COMMUTATIVE (2m,m)-GROUPS

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Abstract. In this paper we prove several results about commutative (2m,m)-groups, and give some examples of nontrivial, finite and infinite, commutative and noncommutative (2m,m)-groups, $m \geq 2$.

§0. INTRODUCTION

Let $m \geq 1$ and let $G \neq \emptyset$. Let $[\ ]: G^{2m} \to G^m$ be a map satisfying the following conditions:

$$[x_1^i [x_{i+1}^{2m+i}] x_{2m+i+1}^{3m}] = [[x_1^{2m}] x_{2m+1}^{3m}], \text{ for each } 1 \leq i \leq m; \quad (1)$$

For each $\underline{a},\underline{b} \in G^m$, there exist $\underline{x},\underline{y} \in G^m$, such that

$$[\underline{a}\ \underline{x}] = \underline{b} = [\underline{y}\ \underline{a}]. \quad (2)$$

Then, the pair $(G;[\ ])$ is called a (2m,m)-group ([1]). Above, (x_1^t) denotes the vector $(x_1,x_2,\ldots,x_t)$, $[x_1^{2m}]$ denotes the image of (x_1^{2m}) under the map $[\ ]$, and $\underline{a}$ denotes a vector from G^m.

Let $(G;[\ ])$ be a (2m,m)-group. Then $(G^m,\circ)$, where $\underline{a}\circ\underline{b} = [\underline{a}\ \underline{b}]$, is a group with a neutral element $(e^m)=(e,\ldots,e)$, $e \in G$ ([2]). The notion of (2m,m)-groups for $m=1$ coincides with the notion of groups. The simplest examples of (2m,m)-groups are the m^{th} products of groups, i.e. if G is a group then $(G,[\ ])$ where $[x_1^m y_1^m] = (x_1y_1,\ldots,x_my_m)$, is a (2m,m)-group. Such groups are called trivial (2m,m)-groups (see [2]). We say that a (2m,m)-group is commutative if the associated group $(G^m,\circ)$ is commutative. The notions of (2m,m)-subgroups and normal (2m,m)-subgroups are introduced and examined in [2].

In this paper we consider commutative (2m,m)-groups for $m \geq 2$, and give some examples of nontrivial (2m,m)-groups.

§1. BASIC RESULTS

Proposition 1.1. *Let $(G;[\ ])$ be a commutative $(2m,m)$-group. Then, for each $1\le i\le m$ and $(x_1^m),(y_1^m)\in G^m$:*

(a) $[e^{m-i}x_1^m e^i] = (x_{i+1}^m, x_1^i)$;

(b) $[x_1^m y_1^m] = [x_1^{i-1}y_i x_{i+1}^m y^{i-1} x_i y_{i+1}^m]$; *and*

(c) $[x_1^m y_1^m] = (a_1^m) \iff [x_{i+1}^m x_1^i y_{i+1}^m y_1^m] = (a_{i+1}^m, a_1^i)$.

Proof. In [2], it is shown that in a $(2m,m)$-group with a neutral element (e^m), $[x_1^i e^m x_{i+1}^m] = (x_1^m)$.

(a) $[e^{m-i}x_1^m e^i] = (e^{m-i},x_1^i)\circ(x_{i+1}^m,e^i) = [x_{i+1}^m e^m x_1^i] =$

$= (x_{i+1}^m, x_1^i)$.

(b) $[x_1^m y_1^m] = [x_1^i[e^{m-1}y_i x_{i+1}^m y_1^{i-1}e]y_{i+1}^m] =$

$= [x^{i-1}e^{m-i+1}e^{i-1}x_i e^{m-1}y_i x_{i+1}^m y_1^{i-1}ey_{i+1}^m] =$

$= \underline{a}\circ\underline{b}\circ\underline{u}\circ\underline{v}\circ\underline{w} = \underline{a}\circ\underline{u}\circ\underline{v}\circ\underline{b}\circ\underline{w}$

$= [x_1^{i-1}y_i x_{i+1}^m y_1^{i-1}x_i y_{i+1}^m]$, where

$\underline{a} = (x_1^{i-1},e^{m-i+1})$, $\underline{b} = (e^{i-1},x_i,e^{m-i})$, $\underline{u} = (e^{i-1},y_i,x_{i+1}^m)$, $\underline{v} = (y_1^{i-1},e^{m-i+1})$ and $\underline{w} = (e^i,y_{i+1}^m)$.

