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HOMOTOPY AND CONGRUENCES OF GENERALIZED QUASIGROUPS

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The notions of a congruence relation and factor structure of a generalized groupoid are considered in this paper and a connection between the homotopy and normal congruences of generalized quasigroups is stated.

0. First we give some definitions. Let Q_1, Q_2, Q_3 be nonempty sets and $A: Q_1 \times Q_2 \to Q_3$ a mapping. The ordered quadruple $(Q_1, Q_2, Q_3; A)$ is called a generalized groupoid or shortly G-grupoid and the mapping A a generalized operation on the sets Q_1, Q_2, Q_3.

A generalized groupoid $(Q_1, Q_2, Q_3; A)$ is called a generalized quasigroup iff the equations

$$A(x,b) = c, \quad A(a,y) = c \tag{0.1}$$

have unique solutions for any $a \in Q_1$, $b \in Q_2$, $c \in Q_3$.

A G-groupoid $(Q_1, Q_2, Q_3; A)$ is said to be homotopically mapped into a G-groupoid $(P_1, P_2, P_3; B)$ iff there exist mappings α, β, γ, where $\alpha: Q_1 \to P_1$, $\beta: Q_2 \to P_2$, $\gamma: Q_3 \to P_3$ such that

$$\gamma A(x,y) = B(\alpha x, \beta y) \tag{0.2}$$

for any $x \in Q_1$ and $y \in Q_2$. The ordered triplet (α, β, γ) is called a homotopy. If α, β, γ are bijections, then (α, β, γ) is an isotopy of the G-groupoids ([2]).

1. Let $(Q_1,Q_2,Q_3;A)$ be a G-groupoid and $\alpha_1,\alpha_2,\alpha_3$ be equivalences on the sets Q_1,Q_2,Q_3 respectively. The ordered triple $(\alpha_1,\alpha_2,\alpha_3)$ of equivalences is said to be a generalized congruence in the G-groupoid iff

$$a\alpha_1 b,\quad c\alpha_2 d \Longrightarrow A(a,c)\alpha_3 B(b,d). \tag{1.1}$$

It can be easily verified that a triplet$(\alpha_1,\alpha_2,\alpha_3)$ of equivalences in a G-groupoid $(Q_1,Q_2,Q_3;A)$ is a generalized congruence iff for every $s\in Q_2$ and $t\in Q_1$

$$a\alpha_1 b \Longrightarrow A(a,s)\alpha_3 A(b,s) \text{ and } c\alpha_2 d \Longrightarrow A(t,c)\alpha_3 A(t,d).$$

Note that if $Q_1 = Q_2 = Q_3$ and $\alpha_1 = \alpha_2 = \alpha_3$, then one obtains a usual congruence in the groupoid.

A generalized congruence $(\alpha_1,\alpha_2,\alpha_3)$ is said to be normal iff for any $s\in Q_2$ and $t\in Q_1$

$$A(a,s)\alpha_3 A(b,s) \Longrightarrow a\alpha_1 b \quad\text{and}\quad A(t,c)\alpha_3 A(t,d) \Rightarrow c\alpha_2 d \tag{1.2}$$

Note that if a G-groupoid $(Q_1,Q_2,Q_3;A)$ has an identity element, then $Q_1 = Q_2 = Q_3$. In that case, by (1.2), $\alpha_1 = \alpha_2 = \alpha_3$, i.e. a normal generalized congruence is a usual normal congruence ([1]).

THEOREM 1. Every normal generalized congruence in a G-quasigroup $(Q_1,Q_2,Q_3;A)$ defines a generalized quasigroup which is a homotopic image of $(Q_1,Q_2,Q_3;A)$.

