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# POST THEOREM FOR VECTOR VALUED SEMIGROUPS

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Abstract. The main result of this paper is the following

THEOREM. *Let* m,k,p *and* q *be integers such that* $m,k,q \geq 1$, $p \geq 0$. *If* $(Q;[\![\ ]\!])$ *is an* $(m+p+q,m+p)$*-semigroup, then there is an* $(m+1,m)$*-semigroup* $(P;[\ ])$ *such that* $Q \subseteq P$ *and*

$$[\![a_1^{m+p+q}]\!] = (b_1^{m+p}) \Longleftrightarrow [a_1^{m+p+q}] = [b_1^{m+p}], \qquad (*)$$

*for any* $a_\nu, b_\lambda \in Q$. (*If* $p=0$, *then we write* $[b_1^m]$ *instead of* $(b_1^m)$.)

0. We give here necessary preliminary definitions and results.

Let n and m be positive integers such that $n-m=k \geq 1$. A mapping

$$[\ ]: (x_1,\ldots,x_n) \longmapsto [x_1\ldots x_n]$$

from $Q^n$ to $Q^m$ [1] is called an associative (n,m)-operation iff the following identity is satisfied for every $j \in \{1,2,\ldots,k\}$:

$$[[x_1^n]x_{n+1}^{n+k}] = [x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}].$$

In this case we say that $(Q;[\ ])$ is an (n,m)-*semigroup*, or a *vector valued semigroup*. (We remark that the notion of vector

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[1] $Q^r$ is the r-th cartezian power of Q; $x_\alpha^\beta$ is an abbreviation for the „string" sequence $x_\alpha x_{\alpha+1}\ldots x_\beta$ if $\alpha \leq \beta$, and it is „empty" if $\alpha > \beta$. Thus, $(x_\alpha^\beta)$ stands for $(x_\alpha, x_{\alpha+1},\ldots,x_\beta)$.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

valued semigroups is defined in [5]).

If [ ] is an (n,m)-operation, nonnecessarily associative, then we can define an (m+sk,m)-operation $[\ ]^s$, for every $s \geq 1$, in the following way: $[x_1^n]^1 = [x_1^n]$ and

$$[x_1^{sk+m}]^s = [x_1^k[x_{k+1}^{sk+m}]^{s-1}] \text{ if } s \geq 2$$

The following „associative law" holds:

*0.1. If* (Q;[ ]) *is an* (m+k,m)-*semigroup*, *then for every* $r,s \geq 1$, $j \in \{1,2,\ldots,sk\}$ *the equality*:

$$[x_1^j y_1^{rk+m} x_{j+1}^{sk}]^{r+s} = [x_1^j [y_1^{rk+m}]^r x_{j+1}^{sk}]^s$$

*is an identity on* (Q;[ ]).

As a corollary we have:

*0.2. If* (Q;[ ]) *is an* (m+k,m)-*semigroup then* (Q;$[\ ]^s$) *is an* (m+sk,m)-*semigroup for any* $s \geq 1$. (In the future, we will omit the index s in $[\ ]^s$. Thus, an (m+1,m)-semigroup (Q;[ ]) induces an (m+k,m)-semigroup (Q;[ ]) for any $k \geq 1$. We note that this simplification in the notation is already used in the formulation of Theorem.)

If (P;[ ]) is an (m+1,m)-semigroup and if $Q \subseteq P$ such that:

$$(a_1^{m+k}) \in Q^{m+k} \Longrightarrow [a_1^{m+k}] \in Q^m,$$

then we say that Q is an (m+k,m)-subsemigroup of (P;[ ]). In this case, the restriction of $[\ ]^k$ on Q induces an (m+k,m)-semigroup (Q;[ ]), called (m+k,m)-*subsemigroup* of (P;[ ]).

Thus, the conclusion of Theorem for p=0 can be stated as follows:

THEOREM 1. *Every* (m+k,m)-*semigroup is an* (m+k,m)-*subsemigroup of an* (m+1,m)-*semigroup*.

We note that the following generalization is a corollary of Theorem 1:

THEOREM 1´. *Every* $(m+sk,m)$-*semigroup is an* $(m+sk,m)$-*subsemigroup of an* $(m+k,m)$-*semigroup*.

