and

(2.1) If $\varphi(v) \neq v$ then $|\varphi(v)| < |v|$;

(2.2) $\varphi(\varphi(v)) = \varphi(v)$.

Let $u = (u_1^{sk+m}, i)$. Then, for each α, $\varphi(u_\alpha) = v_\alpha \in \overline{B}$ is defined, $|\varphi(u_\alpha)| \leq |u_\alpha|$ and $\varphi(\varphi(u_\alpha)) = \varphi(u_\alpha)$. Let $v = (v_1^{sk+m}, i)$.

(i) If for some α, $u_\alpha \neq v_\alpha$, then $|v_\alpha| < |u_\alpha|$, and so, $|v| < |u|$. In this case let $\varphi(u) = \varphi(v)$.

Because $|v| < |u|$ it follows that $\varphi(v)$ is defined, and moreover, (2.1) and (2.2) imply that $|\varphi(u)| = |\varphi(v)| \leq |v| < |u|$, $\varphi(u) \neq u$, and $\varphi(\varphi(u)) = \varphi(\varphi(v)) = \varphi(v) = \varphi(u)$.

(ii) Let $u_\alpha = v_\alpha$ for each α. Then $u = v$. Suppose that there is $j \in \{0,1,2,\ldots,sk\}$ and $r \geq 1$, such that $u_{j+\nu} = (w_1^{rk+m}, \nu)$, for each $\nu \in \mathbb{N}_m$, and let t be the smallest such j. In this case, let

$$\varphi(u) = \varphi(u_1^t\, w_1^{rk+m}\, u_{t+m+1}^{sk+m}, i)\ .$$

Because $|(u_1^t\, w_1^{rk+m}\, u_{t+m+1}^{sk+m}, i)| < |u|$. it follows that $\varphi(u)$ is well defined, and moreover, (2.1) and (2.2) imply that $\varphi(u) \neq u$, $|\varphi(u)| < |u|$ and $\varphi(\varphi(u)) = \varphi(u)$.

(iii) If $\varphi(u)$ can not be defined by (i) or (ii), let $\varphi(u) = u$. In this case, $\varphi(\varphi(u)) = \varphi(u) = u$ and $|\varphi(u)| = |u|$.

The above discusion and (i), (ii) and (iii) complete the inductive step, and so we have defined a map $\varphi: \overline{B} \longrightarrow \overline{B}$. Moreover, we have proved the following:

<u>Lemma 4</u>. (a) For $b \in B$, $\varphi(b) = b$.

(b) For each $u \in \overline{B}$, $|\varphi(u)| \leq |u|$.

(c) For $u \in \overline{B}$, if $\varphi(u) \neq u$, then $|\varphi(u)| < |u|$.

(d) For each $u \in \overline{B}$, $\varphi(\varphi(u)) = \varphi(u)$. ∎

Next, we have the following lemmas.

<u>Lemma 5</u>. Let $u = (u_1^{sk+m}, i) \in \overline{B}$ and let $v_\alpha = \varphi(u_\alpha)$ for $1 \leq \alpha \leq sk+m$. Then:

(a) $\varphi(u) = \varphi(v_1^{sk+m}, i)$; and

(b) $\varphi(u) = \varphi(u_1^{\alpha-1}\, v_\alpha\, u_{\alpha+1}^{sk+m}, i)$ for each $1 \leq \alpha \leq sk+m$.

<u>Proof</u>. (a) If $u_\alpha = v_\alpha$ for each α, then (a) is obvious. If there is α, such that $u_\alpha \neq v_\alpha$, then (a) follows from (i).

(b) If $u_\alpha = v_\alpha$, then (b) is obvious. If $u_\alpha \neq v_\alpha$ then (b) follows from (a), Lemma 4.(d) and (i). ∎

<u>Lemma 6</u>. Let $u = (u_1^{sk+m}, i)$, $j \in \{0,1,\dots,sk\}$, and $u_{j+\alpha} = (v_1^{rk+m}, \alpha)$ for some $r \geq 1$ and each $\alpha \in \mathbb{N}_m$. Then

$$\varphi(u) = \varphi(u_1^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i) .$$

<u>Proof</u>. By induction on the length.