(c) Let $[x_1^m y_1^m] = (a_1^m)$. Then (a) implies that:

$(a_{i+1}^m,a_1^i) = [e^{m-i}a_1^m e^i] = [e^{m-i}x_1^m y_1^m e^i] = [e^{m-i}x_1^m e^i e^{m-i}y_1^m e^i] =$

$= [x_{i+1}^m x_1^i y_{i+1}^m y_1^i]$. ||

Proposition 1.2. *A $(2m,m)$-group $(G;[\ ])$ is commutative if and only if for some $1\le i\le m$,*

$$[x_1^m y_1^m] = [x_1^{i-1}y_i x_{i+1}^m y_1^{i-1}x_i y_{i+1}^m]. \qquad (3)$$

Proof. Proposition 1.1. (b) implies that in a commutative $(2m,m)$-group, (3) is satisfied for every $1\le i\le m$. Conversely, let (3) be satisfied for some i. Then,

$[x_1^m y_1^m] = [e^m x_1^m y_1^m] = [e^{m-i+1}[e^{i-1}x_1^m y_1^{m-i+1}]y_{m-i+2}^m] =$

$= [e^{m-i+1}[e^{i-1}y_1 x_2^m x_1 y_2^{m-i+1}]y_{m-i+2}^m] = [y_1 x_2^m x_1 y_2^m]$

implies that $(x_1^m)\circ(y_1^m) = (y_1^m)\circ(x_1^m)$. ||

In general, in a commutative group $(G^m,\circ)$ with a neutral element (e^m), the identity

$$(x_1^m) = (x_1^i,e^{m-i})\circ(e^i,x_{i+1}^m) \qquad (4)$$

and the quasi identity

$$(x_1^m)\circ(y_1^m) = (z_1^m) \Longrightarrow (x_{i+1}^m,x_1^i)\circ(y_{i+1}^m,y_1^i) = (z_{i+1}^m,z_1^i) \qquad (5)$$

do not hold, which is shown by the following examples.

Example 1.1. Let $G=\{e,a\}$, $(G^2,\circ)$ be the cyclic group generated by (e,a), (e,e) be the neutral element, and $(e,a)\circ(e,a)=(a,a)$. Then $(G^2,\circ)$ satisfies (5) but does not satisfy (4).

Example 1.2. Let $m=2$ and $G=\{e,a,b\}$. It is easy to check that there exists a commutative group $(G^2,\circ)$, such that: (e,e) is the neutral element: $(e,a)^3=(e,b)^3=(e,e)$; $(e,a)^2=(b,b)$; $(e,b)^2=(b,a)$; $(e,a)\circ(e,b)=(a,e)$; $(e,a)\circ(b,a)=(a,b)$; $(b,b)\circ(e,b)=(a,a)$ and $(b,b)\circ(b,a)=(b,e)$. Then $(G^2,\circ)$ satisfies (4) but does not satisfy (5).

It is obvious that if a commutative group $(G^m,\circ)$ satisfies (4), then $(G^m,\circ)$ satisfies:

$$(x_1^m)\circ(y_1^m) = (x_1^{i-1},y_i,x_{i+1}^m)\circ(y_1^{i-1},x_i,y_{i+1}^m). \qquad (6)$$

Proposition 1.1. (c), allows us to describe commutative (2m,m)-groups as algebras with one 2m-ary operation. Let $(G;[\ ])$ be a commutative (2m,m)-group and let

$$[x_1^{2m}] = (g_1(x_1^{2m}),\dots,g_m(x_1^{2m}))$$

where $g_i:G^{2m}\to G$ is a 2m-ary operation, for each $1\le i\le m$. Now, Proposition 1.1. (c) implies that for each $1\le i\le m$,

$$g_i(x_1^m,y_1^m) = g_1(x_i^m,x_1^{i-1},y_i^m,y_1^{i-1}).$$

For a given $f:G^{2m}\to G$, and some i, let $f_i:G^{2m}\to G$ be defined by $f_i(x_1^m,y_1^m)=f(x_i^m,x_1^{i-1},y_i^m,y_1^{i-1})$. Moreover, let $\bar f,\bar f_i:G^{2m}\to G^m$ be defined by $\bar f(x_1^{2m})=(f_1(x_1^{2m}),\dots,f_m(x_1^{2m}))$ and $\bar f_i(x_1^{2m}) = (f_i(x_1^{2m}),\dots,f_m(x_1^{2m}),f_1(x_1^{2m}),\dots,f_{i-1}(x_1^{2m}))$. Above, if we set $f=g_1$, then $g_i=f_i$ and $[\ ]=\bar f$.

Proposition 1.3. *Let* $G \neq \emptyset$ *and* $[\ ]: G^{2m} \to G^m$. *Then:* $(G;[\ ])$ *is a commutative* $(2m,m)$*-group if and only if there exists a map* $f: G^{2m} \to G$ *such that* $[\ ] = \overline{f}$ *and:*

(a) $f(x_1^m, \overline{f}(y_1^m, z_1^m)) = f(\overline{f}(x_1^m, y_1^m), z_1^m)$;

(b) $(\forall \underline{a}, \underline{b} \in G^m)(\exists \underline{c} \in G^m)\ \overline{f}(\underline{a}, \underline{c}) = \underline{b}$; *and*

(c) $f(x_1^m, y_1^m) = f(x_1^{i-1}, y_i, x_{i+1}^m, y_1^{i-1}, x_i, y_{i+1}^m)$.