Also, in the case $p>0$, the following generalization is a corollary of Theorem:

THEOREM 2. *If* $m,p,q$ *and* $k$ *are positive integers such that* $k$ *is a divisor of the both* $p$ *and* $q$, *then for every* $(m+p+q,m+p)$-*semigroup* $(Q;[\![\ ]\!])$ *there is an* $(m+k,m)$-*semigroup* $(P;[\ ])$ *such that* $Q\subseteq P$ *and* (*) *holds for any* $a_\nu,b_\lambda \in Q$.

We have named the subject of this work Post Theorem, because there is an analogy with corresponding Post's Theorem for polyadic groups ([4]). The question whether Post Theorem is true for vector valued groups is a natural one, but we do not know the answer till now.

Further on we assume that $m\geq 2$, since our Theorem reduces to the well known Post Theorems concerning embeddings of polyadic semigroups in (binary) semigroups (see, for example [2]) in the case $p=0$, $m=1$, and to the fact that every vector valued semigroup is a vector valued subsemigroup of a (binary) semigroup ([1]) in the case $m=1$, $p>0$.

We also note that in the proof of our results we use some ideas from the paper [3], where a convenient description of free vector valued semigroups is given.

1. Let $(Q;[\![\ ]\!])$ be an $(m+p+q,m+p)$-semigroup and let $(\overline{Q};\{[\ ]_i^s \mid s\geq 1,\ i\in\{1,2,\ldots,m\}\})$ be the absolutely free universal algebra with a base $Q$, where $[\ ]_i^s$ is an $m+s$-ary operator symbol for any $s\geq 1$ and any $i\in\{1,2,\ldots,m\}$. We will give below a description of this algebra.

If $X$ is a non empty set, then $X^*$ is the set of all finite sequences on $X$ (including the empty sequence). (In other words, $X^*$ is the free monoid (freely) generated by $X$.) If $x=x_1x_2\ldots x_r$, where $x_\nu\in X$, then $r$ is said to be the *dimension* of $x$, and is denoted by $d(x)$. The empty sequence (denoted by 1) has, by definition, dimension zero. Also, we will write $x_1^r$ instead of $x_1x_2\ldots x_r$.

We put $Q_0 = Q$, $\mathbb{N}_m = \{1,2,\ldots,m\}$ and

$$C_s = \{u \in Q_s^* \mid d(u) \geq m+1\}$$

$$Q_{s+1} = Q_s \cup C_s \times \mathbb{N}_m,$$

and

$$\overline{Q} = \bigcup_{s \geq 0} Q_s.$$

Thus we have:

<u>1.1</u>. $u \in \overline{Q}$ <u>iff</u> $u \in Q$ or $u=(v,i)$, <u>where</u> $v \in \overline{Q}^*$, $d(v) \geq m+1$, $i \in \mathbb{N}_m$.

Now, the algebra $(\overline{Q};\{[\ ]_i^s \mid s \geq 1, i \in \mathbb{N}_m\})$ is defined in the following way:

If $s \geq 1$, $u_1,u_2,\ldots,u_{s+m} \in \overline{Q}$, and $i \in \mathbb{N}_m$, then:

$$[u_1^{s+m}]_i^s = (u_1^{s+m}, i).$$

By putting:

$$[u_1^{s+m}]^s = (v_1^m) \Longleftrightarrow [u_1^{s+m}]_i^s = v_i,$$

we obtain the absolutely free vector valued algebra $(\overline{Q};[\ ]^s, s \geq 1)$ with a base Q, where $[\ ]^s$ is an $(m+s,m)$-operation on $\overline{Q}$.

<u>Remark</u>: We will use below the following notations:

(i) a,b,c,d (with or without indexes) will always denote elements of Q.

(ii) x,y,z,u,v,w,t (with or without indexes) will always denote elements of $\overline{Q}^*$.

(iii) $(x,i) \in \overline{Q}$ will always mean that $x \in \overline{Q}^*$ is such that $d(x) \geq m+1$.

(iv) Sometimes, for technical reasons, an element $u_i \in \overline{Q}$ will be denoted by $(u_1^m, i)$, where $u_\nu \in \overline{Q}$, $i \in \mathbb{N}_m$. (Note that $(u_1^m,i) \notin \overline{Q}$ by the construction of $\overline{Q}$.)

We assume that the meaning of „an appearence of u in v", and „w is obtained from v by substitution of an appearence of u in v by t", are clear. Also, the validness of the two properties below are evident.

1.2. _If_ $u \in \overline{Q}$ _and if_ v _is obtained when an appearence of_ $u' \in \overline{Q}$ _in_ u _is substituted by_ $v' \in \overline{Q}$, _then_ $v \in \overline{Q}$.

