ONE-DIMENSIONAL (4,2)-LIE GROUPS

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Abstract. *In this paper we examine the existence of one-dimensional (4,2)-Lie groups, and show that such groups do exist.*

§0. INTRODUCTION

The definition of (n,m)-groups, $n-m=k \geq 0$ is given in [1]. A pair (G,[ ]) where $[\ ]:G^n \to G^m$ is a map satisfying two axioms (associativity and solvability of equations, see [1]), is called an (n,m)-group. For n=4, m=2, these axioms are:

$$[[xyzt]uv] = [x[yztu]v] = [xy[ztuv]]; \text{ and} \tag{0.1}$$

$$(\forall \underline{a},\underline{b} \in G^2)\ (\exists \underline{x},\underline{y} \in G^2) \qquad [\underline{x}\ \underline{a}] = \underline{b} = [\underline{a}\ \underline{y}]. \tag{0.2}$$

In [2] it is proved that if (G,[ ]) is a (4,2)-group then $(G^2,*)$, where $(x,y)*(z,t)=[xyzt]$, is a group with identity element (e,e), $(x,e)*(e,y)=(x,y)$, and the map $\psi:G^2 \to G^2$ defined by $\psi(x,y)=(e,x)*(y,e)$ is an involutive (i.e. $\psi^2=id_G$) automorphism. We say that $(G^2,*)$ is the associated group to (G,[ ]). The converse is also true.

Proposition 0.1. *Let $(G^2,*)$ be a group with identity element (e,e), and let $\psi:G^2 \to G^2$ be an involutive automorphism such that $\psi(x,y)=(e,x)*(y,e)$. Then (G,[ ]), where $[xyzt]=(x,y)*(z,t)$, is a (4,2)-group.*

Proof. The fact that ψ is an involutive automorphism, and

$$\psi(x,y)=(e,x)*(y,e)=(e,x)*(e,e)*(e,e)*(y,e)$$
$$=\psi(x,e)*\psi(e,y)=\psi((x,e)*(e,y)),$$

imply that $(x,e)*(e,y)=(x,y)$.

Now,

$$[[xyzt]uv] = ((x,y)*(z,t))*(u,v) =$$
$$=(x,y)*((z,t)*(u,v))=[xy[ztuv]].$$

Moreover, if $[yztu]=(y,z)*(t,u)=(a,b)$, then

$$
\begin{aligned}
[x[yztu]v] &= [xabv] = (x,a)*(b,v) = \\
&= (x,e)*(e,a)*(b,e)*(e,v) \\
&= (x,e)*\psi(a,b)*(e,v) \\
&= (x,e)*\psi((y,z)*(t,u))*(e,v) \\
&= (x,e)*\psi(y,z)*\psi(t,u)*(e,v) \\
&= (x,e)*(e,y)*(z,e)*(e,t)*(u,e)*(e,v) \\
&= (x,y)*(z,t)*(u,v) \\
&= [[xyzt]uv].
\end{aligned}
$$

Hence, [ ] satisfies (0.1).

The definition of [ ] and the fact that $(G^2,*)$ is a group, imply that [ ] satisfies (0.2). ||

Remark 0.2. Let $(G^2,*)$ and ψ be as in Proposition 0.1. Then, the condition $\psi(x,y)=(e,x)*(y,e)$ is equivalent to the condition

$$(x,e)*\psi(y,e) = (x,y). \tag{0.3}$$

We say that a (4,2)-group is commutative if the associated group is commutative [3].

Let G be a topological space, and let (G,[ ]) be a (4,2)-group. We say that (G,[ ]) is a continuous (4,2)-group if [ ] is a continuous map from G^4 to G^2. We say that (G,[ ]) is a topological (4,2)-group, if the associated group $(G^2,*)$ is a topological group. In other words, (G,[ ]) is a topological (4,2)-group, if the map $\xi:G^4 \to G^2$ defined as $\xi(a,b,c,d)=(a,b)*(c,d)^{-1}$, where $^{-1}:G^2 \to G^2$ is the inverse map for $*$ in G^2, is a continuous map.

If G is a connected topological one-dimensional manifold*), and (G,[ ]) is a topological group, then $(G^2,*)$ is a 2-dimensional Lie group**). In this case we say that (G,[ ]) is a (4,2)-Lie group.