Proof. Let $(\alpha_1,\alpha_2,\alpha_3)$ be a generalized congruence in $(Q_1,Q_2,Q_3;A)$. Define in the ordered triplet $(Q_1/\alpha_1,Q_2/\alpha_2,Q_3/\alpha_3)$ of factor sets a generalized operation $B:Q_1/\alpha_1 \times Q_2/\alpha_2 \to Q_3/\alpha_3$ by

$$B(a^{\alpha_1},b^{\alpha_2}) = \left[A(a,b)\right]^{\alpha_3}. \tag{1.3}$$

If $a^{\alpha_1} = c^{\alpha_1}$ and $b^{\alpha_2} = d^{\alpha_2}$, i.e. $a\alpha_1 c$ and $b\alpha_2 d$, then $A(a,b)\alpha_3 A(c,d)$ and

$$B(c^{\alpha_1},d^{\alpha_2}) = \left[A(c,d)\right]^{\alpha_3} = \left[A(a,b)\right]^{\alpha_3} = B(a^{\alpha_1},b^{\alpha_2}).$$

This shows that B is a mapping, i.e. (1.3) defines a generalized operation and thus $(Q_1/\alpha_1,Q_2/\alpha_2,Q_3/\alpha_3;B)$ is a generalized groupoid. It remains to show that it is a generalized quasigroup.

Consider the equation

$$B(a^{\alpha_1},x^{\alpha_2}) = c^{\alpha_3}, \tag{1.4}$$

where $a^{\alpha_1} \in Q_1/\alpha_1$, $c^{\alpha_3} \in Q_3/\alpha_3$. The equation $A(a,x) = c$ has a unique solution x_o in the G-quasigroup $(Q_1,Q_2,Q_3;A)$. We will show that the class $x_o^{\alpha_2} \in Q_2/\alpha_2$ is a solution of the equation (1.4). Namely, it follows by (1.3) that

$$B(a^{\alpha_1}, x_o^{\alpha_2}) = \left[A(a,x_o)\right]^{\alpha_3} = c^{\alpha_3}.$$

Suppose that $x_1^{\alpha_2}$, $x_1 \neq x_o$, is a solution of (1.4), i.e. $B(a^{\alpha_1}, x_1^{\alpha_2}) = c^{\alpha_3}$ and $B(a^{\alpha_1}, x_o^{\alpha_2}) = c^{\alpha_3}$. Then

$$\left[A(a,x_1)\right]^{\alpha_3} = B(a^{\alpha_1}, x_1^{\alpha_2}) = B(a^{\alpha_1}, x_o^{\alpha_2}) = \left[A(a,x_o)\right]^{\alpha_3},$$

i.e. $A(a,x_1)\alpha_3 A(a,x_o)$. Since $(\alpha_1,\alpha_2,\alpha_3)$ is a normal generalized congruence, it follows by the second implication of (1.2) that $x_1\alpha_2 x_o$, i.e. $x_1^{\alpha_2} = x_o^{\alpha_2}$. Thus the equation (1.4) has a unique solution.

Analogously, the equation $B(x^{\alpha_1}, b^{\alpha_2}) = c^{\alpha_3}$ has a unique solution for every $b^{\alpha_2} \in Q_2/\alpha_2$ and every $c^{\alpha_3} \in Q_3/\alpha_3$.

The quadruple $(Q_1/\alpha_1, Q_2/\alpha_2, Q_3/\alpha_3; B)$ is called a <u>factor structure</u> of $(Q_1,Q_2,Q_3;A)$ under the generalized congruence $(\alpha_1,\alpha_2,\alpha_3)$. As we proved in the previous theorem, a factor quasigroup of a generalized quasigroup is a generalized quasigroup.