1.3. _If_ $(xyz,i) \in \overline{Q}$, _and_ $(y,\nu) \in \overline{Q}$, _where_ $d(xz) \geq 1$, _then_ $u = (x(y,1)(y,2)\ldots(y,m)z,i) \in \overline{Q}$ _as well_[1].

We define two relations $\vdash_1$ and $\vdash_2$ in $\overline{Q}$ as follows.

If $u,v \in \overline{Q}$, then

$\vdash_1$ : $u \vdash_1 v$ iff v is obtained from u when an appearence of $(b_1^{m+p},i)$ is substituted by $(a_1^{m+p+sq},i)$, where $[\![a_1^{m+p+sq}]\!] = (b_1^{m+p})$ in $(Q;[\![\;]\!])$. (If p=0, then $(b_1^m,i)=b_i$.)

$\vdash_2$ : $u \vdash_2 v$ iff v is obtained from u when an appearence of $(xyz,j) \in \overline{Q}$ is substituted by $(x(y,\nu)_{\nu=1}^{m}z,j)$, where $d(y) \geq m+1$

Then, we define a relation ~ by:

~: $u \sim v$ iff $u \vdash_1 v$ or $v \vdash_1 u$ or $u \vdash_2 v$ or $v \vdash_2 u$, i.e. ~ is the symmetric extension of the union of $\vdash_1$ and $\vdash_2$ .

Finally, let ≈ be the reflexive and transitive extension of ~, i.e.

≈: $u \approx v$ iff there exist $w_o, w_1, \ldots, w_r \in \overline{Q}$, such that $u=w_o$, $v=w_r$, $r \geq 0$, and $w_{j-1} \sim w_j$ for each $j \in \{1,2,\ldots,r\}$.

Thus:

1.4. ≈ _is an equivalence relation on_ $\overline{Q}$. (Namely, it is the smallest equivalence relation containing $\vdash_1$ and $\vdash_2$ .)

The following lemma is true:

LEMMA 1. $(b_1^{m+p},i) \approx (c_1^{m+p},i) \Longleftrightarrow (b_1^{m+p},i) = (c_1^{m+p},i)$.

Namely, Lemma 1 is a consequence of Lemma 2, given below. To state Lemma 2, we will denote by Q´ the set

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[1] Sometimes we use the abbreviated notation $(x(y,\nu)_{\nu=1}^{m}z,i)$ for $(x(y,1)\ldots(y,m)z,i)$.

$$Q' = \{(a_1^{m+p}, i) \mid a_\nu \in Q, \ \ i \in \mathbb{N}_m\}.$$

(If p=0, then $Q'=Q$.)

LEMMA 2. <u>There exists a map</u> $\xi : \overline{Q} \to \overline{Q}$ <u>with the properties</u>:

(i) $\xi(u) = u$, <u>for every</u> $u \in Q'$;

(ii) $u \sim v$ <u>and</u> $(\xi(u) \in Q'$ <u>or</u> $\xi(v) \in Q') \Longrightarrow \xi(u) = \xi(v)$.

Let us assume that Lemma 2 is true, and $(b_1^{m+p}, i) \approx (c_1^{m+p}, i)$. Then, there exist $w_0, w_1, \dots, w_r \in \overline{Q}$ such that $w_0 = (b_1^{m+p}, i)$, $w_r = (c_1^{m+p}, i)$ and $w_{j-1} \sim w_j$ for each $j \in \{1, 2, \dots, r\}$. Since $w_0, w_r \in Q'$, $\xi(w_0) = w_0$, $\xi(w_r) = w_r$, and also $\xi(w_0) = \xi(w_1) = \dots = \xi(w_r)$, i.e. $(b_1^{m+p}, i) = w_0 = w_r = (c_1^{m+p}, i)$.

The proof of Lemma 2, that is the construction of the map $\xi$, will be given in the next part of this paper. Here we will show that Theorem is a consequence of Lemma 1.

First we state two propositions.

<u>1.5</u>. $\approx$ <u>is a congruence on the algebra</u> $(\overline{Q}; [\ ]^s, s \geq 1)$.

<u>Proof</u>: It is clear that if $u, v \in \overline{Q}$, $x, y \in \overline{Q}^*$ are such that $u \vdash_\alpha v$, $d(xy) = m+s-1$, $s \geq 1$, then $(xuy, i) \vdash_\alpha (xvy, i)$ for every $i \in \mathbb{N}_m$, and this implies that $\approx$ is a congruence.

