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# SOME EXAMPLES OF FULLY COMMUTATIVE VECTOR-VALUED GROUPS

A b s t r a c t. Fully commutative $(n,m)$-groupoids (quasigroups, semigroups, groups) are considered in [2] and [3]. The following results are shown in [3]. If $k \geqslant 1$, $m \geqslant 2$, then: (i) a fully commutative $(m+k, m)$-group exists on a non-empty finite set $Q$ iff $Q$ contains at most two elements; (ii) every infinite set $Q$ is the carrier of a fully commutative $(m+k, m)$-group. In [4] a convenient description of the free fully commutative $(m+k, m)$-groups is given. We also note that, until now, free fully commutative $(m+k, m)$-groups are the unique known examples of such structures. In this paper we give some natural examples of fully commutative $(m+k,m)$-groups on $\mathbf{C}$ (=the field of complex numbers), and examine some of their properties. We note that $\mathbf{C}$ can be replaced by an arbitrary algebraically closed field.

## §1. *Introduction*

Let $G$ be an arbitrary set. In the $j$-th Cartesian power $G^j$ of $G$ we define a relation $\sim$ as follows:

$$(x_1, \dots ,x_j) \sim (y_1, \dots ,y_j)$$

iff there exists a permutation $\tau$ in the set $\{1, \dots ,j\}$ such that $y_k = x_{\tau(k)}$ for $k=1, \dots ,j$. The set $G^j/\sim$ we shall denote further by $G^{(j)}$, and the elements of that set we shall denote as $(a_1, \dots , a_j) = a_1^j$ for each number $j$.

The following definition of fully commutative groups is given in [3], and it is a continuation of the research into vector-valued algebraic structures (see for example [1]).

The notion of the fully commutative $(n, m)$-groupoid is defined in [2]. Namely, if $n, m$ are positive integers and $Q$ is a non-empty set, then every mapping $[\ \ ] : Q^{(n)} \to Q^{(m)}$ is called a fully commutative $(n, m)$-groupoid on $Q$. A fully commutative $(m+k, m)$-groupoid $(Q;\ [\ \ ])$ is called a fully commutative $(m+k, m)$-group iff the following two axioms are satisfied

(i) (associativity) $[[x_1^{m+k}]\, x_{m+k+1}^{m+2k}] = [x_1^i[x_{i+1}^{i+m+k}]\, x_{i+m+k+1}^{m+2\ k}]$ for each $x_j \in G$ $(j \in \{1, \dots , m+2k\})$ and for each $i \in \{1,2, \dots ,k\}$;

(ii) (solvability) For each $a_1, \dots ,a_m$, $b_1, \dots ,b_k \in G$, there exist unique $x_1^m \in G^{(m)}$ such that

$$[x_1^m b_1^k] = a_1^m.$$

Further on, "$(n, m)$-groupoid", "$(n, m)$-group" will mean "fully commutative $(n, m)$-groupoid" and "fully commutative $(n, m)$-group", respectively.

Let $P_n$ be the set of canonical polynomials of degree $n$ on $\mathbf{C}$, i.e.

$$P_n = \{t^n + a_{n-1}t^{n-1}+\ldots+a_1t+a_0 \mid a_i \in \mathbf{C}\,,\quad i=0, 1, \ldots, n-1\}.$$

The fact that $\mathbf{C}$ is an algebraically closed field implies that there exists a bijection between the mappings from $P_n$ into $P_m$ and $(n, m)$-groupoids on $\mathbf{C}$. Namely, if

$$\varphi : t^n + a_{n-1}\, t^{n-1}+\ldots+a_1t+a_0 \to t^m+b_mt^{m-1}+\ldots+b_1t+b_0\,,$$