Let f_1, f_2, f_3 be the natural mappings: $f_1 = \text{nat}\alpha_1 : Q_1 \to Q_1/\alpha_1$; $f_2 = \text{nat}\alpha_2 : Q_2 \to Q_2/\alpha_2$; $f_3 = \text{nat}\alpha_3 : Q_3 \to Q_3/\alpha_3$. Then by (1.3)

$$f_3A(a,b) = \left[A(a,b)\right]^{\alpha_3} = B(a^{\alpha_1}, b^{\alpha_3}) = B(f_1a, f_2b),$$

which proves that the triplet (f_1,f_2,f_3) is a homotopy. Since f_1,f_2,f_3 are surjections, it follows that the quasigroup $(Q_1/\alpha_1, Q_2/\alpha_2, Q_3/\alpha_3; B)$ is a homotopic image of $(Q_1,Q_2,Q_3;A)$.

THEOREM 2. <u>Every homotopy</u> (α,β,γ) <u>of a</u> G-<u>quasigroup</u> $(Q_1,Q_2,Q_3;A)$ <u>in a</u> G-<u>quasigroup</u> $(P_1,P_2,P_3;B)$ <u>induces a normal generalized congruence in</u> $(Q_1,Q_2,Q_3;A)$.

<u>Proof</u>. The relations $\alpha_1 = \ker\alpha$, $\alpha_2 = \ker\beta$ and $\alpha_3 = \ker\gamma$ are equivalences in the sets Q_1,Q_2,Q_3 respectively. Let $a\alpha_1 b$ and $c\alpha_2 d$ i.e. $\alpha a = \alpha b$ and $\beta c = \beta d$. In this case

$$\gamma A(a,c) = B(\alpha a, \beta c) = B(\alpha b, \beta d) = \gamma A(b,d),$$

i.e. $A(a,c)\alpha_3 A(b,d)$. Thus $(\alpha_1,\alpha_2,\alpha_3)$ is a generalized congruence.

Let $A(a,s)\alpha_3 A(b,s)$, i.e. $\gamma A(a,s) = \gamma A(b,s)$. Since (α,β,γ) is a homotopy, it follows that $B(\alpha a,\beta s) = B(\alpha b,\beta s)$, i.e. $\alpha a=\alpha b$, i.e. $a\alpha_1 b$. The second implication of (1.2) can be proved analogously.

THEOREM 3. <u>Let</u> $H = (\alpha,\beta,\gamma)$ <u>be a homotopy of a quasigroup</u> $(Q_1,Q_2,Q_3;A)$ <u>into a quasigroup</u> $(P_1,P_2,P_3;C)$, $(\alpha_1,\alpha_2,\alpha_3)$ <u>be the generalized congruence induced by the homotopy</u> H (<u>the kernel of the homotopy</u>) <u>and</u> (f_1,f_2,f_3) <u>be the natural homotopy of</u> $(Q_1,Q_2,Q_3;A)$ <u>into its factor quasigroup defined by</u> $(\alpha_1,\alpha_2,\alpha_3)$. <u>Then the quasigroups</u> $(Q_1/\alpha_1,Q_2/\alpha_2,Q_3/\alpha_3;B)$ <u>and</u> $H(Q_1,Q_2,Q_3;A)$ <u>are isotopic</u>.

<u>Proof</u>. Let $g_1:Q_1/\alpha_1 \to \alpha(Q_1)$, $g_2:Q_2/\alpha_2 \to \beta(Q_2)$, and $g_3:Q_3/\alpha_3 \to \gamma(Q_3)$, where $g_1(a^{\alpha_1}) = \alpha(a)$, $g_2(b^{\alpha_2}) = \beta(b)$ and $g_3(c^{\alpha_3}) = \gamma(c)$. Then by (1.3),

$$g_3B(a^{\alpha_1},b^{\alpha_2}) = g_3\Big[A(a,b)\Big]^{\alpha_3} = \gamma A(a,b) = C(\alpha a,\beta b) = C\Big[g_1(a^{\alpha_1}),g_2(a^{\alpha_2})\Big].$$

Thus (g_1,g_2,g_3) is a homotopy.

If $g_1(a_1^{\alpha_1}) = g_1(a_2^{\alpha_1})$, then $\alpha_1(a_1) = \alpha_1(a_2)$, i.e. $a_1^{\alpha_1} = a_2^{\alpha_1}$, from what follows taht the mapping g_1 is injective.