Denote the factor algebra $(\overline{Q}/\approx; [\ ]^s, s \geq 1)$ by $(P; [\ ]^s, s \geq 1)$, and the operation $[\ ]^1$ by $[\ ]$. If $x, y, z \in \overline{Q}^*$ are such that $d(y) \geq m+1$, $d(xz) \geq 1$, then for every $i \in \mathbb{N}_m$ we have

$$(xyz, i) \vdash_2 (x(y, \nu)_{\nu=1}^m z, i),$$

i.e. $(xyz, i) \approx (x(y, \nu)_{\nu=1}^m z, i)$, and this implies that:

<u>1.6</u>. $(P; [\ ])$ <u>is an</u> $(m+1, m)$-<u>semigroup</u>.

Now we are ready to show that Theorem is a consequence of Lemma 1.

First we consider the case p=0. Thus, we have an $(m+q, m)$-semigroup $(Q; [\![\ ]\!])$. Then, $Q' = Q$, and by Lemma 1 we have: $a \approx b \Longrightarrow a = b$. Therefore we can assume that $Q \subseteq P$, and if $[\![ a_1^{m+sq} ]\!] = (b_1^m)$, then $(a_1^{m+sq},i) \approx b_i$ for each $i \in \mathbb{N}_m$, and this implies that $[a_1^{m+sq}] = (b_1^m)$ in $(P;[\ ])$.

Conversely, let $[a_1^{m+sq}] = (b_1^m)$ in $(P;[\ ])$, i.e. $(a_1^{m+sq},i) \approx$ $\approx b_i$ for each $i \in \mathbb{N}_m$, and let $[\![a_1^{m+sq}]\!] = (c_1^m)$ in $(Q;[\![\ ]\!])$. Then we have $c_i \approx (a_1^{m+sq},i)$ and thus $b_i \approx c_i$. By Lemma 1, this implies $b_i = c_i$. This completes the proof of Theorem 1, i.e. of Theorem for $p=0$.

It remains the case $p > 0$.

Let $(Q;[\![\ ]\!])$ be an $(m+p+q,m+p)$-semigroup, and let $\overline{Q}$ and $(P;[\ ])$ be defined as before. We have that $a \approx u \Longrightarrow a = u$, for neither of the relations $a \vdash_{\overline{1}} u$, $u \vdash_{\overline{1}} a$, $a \vdash_{\overline{2}} u$, $u \vdash_{\overline{2}} a$ holds. Thus we can assume that $Q \subseteq P$.

If $[\![a_1^{m+p+sq}]\!] = (b_1^{m+p})$ in $(Q;[\![\ ]\!])$, then $(a_1^{m+p+sq},i) \approx$ $\approx (b_1^{m+p},i)$ for each $i \in \mathbb{N}_m$, and thus we have $[a_1^{m+p+sq}] = [b_1^{m+p}]$ in $(P;[\ ])$. Assume that we also have $[a_1^{m+p+sq}] = [c_1^{m+p}]$ in $(P;[\ ])$. Then $(b_1^{m+p},i) \approx (c_1^{m+p},i)$, and this, by Lemma 1, implies that $(b_1^{m+p},i) = (c_1^{m+p},i)$, i.e. $b_\nu = c_\nu$ for any $\nu \in \{1,2,\ldots,m+p\}$. This completes the proof of Theorem for $p > 0$.

2. Here we will construct a mapping $\xi:\overline{Q} \to \overline{Q}$ such that the conditions of Lemma 2 will be satisfied.

Define the length $|x|$ of an element $x \in \overline{Q}^*$ by

$$|1| = 0, \quad |a| = 1, \quad |(u,i)| = |u|, \quad |tv| = |t| + |v|,$$

where $(u,i) \in \overline{Q}$.

The mapping $\xi:\overline{Q} \to \overline{Q}$ will be defined by induction on the length of elements of $\overline{Q}$ as follows:

(0) $\quad \xi(a) = a.$

Let $u=(x,i) \in \overline{Q}$, where $x = x_1 x_2 \cdots x_s$, $x_1,\ldots,x_s \in \overline{Q}$. Assume that for every $v \in \overline{Q}$, such that $|v|<|u|$, $\xi(v) \in \overline{Q}$ is well defined, and that the following statements hold:

$$|\xi(v)| \le |v|, \quad \xi(\xi(v)) = \xi(v), \quad \xi(v) \ne v \Longleftrightarrow |\xi(v)| < |v| \qquad (2.1)$$

Thus $\xi(x_\nu)$ is a well defined element of $\overline{Q}$ for every $\nu \in \mathbb{N}_s$, and if we put

$$y = \xi(x_1)\cdots\xi(x_s)$$

then by (2.1) $y \neq x$ iff $|y|<|x|$. Assume that $y \neq x$. Then we define $\xi(u)$ by:

(i) $\xi(u) = \xi(y,i)$.

