PROCEEDINGS OF THE CONFERENCE
„ALGEBRA AND LOGIC", CETINJE 1986.

VECTOR VALUED ASSOCIATIVES

N. Celakoski, B. Janeva

Abstract. In this paper the notion of vector valued associative is introduced (as a generalization of the notion of associative [1] as well as of the notion of vector valued semigroup [3]). Some examples of such associatives are given, and an attempt is made to carry over some results already obtained for other vector valued structures. It is shown also that the class of vector valued subassociatives of vector valued semigroups is, in general, a proper subclass of the class of vector valued associatives.

1. DEFINITION OF (F,δ,m)-ASSOCIATIVE

Let F be a (nonepmty) set and $\delta: f \to \delta(f)$, $\rho: f \to \rho(f)$ *be two mappings from F into the set of positive integers. If A is a nonempty set and* $\xi: f \to \overline{f}$ *is a mapping from F into the set* $\overline{F}$ *of vector valued operations such that*

$$\overline{f}: A^{\delta(f)} \to A^{\rho(f)},$$

then a vector valued algebra $(A;\overline{F})$ *is built up. We call* $(A;\overline{F})=\underline{A}$ *an* (F,δ,ρ)-*algebra. Further on we will write F instead of* $\overline{F}$ *and f instead of* $\overline{f}$. *The integers* $\delta(f)$ *and* $\rho(f)$ *are called the length and the dimension of f respectively.*

We will consider in this paper a class of vector valued algebras which satisfy the condition

$$(\forall f \in F) \qquad \delta(f) > m, \qquad (1.1)$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

where $m=\rho(f) \geq 2$ is fixed. The elements of F will be called primary operations of $\underline{A}$; the identity operation will be denoted by 1. (Note that $1 \notin F$).

First we will define polynomial operations (on A) inductively in the following way:

(i) 1 and every $f \in F$ are polynomial operations

(ii) if $g, g_1, \ldots, g_p$ are polynomial operations such that $\delta(g)=\rho(g_1)+\ldots+\rho(g_p)$ [1], then the operation $h=g(g_1 \times \cdots \times g_p)$ is a polynomial operation.

(Here: $h(\underline{x})=g(g_1(\underline{x}_1)g_2(\underline{x}_2)\ldots g_p(\underline{x}_p))$, where $x=\underline{x}_1\ldots\underline{x}_p$ and the length $|\underline{x}_\nu|$ of the string $\underline{x}_\nu$ is $|\underline{x}_\nu|=\delta(g_\nu)$.)

Clearly, $\delta(h)=\delta(g_1)+\ldots+\delta(g_p)$ and $\rho(h)=\rho(g)$.

The set of all polynomial operations of $\underline{A}$ will be denoted by $P_F(A)=P_F$. Obviously,

$$(\forall h \in P_F) \; h \neq 1 \Longrightarrow \rho(h) = m. \tag{1.2}$$

According to (1.1), $\delta(f)-m > 0$ for every $f \in F$. The positive integer

$$i(f) = \delta(f)-m \tag{1.3}$$

will be called the index of the operation f. The index of the identity operation is 0. We denote by P' the set $P_F \setminus \{1\}$ (of nonidentity polynomial operations belonging to P_F).

PROPOSITION 1.1. The set $i(P')$ of the indexes of the nonidentity polynomial operations coincides with the semigroup generated by the set $i(F)=J$ of indexes of the primary operations, i.e.

$$i(P') = \langle i(F)\rangle = \langle J\rangle.$$

---

[1] $\rho(g)$ will denote the dimension, and $\delta(g)$ the length of the polynomial operation g.