Proof. If $(G;[\ ])$ is a commutative $(2m,m)$-group, the conclusion of the Proposition follows from the previous discusion and Proposition 1.1.

Conversely, let f satisfy (a),(b) and (c) and $[\ ] = \overline{f}$. Then (a),(b) and (c) directly imply that $(G^m, \circ)$ is a commutative group, where $(x_1^m) \circ (y_1^m) = [x_1^m y_1^m] = \overline{f}(x_1^m, y_1^m)$. Moreover,

$$f_i(x_1^k, \overline{f}(x_{k+1}^{2m+k}), x_{2m+k+1}^{3m}) = f_i(x_1^k, x_{2m+k+1}^{3m}, \overline{f}_{m-k}(x_{k+1}^{2m+k})) =$$
$$= f_i(x_1^k, x_{2m+k+1}^{3m}, \overline{f}(x_{m+1}^{m+k}, x_{k+1}^m, x_{2m+1}^{2m+k}, x_{m+k+1}^{2m}))$$
$$= f_i(\overline{f}(x_1^k, x_{2m+k+1}^{3m}, x_{m+1}^{m+k}, x_{k+1}^m), x_{2m+1}^{2m+k}, x_{m+k+1}^{2m}))$$
$$= f_i(\overline{f}(x_1^m, x_{m+1}^{m+k}, x_{2m+k+1}^{3m}), x_{2m+1}^{2m+k}, x_{m+k+1}^{2m}))$$
$$= f_i(x_1^m, \overline{f}(x_{m+1}^{m+k}, x_{2m+k+1}^{3m}, x_{2m+1}^{2m+k}, x_{m+k+1}^{2m})) = f_i(x_1^m, \overline{f}(x_{m+1}^{3m})),$$

for each $1 \leq i \leq m$, and each $1 \leq k \leq m$.

Hence $(G;[\ ])$ is a commutative $(2m,m)$-group. ||

The following proposition gives a description of commutative $(2m,m)$-groups similar to the definition of groups as algebras with one binary, one unary, and one nulary operation.

Proposition 1.4. *Let* $G \neq \emptyset$ *and* $[\ ]: G^{2m} \to G^m$. *Then:* $(G;[\ ])$ *is a commutative* $(2m,m)$*-group if and only if there exist* $e \in G$ *and* $g: G \to G$, *such that:*

(a) $[x_1^i [x_{i+1}^{2m+i}] x_{2m+i+1}^{3m}] = [[x_1^{2m}] x_{2m+1}^{3m}]$, *for each* $1 \leq i \leq m$;

(b) $[e^m \underline{x}] = \underline{x}$, *for each* $\underline{x} \in G^m$;

(c) $[x^m (g(x))^m] = (e^m)$, *for each* $x \in G$; *and*

(d) $[x_1^m y_1^m] = [x_1^{m-1} y_m y_1^{m-1} x_m]$, *for each* $(x_1^m), (y_1^m) \in G^m$.

Proof. We have already seen that in a commutative (2m,m)-group there exist e and (a),(b) and (d) are satisfied. To prove (c), let $x\in G$. Then there exists $(y_1^m)\in G^m$, such that $[y_1^m x^m]=(e^m)$, and by Proposition 1.1. (c), we have that $(e^m)=[y_2^m y_1 x^m]= =(y_2^m, y_1)\circ(x^m)=(y_1^m)\circ(x^m)$. This implies $y_1=y_2=\ldots=y_m=y$. Define $g:G\to G$ by $g(x)=y$. Hence, there exists a map g satisfying (c).

Conversely, let $e\in G$ and $g:G\to G$ be given, satisfying (a) to (d). Then, (a) implies that $(G;[\ ])$ is a (2m,m)-semigroup, and (b) and (d) imply that $(G;[\ ])$ is a commutative (2m,m)-semigroup with a neutral element (e^m). Let $(a_1^m)\in G^m$, and let $(b_1^m)= =[(g(a_m))^m(a_m)^{m-1}\ldots(g(a_1))^m(a_1)^{m-1}]$. Then $(a_1^m)\circ(b_1^m)=[a_1^m b_1^m]= =[a_1^{m-1}a_m(g(a_m))^m(a_m)^{m-1}\ldots(g(a_1))^m(a_1)^{m-1}]=[a_1^{m-1}e^m\ldots(g(a_1))^m(a_1)^{m-1}]= =[a_1(g(a_1))^m(a_1)^{m-1}]=(e^m)$, i.e. (b_1^m) is the inverse element for (a_1^m) in $(G^m,\circ)$. Hence, $(G;[\ ])$ is a commutative (2m,m)-group. ||

It is obvious that if $(G,[\ ])$ is a (2m,m)-group, then $\psi:G^m\to G^m$ defined by $\psi(x_1^m)=[ex_1^m e^{m-1}]$, is an automorphism of the group $(G^m,\circ)$ and $\psi^m=id$. (see [2]). The converse is also true.