$$\varphi : (t-z_1)\ldots(t-z_n) \to (t-r_1)\ldots(t-r_m)$$

we can define an $(n, m)$-groupoid $[\ ]_\varphi$ by $[z_1^n]_\varphi = r_1^m$, and inversely if $[\ ] : C^{(n)} \to C^{(m)}$ is an $(n, m)$-groupoid then $[\ ]$ induces a mapping from $P_n$ into $P_m$. Now it is easy to see that $[\ ]$ is an associative $(m+k, m)$-groupoid on $\mathbf{C}$, iff the polynomial $(t-r_1)\ldots(t-r_m)$ is invariant under each permutation of the numbers $z_1, \ldots, z_{m+2k}$, where $r_1^m = [\,[z_1^{m+k}]z_{m+k+1}^{m+2k}]$. Namely,

$$(t-r_1)\ldots(t-r_m) = \psi\,(\psi\,[\,(t-z_1)\ldots(t-z_{m+k})\,]\,(t-z_{m+k+1})\ldots(t-z_{m+2k})),$$

and hence $[\ ]$ is associative iff the polynomial

$$\psi\,(\psi\,[\,(t-z_1)\ldots(t-z_{m+k})\,]\,(t-z_{m+k+1})\ldots(t-z_{m+2k}))$$

is invariant under each permutation of the numbers $z_1, \ldots, z_{m+2k}$.

Using the above connection between polynomials on $\mathbf{C}$ and $(n, m)$-groupoids on $C$, we shall construct in the following section some classes of $(m+k, m)$-groups.

## §2. *Examples*

Example 1. The mapping

$$\psi: t^{m+1}+a_mt^m+\ldots+a_1t+a_0 \to t^m+a_mt^{m-1}+\ldots+a_2t+a_1$$

induces an $(m+1,m)$-group on $\mathbf{C}$. In order to prove the axiom of associativity, we notice that $\psi$ can be extended over $\bigcup_{n=0}^{\infty} P_n$ as follows:

$$\psi\,(t^{m+k}+a_{m+k-1}t^{m+k-1}+\ldots+a_0) = t^m+a_{m+k-1}t^{m-1}+\ldots+a_k\,,$$

and $\psi\,(p) = p$ if $\deg\,(p) \leqslant m$. Now it is obvious that $\psi\,(p_1 \cdot p_2) = \psi\,(\psi(p_1) \cdot \psi(p_2))$ and $\psi\,(\psi\,(p)) = \psi\,(p)$. Thus we obtain

$$\psi\,(\psi\,[(t-z_1)\ldots(t-z_{m+k})\,]\,(t-z_{m+k+1})\ldots(t-z_{m+2k}) =$$
$$= \psi\,(\psi^2\,[\,(t-z_1)\ldots(t-z_{m+k})\,]\,\psi\,[\,(t-z_{m+k+1})\ldots(t-z_{m+2k})\,]\,)=$$
$$= \psi(\,(t-z_1)\ldots(t-z_{m+2k})\,)$$

and this polynomial is invariant under each permutation of the numbers $z_1, \dots, z_{m+2k}$.

The axiom of associativity can be also verified in this way. Let us suppose for example that $m=2$ and $k=1$. Then

$$\psi(t^3+a_2t^2+a_1t+a_0) = t^2+a_2t+a_1,$$

and hence

$$[z_1z_2z_3] = (w_1,w_2) \Leftrightarrow \begin{cases} z_1+z_2+z_3 = w_1+w_2 \\ z_1z_2+z_1z_3+z_2z_3=w_1w_2 \,. \end{cases}$$

Further, if $[z_1z_2z_3] = (w_1,w_2)$. we obtain $[[z_1z_2z_3]\, z_4] = [w_1w_2z_4] = (a,b)$ where $a$ and $b$ satisfy the following conditions

$$a+b = w_1+w_2+z_4 = z_1+z_2+z_3+z_4$$

and

$$ab = w_1w_2+w_1z_4+w_2z_4 = z_1z_2+z_1z_3+z_2z_3+z_1z_4+z_2z_4+z_3z_4.$$

Since $a+b$ and $ab$ are invariant under each permutation of the numbers $z_1,z_2,z_3$ and $z_4$, the associative law is satisfied.