Assume, now, that $y = x$, and that $x$ has the following form:

$$x = x'(y_1,1)(y_2,2)\cdots(y_m,m)x''$$

where $x'$ has the least possible length. Now, $\xi(u)$ is defined by

(ii) $\xi(u) = \xi(x'y_1x'',i)$.

If $x = a_1^{m+p+sq}$, $s \geq 1$ and if $[\![a_1^{m+p+sq}]\!] = (b_1^{m+p})$, then $\xi(u)$ is defined by:

(iii) $\xi(u) = (b_1^{m+p},i)$

(Note that in the case $p=0$ $(b_1^m,i)$ denotes $b_i$.)

If $\xi(u)$ is not defined by either of the cases (0)-(iii), then we put

(iv) $\xi(u) = u$.

Thus $\xi:\overline{Q} \to \overline{Q}$ is a well defined mapping.

We can extend $\xi$ to a mapping $\xi^*:\overline{Q}^* \to \overline{Q}^*$ by the usual way. Namely,

$\xi^*(1) = 1$, $u \in \overline{Q} \Longrightarrow \xi^*(u) = \xi(u)$, $\xi^*(xy) = \xi^*(x)\xi^*(y)$

Further on we will write $\xi$ instead of $\xi^*$.

We say that $x$ is <u>reducible</u> if $\xi(x) \neq x$ or $x=x'(y_1,1)\cdots\cdots(y_m,m)x''$, where $d(y_\nu) \geq m+1$ for $\nu \in \mathbb{N}_m$. Otherwise $x$ is said to be <u>reduced</u>.

The following nine propositions are clear by the definition of $\xi$.

2.1. $\xi(\xi(u)) = \xi(u)$.

2.2. $\xi(u) \neq u \iff |\xi(u)| < |u|$.

2.3. $\xi(xyz) = \xi(x\xi(y)z)$.

2.4. $(xyz,i)\in\overline{Q} \implies \xi(xyz,i) = \xi(x\xi(y)z,i)$.

2.5. If $p > 0$ then:

a) $\xi(u)\in Q \iff u\in Q$;

b) $\xi(a_1^{m+r},i) = (a_1^{m+r},i)$, $r\in \mathbb{N}_p$.

2.6. If $(x,i)\in\overline{Q}$ and $\xi(x,i) = (y,j)\notin Q$, then $i = j$.

2.7. If $(x,i)\in\overline{Q}$ is such that $\xi(x,i)\notin Q$, then $\xi(x,j)\notin Q$ for every $j\in \mathbb{N}_m$.

2.8. Let $v\in\overline{Q}$ be obtained from $u\in\overline{Q}$ in such a way that one appearence of $u'\in\overline{Q}$ is substituted by $v'\in\overline{Q}$. If $\xi(u') = \xi(v')$ then $\xi(u) = \xi(v)$.

2.9. If $x=a_1^{\beta}(y,\lambda)z$, $\beta \geq 0$, $\lambda \geq 2$, $\xi(y,\lambda) = (\overline{y},\lambda)\in Q$, $(x,i)\in\overline{Q}$, then $\xi(x,i)\notin Q'$.

2.10. Let $xz \neq 1$, $(y_\nu,\nu)\in\overline{Q}$ and suppose that $\xi(y_\nu,\nu) = (y_\nu,\nu)$ or $\xi(y_\nu,\nu)\notin Q'$. Then:

$$\xi(x(y_1,1)\cdots(y_m,m)z,i) = \xi(xy_1z,i) \tag{2.2}$$

Proof: Let $\xi(y_\nu,\nu) = (\overline{y}_\nu,\nu)$. By induction on the lengths of the elements of $\overline{Q}^*$ it can be easily seen that (2.2) is true if anyone of the following four conditions is satisfied:

a) $\xi(xy_1\cdots y_mz) \neq xy_1\cdots y_mz$;

b) $x$ is reducible;

c) $\overline{y}_\lambda \neq y_\lambda$ for some $\lambda \geq 2$;

d) $y$ is not reduced.