Proposition 1.5. *Let $(G^m,\circ)$ be a group with a neutral element (e^m) and $(x_1^i,e^{m-i})\circ(e^i,x_{i+1}^m)=(x_1^m)$, and $\psi:G^m\to G^m$ defined by $\psi(x_1^m)=(e,x_1^{m-1})\circ(x_m,e^{m-1})$ be an automorphism of $(G^m,\circ)$, such that $\psi^m=id$. Then $(G;[\ ])$ where $[\ ]$ is defined by $[x_1^m y_1^m]=(x_1^m)\circ(y_1^m)$, is a (2m,m)-group.*

Procf. The definition of $[\ ]$ and the fact that $(G^m,\circ)$ is a group, imply that for each $\underline{a},\underline{b}\in G^m$, there exist $\underline{x},\underline{y}\in G^m$, such that $[\underline{a}\ \underline{x}]=\underline{b}=[\underline{y}\ \underline{a}]$. Since ψ is an automorphism and $\psi^m=id$, it follows that for each $(x_1^m)\in G^m$,

$$(x_1^m) = (x_1,e^{m-1})\circ(e,x_2,e^{m-2})\circ\ldots\circ(e^{m-2},x_{m-1},e)\circ(e^{m-1},x_m).$$

Now let $1\le i\le m-1$. Then:

$$[x_1^i[x_{i+1}^{2m+1}]x_{2m+i+1}^{3m}]=[x_1^i a_1^m x_{2m+i+1}^{3m}]=(x_1^i,a_1^{m-i})\circ(a_{m-i+1}^m,x_{2m+i+1}^{3m})$$

$$=(x_1^i,e^{m-i})\circ\psi^i(a_1^m)\circ(e^i,x_{2m+i+1}^{3m})=(x_1^i,e^{m-i})\circ\psi^i([x_{i+1}^{2m+i}])\circ(e^i,x_{2m+i+1}^{3m})$$

$$=(x_1^i,e^{m-i})\circ\psi^i(x_{i+1}^{m+i})\circ\psi^i(x_{m+i+1}^{2m+i})\circ(e^i,x_{2m+i+1}^{3m})$$

$= (x_1^i, e^{m-i}) \circ (e^i, x_{i+1}^m) \circ (x_{m+1}^{m+i}, e^{m-i}) \circ (e^i, x_{m+i+1}^{2m}) \circ (x_{2m+1}^{2m+i}, e^{m-i}) \circ$

$\circ (e^i, x_{2m+i+1}^{3m}) =$

$= (x_1^m) \circ (x_{m+1}^{2m}) \circ (x_{2m+1}^{3m}) = [x_1^m [x_{m+1}^{3m}]] = [[x_1^{2m}] x_{2m+1}^{3m}]$.

Hence, $(G;[\ ])$ is a $(2m,m)$-group. ||

§2. SPECIAL SUBGROUPS OF COMMUTATIVE (2m,m)-GROUPS

Let $(G,[\ ])$ be a commutative $(2m,m)$-group, and let $D=\{(x^m) \mid x \in G\}$.

Proposition 2.1. *$(D,\circ)$ is a subgroup of $(G^m,\circ)$.*

Proof. Let $(x^m),(y^m) \in D$, and let $[x^m y^m]=(z_1^m)$. Then, Proposition 1.1. (c) implies that $(z_2^m, z_1)=[x^m y^m]=(z_1^m)$, i.e. $z_1=z_2=\ldots=z_m=z$. Hence $(x^m)\circ(y^m)\in D$. It is obvious that $(e^m)\in D$. Proposition 1.4. (c) implies that the inverse element for (x^m) is $((g(x))^m)$, i.e. is in D. Hence $(D,\circ)$ is a subgroup of $(G^m,\circ)$. ||

Using Proposition 2.1, we define a map $+ : G^2 \to G$ by

$$x + y = a \iff [x^m y^m] = (a^m). \tag{7}$$

Proposition 2.2. *$(G,+)$ is a commutative group with a zero e. Moreover, $(G;[\ ])$ satisfies the following implication:*

$$[x_1^m y_1^m] = (a_1^m) \implies \sum_{i=1}^m x_i + \sum_{i=1}^m y_i = \sum_{i=1}^m a_i, \tag{8}$$

where $\sum_{i=1}^m x_i = x_1+x_2+\ldots+x_m$ in $(G,+)$.