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*) *A topological n-dimensional manifold is a paracompact Hausdorff topological space M, such that each $x\in M$ has a neighbourhood homeomorphic to R^n.*

**) *Here we use the positive answer to the Fifth Hilbert problem; i.e. a locally euclidean group is a Lie group (see [6,9]).*

In this paper we examine the existence of one-dimensional (4,2)-Lie groups, and show that such groups do exist, in comparison with the fact that one-dimensional continuous (3,2)-groups do not exist (see [4]). Moreover, we show that there exist much more non-isomorphic one-dimensional (4,2)-Lie groups, than non-isomorphic Lie groups over R^2 and S^1xS^1. The paper is divided into four parts. In 1. we present some necessary notions and results from Lie groups and Lie algebras. In 2. we examine commutative (4,2)-Lie groups over R. In 3. we examine non-commutative (4,2)-Lie groups over R, and in 4. we examine (4,2)-Lie groups over S^1.

§1. SOME RESULTS ABOUT THE LIE GROUPS AND LIE ALGEBRAS

Let g be an r-dimensional abstract Lie algebra over the field R of real numbers. Then, there exists a simply connected r-dimensional Lie group SG, whose Lie algebra is isomorphic to g. The group SG is uniquely determined by g up to local analytic isomorphism (converse of Lie's Third Theorem, see [8]). The above theorem asserts that there is not a bijection between Lie groups and Lie algebras. Many Lie groups may have the same Lie algebra, but among all of these groups there exists only one which is simply connected. It is denoted by SG. It is shown that the enumeration of all posible Lie groups with the same Lie algebra reduces to the problem of finding all discrete subgroups of the centre of SG ([8]). In fact, if (SG,*) is a connected and simply-connected Lie group with g as a Lie algebra, then any Lie group with g as a Lie algebra is isomorphic to a factor group SG/D, where D is a discrete subgroup of the centre of SG.

Since there are only two one-dimensional topological manifolds up to homeomorphism, R and S^1, we consider these two cases separately.

After a simple calculation, one can show that there are exactly two non-isomorphic Lie algebras over the field R ([5], ex. 96). Their correspondind Lie groups over R^2 are:

$$(x,y)*(z,t) = (x+z,y+t) \text{ and} \tag{1.1}$$

$$(x,y)*(z,t) = (x+ze^{-y},y+t). \tag{1.2}$$

Since R^2 is a connected simply-connected manifold, these two groups are the only non isomorphic Lie groups over R^2. The first group is commutative, and the second one is not. Hence, there are two classes of (4,2)-Lie groups over R. The first class contains all the (4,2)-Lie groups whose associated groups over R^2 are isomorphic to the group given by (1.1). The second class contains all the (4,2)-groups whose associated groups over R^2 are isomorphic to the group given by (1.2).

§2. COMMUTATIVE (4,2)-LIE GROUPS OVER R

The group over R^2 given by (1.1) induces a (4,2)-Lie group over R in a trivial way (this (4,2)-group is a trivial (4,2)-group [2]). The corresponding involutive automorphism ψ is given by $\psi(x,y)=(y,x)$. The other (4,2)-Lie groups in this class, can be obtained by a change of coordinates. In fact, the coordinates have to be chosen in such a way, so that the condition (0.3) is valid in the new coordinates. To determine the new coordinates, it suffices to choose a smooth curve $(x(s),y(s))$ as the first coordinate axis which passes through the origin (0,0), and satisfies some additional properties. Using (0.3), we find the correspondence between the two coordinate systems. It is given by

$$h(s,t) = (x(s),y(s))*\psi(x(t),y(t)), \text{ i.e.}$$
$$h(s,t) = (x(s)+y(t),y(s)+x(t)),$$

where $h:R^2 \to R^2$ is a homeomorphism. Note, that some (4,2)-Lie groups obtained by this procedure may be isomorphic. For example, if $x(s)=s$, $y(s)=as$, $a\neq 1$, the (4,2)-Lie group obtained by this procedure, is isomorphic to the (4,2)-group induced by (1.1). So, we must look for another example. Let $x(s)=s$. Then the problem reduces to finding a smooth function $y(s)$ such that the system of equations

$$\begin{aligned} s+y(t) &= p \\ t+y(s) &= q \end{aligned} \tag{2.1}$$

has a unique solution (s,t) for given $p,q \in R$. If t is a unique solution of the equation