(A) Let $\varphi(u_t) = w_t \neq u_t$ for some $1 \leq t \leq j$ or $j+m+1 \leq t \leq sk+m$, or $\varphi(v_q) = z_q \neq v_q$ for some $1 \leq q \leq rk+m$. Then $\varphi(u)$ is defined by (i) and $\varphi(u) = \varphi(w)$ where $w = (w_1^{sk+m}, i)$, $w_\beta = \varphi(u_\beta)$ for each $1 \leq \beta \leq sk+m$. Because $|w| < |u|$, by induction, and using Lemma 5. (a),

$$\begin{aligned}\varphi(u) &= \varphi(w) = \varphi(w_1^j(z_1^{rk+m},1)\dots(z_1^{rk+m},m)w_{j+m+1}^{sk+m}, i) = \\ &= \varphi(w_1^j\, z_1^{rk+m}\, w_{j+m+1}^{sk+m}, i) = \\ &= \varphi(\varphi(u_1)\dots\varphi(u_j)\varphi(v_1)\dots\varphi(v_{rk+m})\varphi(u_{j+m+1})\dots\varphi(u_{sk+m}), i) \\ &= \varphi(u_1^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i) .\end{aligned}$$

Above, $\varphi(u_\gamma)$ was denoted by w_γ, and $\varphi(v_\gamma)$ by z_γ.

(B) Let $\varphi(u_\alpha) = u_\alpha$ and $\varphi(v_\beta) = v_\beta$ for each $1 \leq \alpha \leq j$, $j+m+1 \leq \alpha \leq sk+m$ and $1 \leq \beta \leq rk+m$, and let $\varphi(v_1^{rk+m}, \alpha) \neq (v_1^{rk+m}, \alpha)$ for some $1 \leq \alpha \leq m$. Then $\varphi(v_1^{rk+m}, \alpha)$ must be defined by (ii), since $\varphi(v_\beta) = v_\beta$ for each β. So, there is $0 \leq t \leq rk$ such that $v_{t+\nu} = (w_1^{pk+m}, \nu)$ for each $\nu \in \mathbb{N}_m$. By induction, since $|(v_1^{rk+m}, \nu)| < |u|$, we have that $\varphi(v_1^{rk+m}, \gamma) = \varphi(v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m}, \gamma)$ for each γ. Then, by induction, and using Lemma 5.,

$$\begin{aligned}\varphi(u) &= \varphi(u_1^j\varphi(v_1^{rk+m},1)\dots\varphi(v_1^{rk+m},m)u_{j+m+1}^{sk+m}, i) = \\ &= \varphi(u_1^j(v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m},1)\dots(v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m},m)u_{j+m+1}^{sk+m}, i) \\ &= \varphi(u_1^j\, v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m}\, u_{j+m+1}^{sk+m}, i) = \\ &= \varphi(u_1^j\, v_1^t(w_1^{pk+m},1)\dots(w_1^{pk+m},m)v_{t+m+1}^{rk+m}\, u_{j+m+1}^{sk+m}, i) = \\ &= \varphi(u_1^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i).\end{aligned}$$

Above, we have applied Lemma 6. on w and z where $w = (u_1^j(v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m},1)\dots(v_1^t\, w_1^{pk+m}\, v_{t+m+1}^{rk+m},m)u_{j+m+1}^{sk+m}, i)$ and $z = (u_1^j\, v_1^t(w_1^{pk+m},1)\dots(w_1^{pk+m},m)\, v_{t+m+1}^{rk+m}\, u_{j+m+1}^{sk+m}, i)$; It was possible, since $|w| < |u|$ and $|z| < |u|$.

(C) Let $\varphi(u_\alpha) = u_\alpha$ for each $1 \le \alpha \le sk+m$. Because of the assumption in the Lemma, it is possible to apply (ii). If the given j is the smallest such number, then by (ii) $\varphi(u) = \varphi(u_1^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i)$.