Proof. The fact that $(G,+)$ is a commutative group, follows directly from Proposition 2.1, and the fact that $(G,[\ ])$ is a commutative $(2m,m)$-group.

Now, let $[x_1^m y_1^m]=(a_1^m)$. Then

$$\begin{aligned}((\sum_{i=1}^m a_i)^m) &= [(a_1)^m \ldots (a_m)^m] = (a_1^m)\circ(a_m, a_1^{m-1})\circ\ldots\circ(a_2^m, a_1)\\ &= [x_1^m y_1^m x_m x_1^{m-1} y_m y_1^{m-1} \ldots x_2^m x_1 y_2^m y_1]\\ &= (x_1^m)\circ(x_m, x_1^{m-1})\circ\ldots\circ(x_2^m, x_1)\circ(y_1^m)\circ(y_m, y_1^{m-1})\circ\ldots\circ(y_2^m, y_1)\\ &= [(x_1)^m (x_2)^m \ldots (x_m)^m]\circ[(y_1)^m (y_2)^m \ldots (y_m)^m]\\ &= ((\sum_{i=1}^m x_i)^m)\circ((\sum_{i=1}^m y_i)^m) = ((\sum_{i=1}^m x_i + \sum_{i=1}^m y_i)^m).\end{aligned}$$

Hence $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} x_i + \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} (x_i+y_i)$. ||

Now, let M be a subgroup of (G,+), and let

$$K(M) = \{(x_1^m) \mid (x_1^m)\in G^m, \ \sum_{i=1}^{m} x_i \in M\}. \quad (9)$$

Proposition 2.3. *K(M) is a subgroup of* $(G^m, \circ)$.

Proof. Let $(x_1^m), (y_1^m) \in K(M)$, and let $(x_1^m)\circ(y_1^m)=(a_1^m)$. Then (8) and the fact that M is a subgroup of (G,+) imply that $(a_1^m)\in K(M)$. It is obvious that $(e^m)\in K(M)$. Let $(a_1^m)\in K(M)$. Then, there exists $(b_1^m)\in G^m$ such that $(a_1^m)\circ(b_1^m)=(e^m)$, and (8) implies that $\sum_{i=1}^{m} a_i + \sum_{i=1}^{m} b_i = e$. Since $\sum_{i=1}^{m} a_i$, $e\in M$ and M is a group, it follows that $\sum_{i=1}^{m} b_i \in M$, i.e. $(b_1^m)\in K(M)$. Hence, K(M) is a subgroup of $(G^m, \circ)$. ||

For M={e}, we have the following:

Corollary 2.4. *K({e}) is a subgroup of* $(G^m, \circ)$. ||

Using Proposition 2.2, we denote the image g(x) of $x\in G$ under the inverse map $g:G \to G$ from Proposition 1.4 by -x, i.e. -x=g(x).

Note, that if $(x_1^m)\in K=K(\{e\})$, then $x_m = -\sum_{i=1}^{m-1} x_i$. In this case, denote x_m by $\psi(x_1^{m-1})$. Define

$*:(G^{m-1})^2 \to G^{m-1}$ by

$$(x_1^{m-1})*(y_1^{m-1}) = (a_1^{m-1}) \iff (x_1^{m-1},u)\circ(y_1^{m-1},v) = (a_1^{m-1},w) \quad (10)$$

where $u=\psi(x_1^{m-1})$, $v=\psi(y_1^{m-1})$ and $w=\psi(a_1^{m-1})$.

The proof of the following proposition follows directly from the definition of * and Corollary 2.4.

Proposition 2.5. $(G^{m-1}, *)$ *is a group with a neutral element* (e^{m-1}). ||

Let m be even, i.e. m=2k. Define two maps

$\{\ \}:(G^2)^{2k} \to (G^2)^k$ and $\Delta:(G^2)^2 \to G^2$ by:

$$\{(x_1,x_2)\dots(x_{2m-1},x_{2m})\} = (([x_1^{2m}]_1,[x_1^{2m}]_2),\dots,([x_1^{2m}]_{m-1},[x_1^{2m}]_m)) \quad (11)$$

$$(x,y)\Delta(z,t) = (a,b) \iff \{(x,y)^k(z,t)^k\} = ((a,b)^k). \tag{12}$$

The proofs of the following propositions follow directly from the Proposition 2.2 and the fact that $(G;[\ ])$ is a commutative $(2m,m)$-group.

Proposition 2.6. *$(G^2,\{\ \})$ is a commutative $(2k,k)$-group.* ||

Proposition 2.7. *(G^2,Δ) is a commutative group.* ||

§3. FINITE COMMUTATIVE (2m,m)-GROUPS

Let $(G;[\ ])$ be a finite commutative $(2m,m)$-group.