Proof. Let $h \in P'$, and $h=gf$, where $g \in P'$, $f \in P_F$. Since $\delta(h)=\delta(f)$, $\rho(f)=\delta(g)$ and $\rho(h)=\rho(g)$, it follows that

$$i(h) = \delta(h)-\rho(h) = \delta(f)-\rho(f)+\rho(f)-\rho(g) = i(f) + i(g).$$

If $h=g\times f$ where

$$(g\times f)(x_1^{\delta(g)}, y_1^{\delta(g)}) = (g(x_1^{\delta(g)}), f(y_1^{\delta(f)}))$$

then $i(h) = i(g) + i(f)$.

Inductively, if $h=g(g_1\times\ldots\times g_p)$, where $g \in P'$, $g_1,\ldots,g_p \in P_F$, then

$$i(h) = i(g) + i(g_1)+\ldots+i(g_p). \tag{1.4}$$

We can assume that $i(g) \in \langle J\rangle$ and that either $i(g_\nu)=0$ or $i(g_\nu) \in \langle J\rangle$. This implies that $i(h) \in \langle J\rangle$, i.e. $i(P') \subseteq \langle J\rangle$.

Conversely, let $k \in \langle J\rangle$ and $k \notin J$. Then there exist $i_1,\ldots,i_r \in J$, such that $k=i_1+\ldots+i_r$. Put $h=f_1(f_2\times 1^{i_1})\ldots(f_r\times 1^{i_1+\ldots+i_{r-1}})$ where $i(f_j)=i_j$. Then $i(h) = k$, i.e. $\langle J\rangle \subseteq i(P')$. □

Now we are ready to introduce the concept of vector valued associative.

An (F,δ,m)-algebra $(A;F)$ is called an (F,δ,m)-*associative* iff any two polynomial operations of $\underline{A}$ with the same length are equal, i.e.

$$(g,h \in P_F \ \ \& \ \ \delta(g) = \delta(h)) \Longrightarrow g = h. \tag{1.5}$$

Using the notation $J=i(F)$ and the fact that the dimension of the operations is fixed, an (F,δ,m)-associative $(A;F)$ will be called an m-*dimensional* J-*associative*, or shortly, a (J,m)-*associative*, and it will be denoted by $(A;J)$.

Further on we will often write $[x_1^{\delta(g)}]$ or $[x_1^{k+m}]$ instead of $g(x_1^{\delta(g)})$, where g is a fixed polynomial operation, $k=i(g)$, and $[x]=x$ for the identity operation.

From the definition of the notion of a (J,m)-associative we obtain the following

PROPOSITION 1.2. *If* $(A;J)$ *is a* (J,m)*-associative, then for every* $k\in J$, $(A;[\ ])$ *is an* $(m+k,m)$*-semigroup* ([3]). □

(We say that this $(m+k,m)$*-semigroup is* induced *by the given* (J,m)*-associative.)*

2. EXAMPLES OF (J,m)-ASSOCIATIVES

We will consider two examples of (J,m)-associatives.

Example 1. Let $(A;[\ ])$ be an $(m+d,m)$-semigroup and let $d \mid i(f)$ for every $f\in F$, where F is the set of all $(m+sd,m)$-operations obtained from $[\ ]$ by the general associative law ([3]). Define

$$(\forall f\in F)\qquad f(a_1^{\delta(f)}) = [a_1^{\delta(f)}].$$

Then $(A;[\ ])$ becomes an $(F;\delta,m)$-associative.

Example 2. Let $A=\{a,b,c\}$, $a\neq b\neq c\neq a$ and J be a set of positive integers such that $d=GCD(J)\notin J$. Denote by p the least element of J. Then the set $L=J\setminus\{\alpha p \mid \alpha \geq 1\}$ is nonempty and let q be the least element of L.

Define a set $F=\{f_k \mid k\in J\}$ of vector valued operations on A in the following way:

$(\forall k\in J)\ \delta(f_k) = m+k,\quad \rho(f_k) = m$ and

$$f_k(x_1^{m+k}) = \begin{cases}(b^m) \text{ if } k=q,\ (x_1^{m+k})=(c^{m+q})\\ (a^m) \text{ otherwise.}\end{cases}$$

We are going to show that $(A;F)$ is a (J,m)-associative.