$$t+y(p-y(t)) = q \tag{2.2}$$

then (s,t) where $s=p-y(t)$, is a unique solution of (2.1). Thus, we search for a function y such that for a given $p \in R$, the function $f:R \to R$ given by $f(z)=z+y(p-y(z))$ is a homeomorphism. It is sufficient to take for y a bounded differentiable function, such that $|y'(z)| \le c < 1$. In this case $\lim_{z\to-\infty} f(z)=-\infty$, $\lim_{z\to\infty} f(z)=\infty$, and f is a monotonically increasing function, since $f'(z)= =1+y'(p-y(z)).(-y'(z)) > 0$. For example, such a function is $y(z)=(1/2)\sin(z)$. The implicit function theorem implies that $h:R^2 \to R^2$ defined by

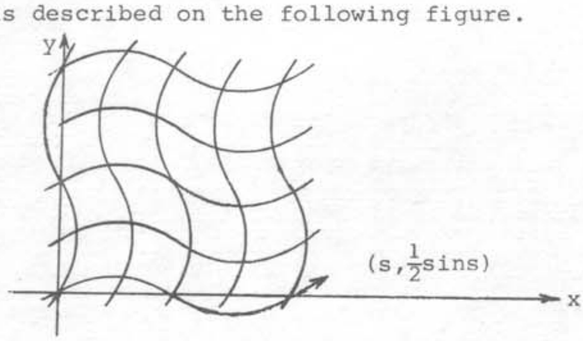
$$h(s,t) = (s+(1/2)\sin(t),\ t+(1/2)\sin(s)) \tag{2.3}$$

is a homeomorphism.

The previous discussion shows that (R,[ ]) where

$$\begin{aligned}[stuv] &= h^{-1}(h(s,t)*h(u,v)) \\ &= h^{-1}((s+(1/2)\sin(t),t+(1/2)\sin(s))* \\ &\qquad *(u+(1/2)\sin(v),v+(1/2)\sin(u))) = \\ &= h^{-1}(s+u+(\sin t+\sin v)/2,t+v+(\sin s+\sin u)/2)\end{aligned} \tag{2.4}$$

is a (4,2)-Lie group. The correspondence between the two coordinate systems is described on the following figure.



The (4,2)-Lie group (R,[ ]) defined by (2.4) is a commutative (4,2)-group but not isomorphic to the (4,2)-Lie group induced by (1.1), because $(R\times\{0\},*)$ is not a subgroup of $(R^2,*)$ for the first, and is a subgroup of $(R^2,*)$ for the second (4,2)-Lie group.

<u>Question 2.1</u>. Is it possible to describe all commutative (4,2)-Lie groups over R by the procedure given above? If the answer is positive, then is there a method for finding all non-isomoprhic commutative (4,2)-Lie groups over R? If the answer is negative, what are the other examples of commutative (4,2)-Lie groups over R?

<u>Question 2.2</u>. How the commutative (4,2)-Lie groups over R classify the commutative Lie groups over R^2?

§3. NON-COMMUTATIVE (4,2)-LIE GROUPS OVER R

Here we examine the (4,2)-Lie groups whose associated groups over R^2 are isomorphic to the group given by (1.2). In this case, for the chosen coordinates, we do not know of any other involutive automorphism but the identity. In order to find an involutive automorphism of the group over R^2 defined by (1.2), we are going to give an alternative presentation of this group. Suppose that in a neighbourhood of (0,0) the seeking group satisfies the following conditions:

$$(x,0)*(0,y) = (x,y); \tag{3.1}$$

$$\psi(x,0)=(0,x), \text{ where } \psi \text{ is an involutive automorphism;} \tag{3.2}$$

$$(x,0)*(y,0) = (x+y,0); \text{ and} \tag{3.3}$$

$$(0,y)*(z,0) = (z+\phi(y,z),y-\phi(y,z)) \tag{3.4}$$

where $\phi(y,z)$ is a differentiable function of two variables.