Proposition 3.1. *If $(G^m,\circ)$ is a cyclic group, then m is divisible by the number of elements of G, $|G|$ $(m\geq 2)$.*

Proof. Let (x_1^m) be a generator for $(G^m,\circ)$ and let $|G|=n$. Then the subgroup D of $(G^m,\circ)$ is cyclic and has n elements. Let $(a^m)\in D$ be its generator, and let $(a^m)=(x_1^m)^t$ for some t. Since $(a^m)^n=(e^m)=(x_1^m)^{nt}$, it follows that the order n^m of (x_1^m) is a divisor of nt, i.e. n^{m-1} is a divisor of t. Proposition 1.1. (b) implies that $(x_{i+1}^m,x_1^i)^{t+1}=[x_{i+1}^m(x_1^m)^t x_1^i]=[x_{i+1}^m a^m x_1^i]=(x_{i+1}^m,x_1^i)\circ(a^m)$, which together with the fact that $(G^m,\circ)$ is a group, implies that $(x_{i+1}^m,x_1^i)^t=(a^m)$, for each i. Then Proposition 1.1. (b) and the facts that $|D|=n$ and n is a divisor of t, imply that $(a^m)^m= =(x_1^m)^t\circ(x_m,x_1^{m-1})^t\circ\ldots\circ(x_2^m,x_1)^t=[x_1^m x_m x_1^{m-1}\ldots x_2^m x_1]^t=[(x_1)^m(x_2)^m\ldots \ldots(x_m)^m]^t=(e^m)$. Hence, the order n of (a^m) is a divisor of m. ||

For $m=2$ and $m=3$ we have the following.

Proposition 3.2. *If $(G;[\ ])$ is a finite commutative $(4,2)$-group and $(G^2,\circ)$ is a cyclic group, then $|G|=1$.*

Proof. Since $|G|$ is a divisor of 2 it follows that $|G|=1$ or $|G|=2$. Let $G=\{e,a\}$. Then (e,a) and (a,e) are generators for the group $(G^2,\circ)$, and $(a,a)=(e,a)\circ(e,a)$. Since $(a,a)=(a,e)\circ(e,a)$, it follows that $e=a$, i.e. $|G|=1$. ||

Proposition 3.3. *If $(G;[\ ])$ is a finite commutative $(6,3)$-group and $(G^3,\circ)$ is a cyclic group, then $|G|=1$.*

Proof. Since $|G|$ is a divisor of 3, it follows that $|G|=1$ or $|G|=3$. Let $G=\{e,a,b\}$. Then the subgroup K of G^3 of Corollary

2.4 is a cyclic group, and moreover, $K=\{(e^3),(a^3),(b^3),(e,a,b),(e,b,a),(a,e,b),(a,b,e),(b,e,a),(b,a,e)\}$, because $(G,+)$ is a cyclic group with a neutral element e. Generators of K are elements of the form (x_1^3) where $x_1\neq x_2\neq x_3\neq x_1$, and moreover, $(e,a,b)^3=$ $=(b,e,a)^3=(a,b,e)^3$ and $(e,b,a)^3=(a,e,b)^3=(b,a,e)^3$. Hence, $(e,a,b)^3$ and $(e,b,a)^3$ are generators for subgroups of K of order 3, i.e.

$(e,a,b)^3,(e,b,a)^3\in\{(a,a,a),(b,b,b)\}$.

If $(e,a,b)^2\in\{(a,b,e),(b,e,a)\}$, then $(e,a,b)^6=(a,b,e)^3=$ $=(e,a,b)^3$ or $(e,a,b)^6=(b,e,a)^3=(e,a,b)^3$, which implies that $(e,a,b)^3=(e,e,e)$ i.e. $e=a=b$.

If $(e,a,b)^2=(e,a,b)$, then $e=a=b$.

It is obvious that $(e,a,b)^3 = (a,e,b)^3$ implies $e=a=b$.

So, we are left with two cases:

Case 1. $(e,a,b)^3=(a,a,a)$ and Case 2. $(e,a,b)^3=(b,b,b)$.

In the Case 1 we have three cases.

Case 1.1. $(e,a,b)^2=(b,a,e)$. Then $(a,a,a)=(e,a,b)\circ(b,a,e)=$ $=(e,a,e)\circ(b,a,a)$ and $(a,a,a)=(a,e,a)\circ(e,a,e)$, which imply that $e=a=b$.

Case 1.2. $(e,a,b)^2=(e,b,a)$. Then $(a,a,a)=(e,a,b)^3=$ $=(e,a,b)\circ(e,b,a)=(e,a,a)\circ(e,b,b)$ and $(a,a,a)=(a,e,e)\circ(e,a,a)$, which imply that $e=a=b$.