In the case when none of a), b), c), d) is true, then (2.2) follows by the definition of $\xi$.

2.11. Let $[\![a_1^{m+p+sq}]\!] = (b_1^{m+p})$ in $(Q;[\![\ ]\!])$, and suppose that $v \in \overline{Q}$ is obtained from $u \in \overline{Q}$ when an appearence of $a_1^{m+p+sq}$ in u is replaced by $b_1^{m+p}$. Then

$$\xi(u) \in Q' \text{ or } \xi(v) \in Q' \Longrightarrow \xi(u) = \xi(v).$$

Proof: There exists an $\alpha \geq 0$ and $u_o, u_1, \ldots, u_\alpha$, $v_o, v_1, \ldots, v_\alpha \in \overline{Q}$ such that

$$
\begin{aligned}
&u = u_o, \ v = v_o \\
&u_\lambda = (x_\lambda u_{\lambda+1} z_\lambda, i_\lambda), \quad v_\lambda = (x_\lambda v_{\lambda+1} z_\lambda, i_\lambda), \ 0 \leq \lambda < \alpha \qquad (2.3) \\
&u_\alpha = (x_\alpha a_1^{m+p+sq} z_\alpha, i_\alpha), \quad v_\alpha = (x_\alpha b_1^{m+p} z_\alpha, i_\alpha).
\end{aligned}
$$

It is clear that if one of the following conditions

a) $\xi(x_o \cdots x_\alpha z_\alpha \cdots z_o) \neq x_o \cdots x_\alpha z_\alpha \cdots z_o$,

b) $x_\lambda$ is reducible for some $\lambda$,

is satisfied, then we can obtain a sequence of elements of $\overline{Q}$: $\overline{u} = \overline{u}_o$, $\overline{u}_1, \ldots, \overline{u}_\alpha$, $\overline{v} = \overline{v}_o, \overline{v}_1, \ldots, \overline{v}_\alpha$ such that (2.3) is satisfied, and moreover

$$\xi(u) = \xi(\overline{u}), \ \xi(v) = \xi(\overline{v}), \ |\overline{u}| < |u|, \quad |\overline{v}| < |v|,$$

which implies $\xi(u) = \xi(v)$ by induction.

Thus we can assume that:

a′) $\xi(x_\lambda) = x_\lambda$, $\quad \xi(z_\lambda) = z_\lambda$ for any $\lambda$, and

b′) $x_\lambda$ is reduced for any $\lambda$.

If there exists a $\lambda$ such that $\xi(u_\lambda) = \xi(v_\lambda)$, then by 2.8 we have $\xi(u) = \xi(v)$.

Consider the case $\alpha = 0$, i.e.

$$u = (xa_1^{m+p+sq} z, i), \quad v = (xb_1^{m+p} z, i)$$

and $[\![a_1^{m+p+sq}]\!] = (b_1^{m+p})$ in $(Q;[\![\ ]\!])$.

If z is reducible then we can again obtain two elements

$$\overline{u} = (xa_1^{m+p+sq} z', i), \quad \overline{v} = (xb_1^{m+p} z', i)$$

such that

$$\xi(u) = \xi(\bar{u}),\ \xi(v) = \xi(\bar{v}),\quad |\bar{u}| < |u|,\ |\bar{v}| < |v|$$

and the proof follows by induction. So, we can assume that z is reduced.

Now, $\xi(u) \in Q'$ or $\xi(v) \in Q'$, iff $x = c_1^\beta$, $z = d_1^\gamma$ where $\beta+\gamma = rq$, $r \geq 0$. Then we have:

$$\begin{aligned} \xi(u) &= ([\![c_1^\beta a_1^{m+p+sq} d_1^\gamma]\!], i) = \\ &= ([\![c_1^\beta [\![a_1^{m+p+sq}]\!] d_1^\gamma]\!], i) = \\ &= ([\![c_1^\beta b_1^{m+p} d_1^\gamma]\!], i) = \\ &= \xi(v). \end{aligned}$$

There remains the case $\alpha > 0$. By the same argument as in the case $\alpha = 0$ we can assume that $z_\alpha$ is reduced. Also as in the case $\alpha = 0$ we can conclude that

$$\xi(u_\alpha) \neq u_\alpha \quad \text{iff} \quad \xi(u_\alpha) = \xi(v_\alpha) \in Q',$$

and by <u>2.8</u> we will have $\xi(u) = \xi(v)$.