As a consequence of (3.2) and (3.3), we have

$$\begin{aligned}(0,x)*(0,y) &= \psi(x,0)*\psi(y,0) = \\ &= \psi((x,0)*(y,0)) = \psi(x+y,0) = (0,x+y).\end{aligned} \tag{3.5}$$

Now (3.1) to (3.5) imply that

$$\begin{aligned}(x,y)*(z,t) &= (x,0)*(0,y)*(z,0)*(0,t) \\ &= (x,0)*(z+\phi(y,z),0)*(0,y-\phi(y,z))*(0,t) \\ &= (x+z+\phi(y,z),y+t-\phi(y,z)).\end{aligned}$$

The associativity of (*) implies that

$$\phi(\alpha+\beta,\gamma) = \phi(\beta,\gamma) + \phi(\alpha,\gamma+\phi(\beta,\gamma)); \tag{3.6}$$

$$\phi(\alpha,\beta+\gamma) = \phi(\alpha,\beta) + \phi(\alpha-\phi(\alpha,\beta),\gamma); \tag{3.7}$$

$$\phi(\alpha,0) = 0; \text{ and} \tag{3.8}$$

$$\phi(0,\gamma) = 0. \tag{3.9}$$

Suppose that $\phi(\alpha,\beta)=\alpha\beta W(\alpha,\beta)$. Then, when $\alpha \to 0$, $\gamma \neq 0$ (3.6) implies that

$$W(\beta,\gamma)+\beta \frac{\partial W(\beta,\gamma)}{\partial\beta} = (1+\beta W(\beta,\gamma))\cdot W(0,\gamma(1+\beta W(\beta,\gamma))). \tag{3.10}$$

Let us suppose that $W(0,0)=\theta$. Taking $\gamma \to 0$ in (3.10), for $\beta \neq 0$ we have

$$W(\beta,0) + \beta\frac{\partial W(\beta,0)}{\partial\beta} = (1+\beta W(\beta,0))\cdot\theta, \text{ i.e.}$$

$$W(\beta,0) = -\frac{1}{\beta} + \frac{C}{\beta}e^{\beta\theta}.$$

Now, (3.9) implies that W does not have singularity for $\beta = 0$. Hence C=1, i.e.

$$W(\beta,0) = \frac{1}{\beta}(e^{\beta\theta}-1), \tag{3.11}$$

for $\beta \neq 0$.

Symmetrically we have:

$$W(\alpha,\gamma)+\alpha \frac{\partial W(\alpha,\gamma)}{\partial\gamma} = (1-\gamma W(\alpha,\gamma)).W(\alpha(1-\beta W(\alpha,\gamma)),0) \tag{3.12}$$

and

$$W(0,\gamma) = \frac{1}{\gamma}(1-e^{-\gamma\theta}) \tag{3.13}$$

for $\gamma \neq 0$.

Substituting (3.11) in (3.12), we obtain

$$W(\alpha,\gamma)+\gamma\cdot\frac{\partial W(\alpha,\gamma)}{\partial\gamma} = \frac{1}{\alpha}\left[e^{\alpha\theta(1-\gamma W(\alpha,\gamma))}-1\right], \text{ i.e.}$$

$$\frac{\partial\phi(\alpha,\gamma)}{\partial\gamma} = e^{\alpha\theta}e^{-\theta\phi(\alpha,\gamma)} - 1. \tag{3.14}$$

Symmetrically, substituting (3.13) in (3.10), we obtain

$$\frac{\partial\phi(\alpha,\gamma)}{\partial\alpha} = 1 - e^{-\theta\gamma}e^{\theta\phi(\alpha,\gamma)}. \tag{3.15}$$

Since $\frac{\partial}{\partial\gamma}\frac{\partial\phi(\alpha,\gamma)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\frac{\partial\phi(\alpha,\gamma)}{\partial\gamma}$, it follows that the system of partial differential equations (3.14), (3.15) has a solution. By solving that system, one obtains

$$\phi(\alpha,\gamma) = \alpha-\gamma + \frac{1}{\theta}\ln(e^{\theta\gamma}+e^{-\theta\alpha}-1).$$

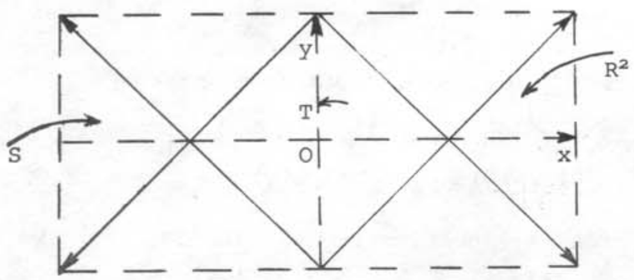
Hence, the operation (*) locally is given by:

$$(\alpha,\beta)*(\gamma,\delta) = \\ = (\alpha+\beta + \frac{1}{\theta}\ln(e^{\theta\gamma}+e^{-\theta\beta}-1),\gamma+\delta - \frac{1}{\theta}\ln(e^{\theta\gamma}+e^{-\theta\beta}-1)). \quad (3.16)$$