Case 1.3. $(e,a,b)^2=(a,e,b)$. Then $(a,a,a)=(e,a,b)\circ(a,e,b)=$ $=(e,e,b)\circ(a,a,b)=(e,e,b)^2\circ(a,a,e)$ and $(a,a,a)=(a,a,e)\circ(e,e,a)$, which imply that $(e,e,b)^2=(e,e,a)$. Now, $(a,e,b)^2\in\{(e,a,b),(a,b,e),(b,e,a)\}$. If $(a,e,b)^2=(e,a,b)$, then $(e,a,b)^3=(a,e,b)^3$, which implies that $e=a=b$. If $(a,e,b)^2=(a,b,e)$, then $(a,e,e)\circ(e,b,e)=$ $=(a,b,e)=(a,e,b)^2=(a,e,e)^2\circ(e,e,b)^2=(a,e,e)^2\circ(e,e,a)=(a,e,e)(a,e,a)$, i.e. $e=a=b$. If $(a,e,b)^2=(b,e,a)$, then $(a,e,b)^3=(b,b,b)$, which implies that $(e,b,e)\circ(b,e,b)=(b,b,b)=(a,e,b)^3=(a,e,b)\circ(b,e,a)=$ $=(a,e,a)\circ(b,e,b)$, i.e. $e=a=b$.

The Case 2 is symmetric to the Case 1. Hence, $|G|=1$. ||

Question 3.1. Is there a (2m,m)-group with more than one element and $m \geq 2$, such that the associated group is cyclic?

§4. EXAMPLES OF (2m,m)-GROUPS

Let (G,+) be a commutative group with zero 0, and let H be a subgroup of G. For each class x+H we choose an element from x+H, denoted by $\overline{x}$, i.e. $\overline{\ }:G \to G$ is a retraction of G, and moreover,

$$\overline{x}+H = x+H; \text{ and } \overline{x}=\overline{y} \iff x+H = y+H. \tag{13}$$

Then, $(x+\overline{y})+H=x+H + y+H=(x+y)+H=(\overline{x+y})+H$ implies $\overline{x+\overline{y}}=\overline{x+y}$.

Now, let $f:G^{2m} \to G$ be a map defined by

$$f(x_1^m,y_1^m) = x_1+y_1 - \sum_{i=2}^{m} (\overline{x_i}+\overline{y_i}) + \sum_{i=2}^{m} (\overline{x_i+y_i}), \text{ i.e.} \tag{14}$$

$$f(x_1^m,y_1^m) = x_1+y_1 + \sum_{i=2}^{m} ((\overline{x_i+y_i}) - \overline{x_i} - \overline{y_i}). \tag{14'}$$

It is obvious that for each $1 \leq i \leq m$,

$f(x_1^{i-1},y_i,x_{i+1}^m,y_1^{i-1},x_i,y_{i+1}^m) = f(x_1^m,y_1^m)$.

Define $[\ ]:G^{2m} \to G^m$ by $[x_1^{2m}]=\overline{f}(x_1^{2m})$. (See Proposition 1.3).

Proposition 4.1. *$(G;[\ ])$ is a commutative $(2m,m)$-group. Moreover, if $\emptyset \neq H \neq G$, $\overline{0}=0$, and there are $a,b \in G$ such that $\overline{a+b} \neq \overline{a}+\overline{b}$, then $(G;[\ ])$ is not a trivial $(2m,m)$-group.*

Proof. We have seen that f satisfies (c) from Proposition 1.3. Next, $f(x_1^m,\overline{f}(y_1^m,z_1^m)) =$

$$= x_1+f_1(y_1^m,z_1^m) + \sum_{i=2}^{m} ((\overline{x_i+f_i(y_1^m,z_1^m)}) - \overline{x}_i - \overline{f_i(y_1^m,z_1^m)})$$

$$= x_1+y_1+z_1 + \underline{\sum_{i=2}^{m} ((\overline{y_i+z_i})-\overline{y}_i-\overline{z}_i) - \sum_{i=2}^{m} (\overline{x}_i+(y_i+z_i+\sum_{\substack{j=1\\ j\neq 1}}^{m}((\overline{y_j+z_j})-\overline{y}_j-\overline{z}_j))}$$

$$+ \sum_{i=2}^{m} (\overline{x_i+y_i+z_i + \sum_{\substack{j=1\\ j\neq 1}}^{m} ((\overline{y_j+z_j})-\overline{y}_j-\overline{z}_j)}) =$$

$$= x_1+y_1+z_1 - \sum_{i=2}^{m} (\overline{x}_i+\overline{y}_i+\overline{z}_i) + \sum_{i=2}^{m} (\overline{x_i+y_i+z_i}).$$

Similarly,

$$f(\overline{f}(x_1^m,y_1^m),z_1^m) = x_1+y_1+z_1 - \sum_{i=2}^{m} (\overline{x}_i+\overline{y}_i+\overline{z}_i) + \sum_{i=2}^{m} (\overline{x_i+y_i+z_i}).$$

Hence, f satisfies (a) from Proposition 1.3.