Thus, we can assume that $\xi(u_\alpha) = u_\alpha$, and then we will also have $\xi(v_\alpha) = v_\alpha$.

The fact that $\xi(u_0) \in Q'$ or $\xi(v_0) \in Q'$ and $\alpha > 0$ implies that $\xi(u_0) \neq u_0$, $\xi(v_0) \neq v_0$. Let $\beta$ be the largest number such that $\xi(u_\beta) \neq u_\beta$ or $\xi(v_\beta) \neq v_\beta$. Then we have $\beta < \alpha$ and

$$\xi(u_{\beta+1}) = u_{\beta+1},\quad \xi(v_{\beta+1}) = v_{\beta+1}.$$

Since it is assumed that $x_\beta$ is reduced, from the equalities

$$u_\beta = (x_\beta u_{\beta+1} z_\beta, i_\beta),\quad v_\beta = (x_\beta v_{\beta+1} z_\beta, i_\beta)$$

it follows that $\xi(u_\beta) \neq u_\beta$ or $\xi(v_\beta) \neq v_\beta$, iff one of the following three statements hold:

1) $x_\beta = x'(t_1,1)\cdots(t_{\nu-1},\nu-1)$, $i_{\beta+1} = \nu > 1$, $z_\beta = (t_{\nu+1},\nu+1)\cdots(t_m,m)z'$;

2) $i_{\beta+1} = 1$, $z_\beta = (t_2,2)\cdots(t_m,m)z'$;

3) $z_\beta = z'(t_1,1)\cdots(t_m,m)z''$ and $x_\beta u_{\beta+1}z'$, $x_\beta v_{\beta+1}z'$ are reduced.

In the case 1) we have

$$\xi(u_\beta) = \xi(x't_1z',i_\beta) = \xi(v_\beta)$$

which implies $\xi(u_o) = \xi(v_o)$ by <u>2.8</u>.

In the case 2) we have

$$\xi(u_\beta) = \xi(\overline{u}_\beta),\ \xi(v_\beta) = \xi(\overline{v}_\beta),$$
$$|\overline{u}_\beta|<|u_\beta|,\ |\overline{v}_\beta| = |v_\beta|,$$

where

$$\overline{u}_\beta = (x_\beta x_{\beta+1}u_{\beta+2}z_{\beta+1}z',i_\beta),\quad \overline{v}_\beta = (x_\beta x_{\beta+1}v_{\beta+2}z_{\beta+1}z',i_\beta)$$

and the conclusion follows by induction.

In the case 3) we have the same situation as in 2) where

$$\overline{u}_\beta = (x_\beta u_{\beta+1}z't_1z'',i_\beta),\quad \overline{v}_\beta = (x_\beta u_{\beta+1}z't_1z'',i_\beta).$$

This completes the proof of <u>2.11</u>.

As a corollary from <u>2.8</u> we obtain the following proposition:

<u>2.12</u>. <u>If</u> $u,v\in\overline{Q}$ <u>are such that</u> $u\vdash_{\overline{1}} v$, <u>then</u> $\xi(u) = \xi(v)$.

To complete the proof of Lemma 2 we need the following proposition:

<u>2.13</u>. <u>Let</u> $u,v\in\overline{Q}$ <u>and</u> $u\vdash_{\overline{2}} v$. <u>If</u> $\xi(u)\in Q'$ <u>or</u> $\xi(v)\in Q'$, <u>then</u> $\xi(u) = \xi(v)$.

<u>Proof</u>: From $u\vdash_{\overline{2}} v$ it follows that there exist an $\alpha \geq 0$, $u_\lambda,v_\lambda\in\overline{Q}$ such that

$u = u_o,\ \ v = v_o$

$u_\lambda = (x_\lambda u_{\lambda+1}z_\lambda,i_\lambda),\quad v_\lambda = (x_\lambda v_{\lambda+1}z_\lambda,i_\lambda),\quad 0 \leq \lambda < \alpha$

$u_\alpha = (x_\alpha yz_\alpha,i_\alpha),\quad v_\alpha = (x_\alpha(y,1)\cdots(y,m)z_\alpha,i_\alpha).$

By the same arguments as in the proof of <u>2.11</u> we can assume that:

a) $\xi(y) = y$, $\xi(x_\lambda) = x_\lambda$, $\xi(z_\lambda) = z_\lambda$, for every $\lambda$;

b) $x_\lambda$ is reduced for every $\lambda$.