An elementary calculation, shows that (3.16) defines a local (4,2)-Lie group. We also note that $\theta \to 0$ leads to the operation defined by (1.1). So, we consider the special case, when $\theta = 1$, i.e.

$$(\alpha,\beta)*(\gamma,\delta) = \\ = (\alpha+\beta+\ln(e^{\gamma}+e^{-\beta}-1),\gamma+\delta-\ln(e^{\gamma}+e^{-\beta}-1)). \quad (3.17)$$

Now, we are going to give a globalization of the local Lie group over R^2. We use the following notations: Let $T=\{(a-\infty,b+\infty) \mid a,b\in R\}/\sim$, where $(a-\infty,b+\infty)\sim(c-\infty,d+\infty) \iff a+b=c+d$; let $S=\{(a+\pi i,b-\pi i) \mid a,b\in R\}$; let $e^{a-\infty}=0$, $e^{a+\pi i}=-e^a$, $\ln 0=-\infty$, and $\ln(-1)=\pi i$. It is obvious that R^2, T and S are pairwise disjoint, where T is homeomorphic to R and S is homeomorphic to R^2. Let $G=R^2\cup T\cup S$. We can identify G with RxR by identifying R^2 with $(0,\infty)$xR; T with $\{0\}$xR and S with $(-\infty,0)$xR, as the following figure shows.



This makes G a topological space, and G is homeomorphic to R^2. Define an operation * on G, by (3.17), with the notation used above. An elementary calculation shows that (G,*) is a Lie group. Now, we define $\psi: G \to G$ by:

$$\psi(x,y) = (x+\ln(e^y+e^{-x}-1), y-\ln(e^y+e^{-x}-1))$$

for $(x,y) \in G\setminus T$; and

$$\psi(a-\infty, b+\infty) = (a+\ln(e^b+e^{-a}), b-\ln(e^b+e^{-a})).$$

It can be verified that ψ is an involutive automorphism of (G,*).

Similarly as in 2, the existence of non-commutative (4,2)-Lie groups reduces to the existence of continuous functions $s \to (x(s),y(s))$ from R into G, such that the map $h: R^2 \to G$ defined by:

$$h(s,t) = (x(s),y(s))*\psi(x(t),y(t))$$

is a homeomorphism. We consider a class of functions, $s \to (s,\lambda s)$ for $\lambda \in R$. Next, we are going to determine those λ which make h a homeomorphism. For these functions,

$$h(s,t) = (s,\lambda s)*\psi(t,\lambda t) = (\xi(s,t),\mu(s,t)$$

where $\xi(s,t)=(1+\lambda)s+\ln(e^{(1+\lambda)t}-e^t+e^{-\lambda s})$, and $\mu(s,t)=(1+\lambda)t-\ln(e^{(1+\lambda)t}-e^t+e^{-\lambda s})$. In order h to be a homeomorphism, it is necessary for the equations

$$\xi(s,t) = a, \quad \mu(s,t) = b \tag{3.18}$$

to have a unique solution $(s,t) \in R^2$, for a given $(a,b) \in G$. The system (3.18) is equivalent to the system

$$\begin{aligned} &(1+\lambda)(s+t) = a+b \\ &1-e^{-\lambda t} + e^{-\lambda s-(1+\lambda)t} = e^{-b}. \end{aligned} \tag{3.19}$$

Since $a+b \in R$ for $(a,b) \in G$, substituting s from the first equation in the second one, we have to show that

$$-e^{-\lambda t} + e^{-\frac{\lambda}{\lambda+1}(a+b)}.e^{-t} = e^{-b}+1 \tag{3.20}$$

has a unique solution $t \in R$, for given a+b, $e^{-b}+1 \in R$. Hence we have to determine λ, such that the function $f: R \to R$, defined by