Namely, let $h\in P'$ be a polynomial operation with a positive index $k\in J$. Then the following implication holds:

$$h \neq f_q \text{ or } \underline{x} \neq (c^{m+q}) \Longrightarrow h(\underline{x}) = (a^m). \tag{2.1}$$

We will show (2.1) by induction of the construction of polynomial operations.

If $h\in F$, then (2.1) is satisfied by the definition of F.

Assume that $h=g(g_1\times\ldots\times g_r)$. Then $g\neq 1$ and put

$k=i(h) = i(g) + i(g_1)+\dots+i(g_r)$. Therefore

$$k = q \Longleftrightarrow g = f_q, \quad g_1=\dots=g_r=1. \qquad (2.2)$$

Namely it is clear that the implication $\Longleftarrow$ is true. Let $k=q$. Then $i(g) \leq q$, $i(g_\nu) \leq q$ and $i(g) \neq 0$, which implies that $i(g)=q$, $i(g_\nu)=0$.

Let $g\neq f_q$ or $g_\lambda\neq 1$ for some λ. In the first case we have $h(\underline{x})=g(\underline{y})=(a^m)$. In the second case, i.e. if $g=f_q$ and $g_\lambda\neq 1$ for some λ, we have $h(\underline{x})=g(\underline{y})$, where $\underline{y}\neq(c^{m+q})$, and therefore $g(\underline{y})=(a^m)$.

Clearly, (2.1) implies that $(A;F)$ is a (J,m)-associative.

3. (K,m)-SUBASSOCIATIVES OF (m+d,m)-SEMIGROUPS

Let $(A;J)$ be a (J,m)-associative and $J\subseteq L\subseteq \langle J\rangle$. Then, for every $\ell\in L$, by the given (J,m)-associative $(A;J)$, an $(\ell+m,m)$-operation is induced such that $(A;L)$ is an (L,m)-associative. Note that $(A;L)$ is not essentially different from $(A;J)$. Therefore, further on, we will consider only (K,m)-associatives where K is a subsemigroup of the additive semigroup of positive integers.

In this case, if $M\subseteq K$, then a given (K,m)-associative $(A;K)$ induces a corresponding (M,m)-associative wich is called an M-restriction of $(A;K)$. Specially, if $k\in K$, then we have $(m+k,m)$-semigroup $(A;[\![\ ]\!])$ induced by the (K,m)-associative $(A;K)$.

We note that an $(m+k,m)$-semigroup is in fact a $(\langle k\rangle,m)$-associative.

A (K,m)-associative $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+d,m)$-semigroup $(Q;[\ ])$ iff $d \mid GCD(K)$, $A\subseteq Q$ and

$$(\forall a_1^{k+m}\in A^{k+m})\ [\![a_1^{k+m}]\!] = [a_1^{k+m}]. \qquad (3.1)$$

PROPOSITION 3.1. *A (K,m)-associative $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+d,m)$-semigroup iff $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+1,m)$-semigroup.*

<u>Proof</u>. If $(A;[\![\ ]\!])$ is a (K,m)-subassociative of a $(d+m,m)$-semigroup $(Q;[\ ]')$, then $(Q;[\ ]')$ is a $(d+m,m)$-subsemigroup of an $(m+1,m)$-semigroup $(P;[\ ])$ ([4]). Thus, $(A;[\![\ ]\!])$ is a (K,m)-subassociative of a an $(m+1,m)$-semigroup $(P;[\ ])$.

Conversely, if $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+1,m)$-semigroup $(Q;[\ ]')$, then the operation $[\ ]$ defined by

$$(\forall x_\nu \in Q)\ [x_1^{d+m}] = [x_1^{d+m}]',$$

is the $(m+d,m)$-operation induced by $[\ ]'$ on Q, and $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+d,m)$-subsemigroup $(Q;[\ ])$.