For given $(a_1^m),(b_1^m)\in G^m$, let $(c_1^m)\in G^m$ be defined by $c_i = b_i-a_i + \sum_{j\neq i} \overline{a}_j - \sum_{j\neq i} \overline{b}_j + \sum_{j\neq i} (\overline{b_j-a_j})$. Then, an easy computation shows that for each $1\leq i\leq m$, $f_i(a_1^m,c_1^m)=b_i$. Hence, $f(a_1^m,c_1^m)= =(b_1^m)$, i.e. f satisfies (b) from Proposition 1.3. This, completes the proof that (G,[ ]) is a commutative (2m,m)-group.

Now, let $a,b\in G$ be the elements satisfying $\overline{a+b}\neq\overline{a}+\overline{b}$. Then, $f_1(a,0^{m-1},b,0^{m-1})=a+b-(m-1)\overline{0}-(m-1)\overline{0}+(m-1)\overline{0}=a+b$, and $f_i(a,0^{m-1},b,0^{m-1})=-\overline{a}-\overline{b}+\overline{a+b}$, for $i\neq 1$, i.e.

$$[a0^{m-1}b0^{m-1}] = (a+b,\overline{a+b}-\overline{a}-\overline{b},\dots,\overline{a+b}-\overline{a}-\overline{b}).$$

Hence, (G,[ ]) is not a trivial (2m,m)-group. ||

Proposition 4.2. *If $(G_1;[\ ]')$, $(G_2;[\ ]'')$ are (2m,m)-groups, then $(G_1\times G_2;[\ ])$, where*

$$[(x_1,y_1)\dots(x_{2m},y_{2m})]=(([x_1^{2m}]_i',[y_1^{2m}]_i''))_{i=1}^m,$$ *is a (2m,m)-group.*

Moreover, if one of the groups $(G_1;[\ ]'),(G_2;[\ ]'')$ is not a trivial or commutative (2m,m)-group, then $(G_1\times G_2;[\ ])$ is not a trivial or commutative (2m,m)-group.

Proof. Follows directly from the definition. ||

Using Propositions 4.1 and 4.2 we can construct examples of nontrivial finite and infinite, commutative and not commutative (2m,m)-groups.

Example 4.1. Let (G,+) be the group $(Z_4,+)$, and $H=\{0,2\}$. Define: $\overline{0}=0=\overline{2}$, and $\overline{1} = 1 = \overline{3}$. Then (G;[ ]) with

$$[xyzt] = (x+z-\overline{y}-\overline{t}+\overline{y+t},y+t-\overline{x}-\overline{z}+\overline{x+z})$$

is a nontrivial commutative (2m,m)-group. For example

$$[1\ 0\ 3\ 0] = (0,2).$$

Now, let (G,+), H, $^{-}$:G → G, and (G,[ ]) be as in Proposition 4.1, and let (G,⊕) be the commutative group obtained by Proposition 2.2 from (G,[ ]).

Proposition 4.3. *(i) (H,⊕) is a subgroup of (G,⊕).*

(ii) If $\overline{0}=0$, then (H,⊕)=(H,+).

(iii) If $\overline{0}=0$, then (G,+)/(H,+) = (G,⊕)/(H,⊕).

Proof. For h∈H, $\overline{h}=\overline{0}\in H$.

(i) Let u,v∈H. Then $u\oplus v=[u^m v^m]_1=u+v-(m-1)\overline{0}\in H$. The identity in (G,⊕) is $(m-1)\overline{0}$, but $(m-1)\overline{0}\in H$, since $\overline{0}\in H$. For h∈H, $u=-h+2(m-1)\overline{0}\in H$, and $h\oplus u=h+u-(m-1)\overline{0}=h-h+2(m-1)\overline{0}-(m-1)\overline{0}=(m-1)\overline{0}$. Hence, (H,⊕) is a subgroup of (G,⊕).

(ii) If $\overline{0}$ = 0, then u⊕v=u+v-0=u+v, for u,v∈H.

(iii) If $\overline{0}$ = 0, then for x∈G and u∈H we have: $x\oplus u = x+u-(m-1)\overline{x}-(m-1)\overline{u}+(m-1)(\overline{x+u})=x+u-(m-1)0=x+u$, using the facts that $\overline{u} = \overline{0} = 0$, and $\overline{x+u}=\overline{x}$. ||

The converse of Proposition 4.3 (ii) does not hold; this is shown by the following example.

Example 4.2. Let G={0,1,2,3}, (G,+)=(Z_4,+), H={0,2}, $0=\overline{2}=2$, $\overline{1}=\overline{3}=1$, and m=3. Then, for u,v∈H we have: u+v=u+v-v-u+u+v-u-v+u+v=u+v. Hence (H,⊕)=(H,+), but $\overline{0} \neq 0$.

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