$f(z)=-e^{-\lambda z}+Ce^{-z}$, $(C>0,\ \lambda\neq -1)$ is a homeomorphism. This is possible for $\lambda\in(-\infty,0)\setminus\{-1\}$. In this case, indeed, $\lim_{z\to\infty} f(z) = -\infty$, $\lim_{z\to-\infty} f(z) = +\infty$, and $f'(z)=-[e^{-\lambda z}(-\lambda)+Ce^{-z}] < 0$. In particular, for $\lambda=-2$, we can find $h^{-1}:G \to R$ explicitly. Namely,

$$h^{-1}(a,b) = (-a-b-\ln(f(a,b)),\ \ln(f(a,b)))$$

where $f(a,b) = \sqrt[3]{g(a,b)+\sqrt{k(a,b)}} + \sqrt[3]{g(a,b)-\sqrt{k(a,b)}}$, where $g(a,b) = \frac{1}{2}e^{-2(a+b)}$; and

$$k(a,b) = (g(a,b))^2 - \left(\frac{1-e^{-b}}{3}\right)^3.$$

Remark 3.1. The above formulas are obtained by solving the third order equation over R

$$z^3 + (1+e^{-b})z = e^{-2(a+b)},$$

which is equivalent to (3.20) for $\lambda=-2$ and $z=e^t$.

Next, the (4,2)-operation [ ] on R is defined by

$$[stuv] = h^{-1}(h(s,t)*h(u,v)),\text{ i.e.}$$

$$[stuv] = (s+t+u+v-\ln(A+B),\ \ln(A+B)), \tag{3.21}$$

where $A = \sqrt[3]{g(-s-t,-u-v) + \sqrt{(g(-s-t,-u-v))^2-(w/3)^3}}$,

$B = \sqrt[3]{g(-s-t,-u-v) - \sqrt{(g(-s-t,-u-v))^2-(w/3)^3}}$, and

$w = e^{2v} - e^{2u+v} + e^{u+v+2t} - e^{u+v+t+2s}$.

Note that A+B is always positive.

The above discussion shows that (R,[ ]) is a non-commutative (4,2)-topological group, which can also be verified directly, using (3.21). This group is not a Lie group because A and B are not differentiable functions at any points of R^2, although the induced group $(R^2,*)$ is continuously isomorphic to a Lie group over R^2.

Question 3.2. Is it possible to describe all non-commutative (4,2)-Lie groups over R by the above procedure? If the answer is positive, then is there a method for finding all non-commutative non-isomorphic (4,2)-Lie groups? If the answer is negative, what are the other examples of non-commutative (4,2)-Lie groups?

Question 3.3. How the non-commutative (4,2)-Lie groups over R classify the non-commutative Lie groups over R^2?

§4. (4,2)-LIE GROUPS OVER S^1

Now, we consider the one-dimensional manifold S^1, i.e. the circle.

Proposition 4.1. *If (S^1,[ ]) is a (4,2)-Lie group, then the associated group is a commutative Lie group.*

Proof. If the associated group ($S^1 x S^1$,*) is a non-commutative Lie group, then it follows that ($S^1 x S^1$,*) is isomorphic to a factor group of (R^2,*) defined by (1.2), over a discrete subgroup of its centre. But the centre of (R^2,*) is trivial. Hence the Lie group ($S^1 x S^1$,*) has to be isomorphic to (R^2,*), or in particular $S^1 x S^1$ has to be homeomorphic to R^2, but this is not the case, i.e. $S^1 x S^1$ is not homeomorphic to R^2. ||

Now, we are going to construct a countable family of non-isomorphic commutative (4,2)-Lie groups over S^1. Let n be a positive integer. Then it is easy to check that $H=\{2\pi nk \mid k \in Z\}$ is a normal (4,2)-subgroup of (R,[ ]) defined by (2.4), and the factor group R/H is a commutative (4,2)-Lie group over S^1. For different n's, we obtain non-isomorphic commutative (4,2)-Lie groups over S^1.

Except this, the factor group of (R,[ ]) by the normal (4,2)-subgroup Z, is a commutative (4,2)-Lie group over S^1, which is not isomorphic to any of the above (4,2)-groups on S^1.

Question 4.2. Is it possible to describe all commutative (4,2)-Lie groups over S^1 by the above procedure? If the answer is positive, is there a method for finding all commutative non-isomorphic (4,2)-Lie groups? If the answer is negative, are there any other examples of commutative (4,2)-Lie groups?

Question 4.3. How the commutative (4,2)-Lie groups over S^1 classify the commutative Lie groups over $S^1 x S^1$?

R E F E R E N C E S

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