If not, let t be the smallest such number. Then for each $\gamma \in \mathbb{N}_m$, $u_{t+\gamma} = (z_1^{pk+m}, \gamma)$, and because $t < j$ and $u_{j+\alpha} \ne u_{t+m}$ for each $1 \le \alpha \le m-1$, it follows that $t+m \le j$. Then by induction and (ii),

$$\varphi(u) = \varphi(u_1^t\, z_1^{pk+m}\, u_{t+m+1}^j (v_1^{rk+m},1)\dots(v_1^{rk+m},m) u_{j+m+1}^{sk+m}, i) =$$
$$= \varphi(u_1^t\, z_1^{pk+m}\, u_{t+m+1}^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i) =$$
$$= \varphi(u_1^t (z_1^{pk+m},1)\dots(z_1^{pk+m},m) u_{t+m+1}^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i) =$$
$$= \varphi(u_1^j\, v_1^{rk+m}\, u_{j+m+1}^{sk+m}, i)\ .$$

Above, we have applied Lemma 6. on $w = (u_1^t z_1^{pk+m} u_{t+m+1}^{sk+m}, i)$ and $w' = (u_1^j v_1^{rk+m} u_{j+m+1}^{sk+m}, i)$; It was possible since $|w| < |u|$ and $|w'| < |u|$. ∎

Now, let $Q = \varphi(\overline{B})$. By Lemma 4.(d),

$Q = \{u \mid u \in \overline{B},\ \varphi(u) = u\}$.

Define a map $[\ ] : Q^n \longrightarrow Q^m$, by

(2.3) $\quad [u_1^n] = (v_1^m) \iff v_i = \varphi(u_1^n, i)$ for each $i \in \mathbb{N}_m$.

Because $u_j \in Q$, it follows that $(u_1^n, i) \in \overline{B}$, and so: $\varphi(u_1^n, i) \in Q$ for each $i \in \mathbb{N}_m$. Hence $[\ ]$ is well defined.

<u>Theorem</u> 7. $(Q; [\ ])$ is a free (n,m)-semigroup with a basis B.

<u>Proof</u>. (A) Let $[x_1^j [x_{j+1}^{j+n}]\, x_{j+n+1}^{n+k}] = (a_1^m)$, and $[x_{j+1}^{j+n}] = (b_1^m)$. Then $b_\alpha = \varphi(x_{j+1}^{j+n}, \alpha)$ and $a_i = \varphi(x_1^j\, b_1^m\, x_{j+n+1}^{n+k}, i)$ for each $\alpha, i \in \mathbb{N}_m$. Lemmas 5. and 6. imply that $a_i = \varphi(x_1^j (x_{j+1}^{j+n},1)\dots(x_{j+1}^{j+n},m) x_{j+n+1}^{n+k}, i) =$
$= \varphi(x_1^{n+k}, i)$ for each $i \in \mathbb{N}_m$.

On the other side, let $[[x_1^n] x_{n+1}^{n+k}] = (c_1^m)$ and $[x_1^n] = (d_1^m)$. Similarly as above, Lemmas 5. and 6. imply that for each $i \in \mathbb{N}_m$, $c_i = \varphi(x_1^{n+k}, i)$, i.e. $a_i = c_i$. Hence, for each $1 \le j \le k$, $[[x_1^n]\, x_{n+1}^{n+k}] = [x_1^j [x_{j+1}^{j+n}]\, x_{j+n+1}^{n+k}]$, i.e. $(Q; [\ ])$ is an (n,m)-semigroup.

(B) Because $\varphi(b) = b$ for each $b \in B$, it follows that $B \subseteq Q$. Let $u = (u_1^{sk+m}, i) \in Q$, and suppose that for each $1 \leq \alpha \leq sk+m$, $u_\alpha \in \langle B \rangle$, where $\langle B \rangle$ is the (n,m)-subsemigroup of $(Q;[\,])$ generated by B. Since $u_\alpha \in \langle B \rangle$, it follows that $[u_1^{sk+m}] = (a_1^m) \in \langle B \rangle^m$, i.e. $a_i \in \langle B \rangle$ for each $i \in \mathbb{N}_m$. But $a_i = \varphi(u_1^{sk+m}, i) = \varphi(u) = u$, since $u \in Q$, i.e. $u \in \langle B \rangle$. Hence, $\langle B \rangle = Q$, i.e. $(Q;[\,])$ is generated by B. Here we have used (1.2) and Proposition 1.