PROPOSITION 3.2. <u>A</u> (K,m)-<u>associative</u> <u>is</u> <u>a</u> (K,m)-<u>subassociative</u> <u>of</u> <u>an</u> $(m+1,m)$-<u>semigroup</u> <u>iff</u> $d=GCD(K)\in K$.

<u>Proof</u>. If $d\in K$, then the (K,m)-associative $(A;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+d,m)$-semigroup $(A;[\ ])$ induced by the (K,m)-associative $(A;[\![\ ]\!])$.

Conversely, we will show that if $d\notin K$, then the (K,m)-associative $(A;F)$ of Example 2 is not a (K,m)-subassociative of an $(m+d,m)$-semigroup.

Suppose that there exists an $(m+d,m)$-semigroup $(P;[\ ])$, such that $A\subseteq P$ and

$$(\forall x_\nu \in A)(\forall k\in K) f_k(x_1^{k+m}) = [x_1^{k+m}].$$

Let q be as in Example 2 and $q=tp+r$, where $d|r$ and $r > 0$. Then

$$\begin{aligned}(b^m) &= f_q(c^{q+m}) = [c^{tp+r+m}] = [[c^{tp+m}]c^r] = \\ &= [a^m c^r] = [f_{tp}(a^{tp+m})c^r] = [[a^{tp+m}]c^r] = \\ &= f_q(a^{tp+m}c^r) = (a^m)\end{aligned}$$

which contradicts the fact that $b\neq a$.

Thus, $(A;F)$ is not a (K,m)-subassociative of an $(m+d,m)$-semigroup, and, by Proposition 3.1, $(A;F)$ is not a (K,m)-subassociative of an $(m+1,m)$-semigroup as well. □

COROLLARY 3.3. If $d \notin GCD(K)$, then the class of (K,m)-subassociatives of semigroups is a proper subclass of the class of (K,m)-associatives. □

It is desirable to have an axiom system for the class of (K,m)-subassociatives of $(m+1,m)$-semigroups. Such a system is described in [2] when $m=1$, but we do not know a convenient axiom system of (K,m)-subassociatives of $(m+1,m)$-semigroups in the case $m \geq 2$.

We can only state the following proposition, which does not give enough informations for this class of algebras.

PROPOSITION 3.4. Let $(Q;[\![\ ]\!])$ be a (K,m)-associative and let $(P;[\ ])$ be the free $(m+1,m)$-semigroup with a base Q. Denote by $\approx$ the minimal congruence on P, such that

$$[\![a_1^{m+k}]\!] = (b_1^m) \text{ in } (Q;[\![\ ]\!]) \Longrightarrow [a_1^{m+k}] \approx (b_1^m)$$

Then $(Q;[\![\ ]\!])$ is a (K,m)-subassociative of an $(m+1,m)$-semigroup iff the following statement is satisfied

$$a,b \in Q \Longrightarrow (a \approx b \Longrightarrow a = b). \square$$

(Above, $(x_1^m) \approx (y_1^m)$ means $x_\nu \approx y_\nu$ for every $\nu \in \{1,2,\dots,m\}$).

We note that a satisfactory description of free vector valued semigroups is given in [5].

R E F E R E N C E S

[1] Ѓ.Чупона: За асоцијативите, Contributions, I 1-Sect. nat Sc. and Math., Maced. Acad. Sc. and Arts, (1969), 11-20

[2] N.Celakoski: On semigroup associatives, Maced. Acad. Sc. and Arts, Contributions IX 2-Sect. nat. Sc. and Math., (1977), 5-19

[3] G.Čupona: Vector valued semigroups, Semigroup Forum, Vol. 26 (1983), 65-74

[4] G.Čupona, S.Markovski, B.Janeva: Post theorem for vector valued semigroups, This volume

[5] D.Dimovski: Free vector valued semigroups, This volume

N.Celakoski, Mašinski fakultet, Skopje

B.Janeva, Prirodno-matematički fakultet, Skopje