Let $a' \in \alpha(Q\ )$. Then there exists $a \in Q_1$ such that $\alpha a = a'$. Since $g_1(a^{\alpha}) = \alpha(a) = a'$, it follows that the mapping g_1 is surjective and thus it is bijective. Analogically, g_2 and g_3 are bijections. Therefore (g_1,g_2,g_3) is an isotopy.

The above results are a generalization of the theorem on homomorphisms applied on the generalized operations.

Let $(\alpha_1,\alpha_2,\alpha_3)$ be a normal generalized congruence in a quasigroup $(Q_1,Q_2,Q_3;A)$. An ordered triplet $(a^{\alpha_1},b^{\alpha_2},c^{\alpha_3})$, where $a^{\alpha_1}\in Q_1/\alpha_1$, $b^{\alpha_2}\in Q_2/\alpha_2$ and $c^{\alpha_3}\in Q_3/\alpha_3$, is called a <u>class</u> of the generalized congruence $(\alpha_1,\alpha_2,\alpha_3)$.

THEOREM 4. <u>A class</u> $(a^{\alpha_1},b^{\alpha_2},c^{\alpha_3})$ <u>is a</u> G-<u>subquasigroup of</u> $(Q_1,Q_2,Q_3;A)$ <u>iff there exist</u> $a_1\in a^{\alpha_1}$, $b_1\in b^{\alpha_2}$, $c_1\in c^{\alpha_3}$ <u>such that</u> $A(a_1,b_1) = c_1$.

Proof. Let there exist $a_1 \in a^{\alpha_1}$, $b_1 \in b^{\alpha_2}$ and $c_1 \in c^{\alpha_3}$, such that $A(a_1,b_1) = c_1$ and let $a' \in a^{\alpha_1}$, $b' \in b^{\alpha_2}$. Since $(\alpha_1,\alpha_2,\alpha_3)$ is a generalized congruence, it follows that $A(a',b')\alpha_3 A(a_1,b_1)$, i.e. $A(a',b')\ c_1^{\alpha_3} = c^{\alpha_3}$. Thus $(a^{\alpha_1},b^{\alpha_2},c^{\alpha_3};A)$ is a G-subgroupoid.

The equation $A(a',x) = c'$ with $a' \in a^{\alpha_1} \subsetneq Q_1$ and $c' \in c^{\alpha_3} \subsetneq Q_3$ has a unique solution x_0 in $(Q_1,Q_2,Q_3;A)$.

We will show that $x_0 \in b^{\alpha_2} \in Q_2$. Because of $a'\alpha_1 a_1$ and $x_0\alpha_2 x_0$, one obtains that $A(a_1,x_0)\alpha_3 A(a',x_0) = c'$, i.e. $A(a_1,x_0) \in c^{\alpha_3}$. Since $A(a_1,b_1) = c_1 \in c^{\alpha_3}$, it follows that $A(a_1,x_0)\alpha_3 A(a_1,b_1)$, i.e. $x_0\alpha_2 b_1$ or $x_0 \in b^{\alpha_2}$. Thus the equation $A(a',x) = c'$ has a unique solution $x_0 \in b^{\alpha_2}$.

Analogically, the equation $A(y,b') = c'$, where $b' \in b^{\alpha_2} \subsetneq Q_2$, $c' \in c^{\alpha_3} \subsetneq Q_3$, has a unique solution $y_0 \in a^{\alpha_1}$.

R E F E R E N C E S

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ХОМОТОПИЈА И КОНГРУЕНЦИИ НА ОБОПШТЕНИ КВАЗИГРУПИ

Кирил Стојменовски

Р е з и м е

Во работата се разгледува поимот конгруенција и факторструктура на обопштен групоид, и се утврдува врската помеѓу хомотопијата и нормалните конгруенции на обопштени квазигрупи.