(C) Let $(G;[\![\,]\!])$ be an (n,m)-semigroup and let $f: B \longrightarrow G$ be a map. Define a map $g: Q \longrightarrow G$, by induction, as follows: for $b \in B$ let $g(b) = f(b)$; and

$$g(u_1^{sk+m}, i) = x_i \iff (x_1^m) = [\![g(u_1)\ldots g(u_{sk+m})]\!].$$

This map is well defined, since $(u_1^{sk+m}, i) = (v_1^{rk+m}, j)$ for elements from Q implies that $i = j$, $s = r$, and $u_\alpha = v_\alpha$ for each $1 \leq \alpha \leq sk+m$. Let $h: \overline{B} \longrightarrow G$ be the map $g \circ \varphi$, i.e. $h(u) = g(\varphi(u))$. It is clear that $h|_Q = g$. Now we are going to show by induction, that $h(u_1^{sk+m}, i) = [\![g(u_1)\ldots g(u_{sk+m})]\!]_i$ for each $(u_1^{sk+m}, i) \in \overline{B}$ with $u_\alpha \in Q$. Since $u_\alpha \in Q$, it follows that $\varphi(u_1^{sk+m}, i)$ is not defined by (i).

If $\varphi(u_1^{sk+m}, i) = (u_1^{sk+m}, i)$, then

$$h(u_1^{sk+m}, i) = g(u_1^{sk+m}, i) = [\![g(u_1)\ldots g(u_{sk+m})]\!]_i .$$

If $\varphi(u_1^{sk+m}, i) \neq (u_1^{sk+m}, i)$, then $\varphi(u_1^{sk+m}, i)$ is defined by (ii). Let $p = rk+m$ and $u_{j+\nu} = (v_1^p, \nu)$ for each $\nu \in \mathbb{N}_m$. Then

$$\begin{aligned}
g(\varphi(u_1^{sk+m}, i)) &= g(\varphi(u_1^j\, v_1^p\, u_{j+m+1}^{sk+m}, i) = \\
&= [\![g(u_1)\ldots g(u_j)g(v_1)\ldots g(v_p)g(u_{j+m+1})\ldots g(u_{sk+m})]\!]_i = \\
&= [\![g(u_1).g(u_j)[\![g(v_1).g(v_p)]\!]g(u_{j+m+1}).g(u_{sk+m})]\!]_i = \\
&= [\![g(u_1)\ldots g(u_j)[\![g(v_1)\ldots g(v_p)]\!]_1 \ldots \\
&\quad \ldots [\![g(v_1)\ldots g(v_p)]\!]_m g(u_{j+m+1})\ldots g(u_{sk+m})]\!]_i = \\
&= [\![g(u_1)\ldots g(u_j)g(v_1^p, 1)\ldots g(v_1^p, m)g(u_{j+m+1})\ldots \\
&\quad \ldots g(u_{sk+m})]\!]_i = \\
&= [\![g(u_1).g(u_j)g(u_{j+1}).g(u_{j+m})g(u_{j+m+1}).g(u_{sk+m})]\!]_i \\
&= h(u_1^{sk+m}, i).
\end{aligned}$$

The above implies that g is an (n,m)-homomorphism, since
$g([u_1^{k+m}]_i) = g(\varphi(u_1^{k+m},i)) = h(u_1^{k+m},i) = [\![g(u_1)\dots g(u_{k+m})]\!]_i$,
i.e. $g^m([u_1^{k+m}]) = [\![g(u_1)\dots g(u_{k+m})]\!]$. ∎

Remark 8. From the construction of Q, it follows that $\bar{B}$ is a free algebra with signature $\{[\ ]_i^s \mid s \geq 1,\ i \in \mathbb{N}_m\}$ generated by B, where $[\ ]_i^s$ denotes an $sk+m$ operation. If $u \in \bar{B}$, then we can say that $\varphi(u)$ is an "irreducible representative" of u. The definition of φ implies that $\varphi(u)$ is obtained from u by a finitely many transformations of type

$$(\dots(v_1^{rk+m},1)\dots(v_1^{rk+m},m)\dots,i) \rightsquigarrow (\dots v_1^{rk+m}\dots,i)\ ,$$

and Lemmas 4.,5., and 6. imply that $\varphi(u)$ does not depend on the order of those transformations.

References

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