If $\xi(y,\nu) \notin Q'$ or $\xi(y,\nu) = (y,\nu)$ then by 2.10 we have $\xi(u_\alpha) = \xi(v_\alpha)$, and, by 2.8, $\xi(u) = \xi(v)$.

Thus we can assume that $\xi(y,\nu) = (b_1^{m+p},\nu) \in Q'$ and $\xi(y,\nu) \neq (y,\nu)$

Let y be reducible and let $y = y'(y_1,1)\cdots(y_m,m)y''$, where $y'$ is reduced. Then, $x_\alpha y'$ is reduced, for if it were reducible, then we would have

$$x_\alpha = x'(t_1,1)\cdots(t_{\gamma-1},\gamma-1), \quad y' = (t_\gamma,\gamma)\cdots(t_m,m)y''', \ \gamma \geq 2,$$

but this is impossible by 2.9.

Thus we have:

$$\xi(u_\alpha) = \xi(\overline{u}_\alpha), \quad \xi(v_\alpha) = \xi(\overline{v}_\alpha)$$

where

$$\overline{u}_\alpha = (x_\alpha y' y_1 y'' z_\alpha, i_\alpha), \quad \overline{v}_\alpha = (x_\alpha (y' y_1 y'',1)\cdots(y' y_1 y'',m) z_\alpha, i$$

and this implies there exist $\overline{u},\overline{v} \in \overline{Q}$ such that

$$\xi(u) = \xi(\overline{u}), \quad \xi(v) = \xi(\overline{v}), \quad |\overline{u}| < |u|, \ |\overline{v}| < |v|, \quad \overline{u} \vdash_2 \overline{v}.$$

Thus, the conclusion follows by induction.

Therefore we can assume that y is reduced. Then $\xi(y,\nu) \in Q'$, $\xi(y,\nu) \neq (y,\nu)$ is possible only if $y = a_1^{m+p+sq}$, $s \geq 1$. If $[a_1^{m+p+sq}] = (b_1^{m+p})$ in $(Q;[\ ])$, and if we put

$$\overline{v}_\alpha = (x_\alpha b_1^{m+p} z_\alpha, i_\alpha)$$
$$\overline{v}_\lambda = (x_\lambda \overline{v}_{\lambda+1} z_\lambda, i_\lambda)$$

we obtain that $\xi(v) = \xi(\overline{v}_0)$, and by 2.11 we have $\xi(u) = \xi(\overline{v}_0)$.

This completes the proof of 2.13.

Finally conclude that (i) of Lemma 2 is a corollary of 2.5 b), and (ii) is a corollary of 2.12 and 2.13.

3. We make a few more remarks.

The (m+1,m)-semigroup (P;[ ]), obtained in 1, has a universal property of this kind:

If (P´;[ ]´) is any (m+1,m)-semigroup, such that $Q \subseteq P'$ and

$$[\![a_1^{m+p+q}]\!] = (b_1^{m+p}) \text{ in } (Q;[\![\ ]\!]) \Longleftrightarrow$$
$$\Longleftrightarrow [a_1^{m+p+q}]' = [b_1^{m+p}]' \text{ in } (P';[\ ]')$$

for all $a_\nu, b_\nu \in Q$, then there exists a unique homomorphism $\xi:(P;[\ ]) \to (P';[\ ]')$, such that $\xi(a)=a$ for all $a \in Q$.

It should be noted that when Theorem 1´ and Theorem 2 are considered, the (m+k,m)-semigroup used there has not this universal property. Nevertheless, by slightly modified construction of $\overline{Q}$, one can get an (m+k,m)-semigroup with wanted universal property. Namely, we first construct the absolutely free vector valued algebra $\overline{Q}'$ of type $\{[\ ]^s \mid s \geq 1\}$ freely generated by Q, where $[\ ]^s$ is a symbol for an (m+sk,m)-operation for all $s \geq 1$. Further, we define the relations $\vdash_{\overline{1}}$, $\vdash_{\overline{2}}$, $\approx$ in the same manner as in 1. In such a way the obtained (m+k,m)-semigroup (P´;[ ]´) is the wanted universal (m+k,m)-semigroup for the given (m+p+q, m+p)-semigroup (Q;[[ ]]). We can realize a proof of this fact almost without any changes, but we actually do not need such a proof, since the universal property of (P´,[ ]´) is clear if we have had proved Lemma 1 for k = 1.

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