In order to verify the axiom of solvability, let us suppose that $w_1,\dots,w_m, z \in \mathbf{C}$ are given numbers. We should show the existence of $z_1,\dots,z_m \in \mathbf{C}$ such that $[z_1^m z] = w_1^m$. First, let $a_1,\dots,a_m \in \mathbf{C}$ be such that

$$t^m+a_mt^{m-1}+\dots+a_2t+a_1 = (t-w_1)\dots(t-w_m),$$

and then $a_0$ can uniquely be determined such that $z_0$ is a root of the polynomial $t^{m+1}+a_mt^m+\dots+a_1t+a_0$. If $z_0{}^m$ is the complete sequence of roots of this polynomial, then we have $[z_0^m] = w_1^m$. This implies that $(\mathbf{C}, [\ ])$ is an $(m+1,m)$-group.

This $(m+1,m)$-group, induces an $(m+k,m)$-group on $\mathbf{C}$. The corresponding mapping is:

$$\psi(t^{m+k}+a_{m+k-1}t^{m+k-1}+\dots+a_1t+a_0) = t^m+a_{m+k-1}\,t^{m-1}+\dots+a_k\,.$$

In special case when $m=k=1$, we obtain $[z_1z_2] = z_1+z_2$, and that is why the above $(m+k,m)$-group is called additive, denoted by $[\ ]_+$.

The proofs in the following examples are analogous, and we shall omit them.

Example 2. Let $\alpha \in \mathbf{C}$, $\alpha \neq 0$. The mapping

$$\psi: t^{m+1}+a_mt^m+\dots+a_0 \to t^m+a_mt^{m-1}+\dots+a_1+\alpha$$

induces an $(m+1,m)$-group on $\mathbf{C}$. This group induces an $(m+k,m)$-group on $\mathbf{C}$, and:

$$\psi: t^{m+k}+a_{m+k+1}t^{m+k-1}+\dots+a_0 \to t^m+a_{m+k-1}t^{m-1}+\dots+a_k+k\alpha$$

is the corresponding polynomial mapping.

Example 3. Let $\beta \in \mathbf{C}$, $\beta \neq 0$. The mapping

$$\psi : t^{m+1}+a_m t^m+\ldots+a_0 \to t^m+(a_{m-1}t^{m-1}+\ldots+a_0)\,\beta$$

induces an $(m+1,m)$-group on $\mathbf{C}\setminus\{0\}$. This group induces an $(m+k, m)$-group on $\mathbf{C}/\{0\}$, and the corresponding polynomial mapping is

$$\psi : t^{m+k}+a_{m+k-1}t^{m+k-1}+\ldots+a_0 \to t^m+\beta^k(a_{m-1}t^{m-1}+\ldots+a_0)\,.$$

Specially i f $\beta=-1$, we obtain an $(m+k, m)$-group on $\mathbf{C}\setminus\{0\}$, which is a generalization of the usual multiplication. Indeed, if $m=k=1$ we obtain $[z_1 z_2]=z_1 \cdot z_2$. More generally, if $[z_1^{m+k}]=w_1^m$, then

$$z_1 \cdot z_2 \ldots z_{m+k} = w_1 \cdot w_2 \ldots w_m\,.$$

So the above $(m+k,m)$-group for $\beta=-1$ will be called multiplicative group, and denoted by [ ].

Example 4. Let $\gamma \in \mathbf{C}$, $\gamma \neq 0$. The mapping

$$\psi : t^{m+1}+a_m t^m+\ldots+a_0 \to t^m+a_m t^{m-1}+\ldots+a_1+\gamma a_0$$

induces an $(m+1, m)$-group on $\mathbf{C}\setminus\{1/\gamma\}$. This group induces an $(m+k, m)$-group on $\mathbf{C}\setminus\{1/\gamma\}$, and:

$$\psi : t^{m+k}+a_{m+k-1}\, t^{m+k-1}+\ldots+ a_0 \to$$

$$\to t^m + a_{m+k-1}t^{m-1}+\ldots+a_k+\gamma a_{k-1}+\gamma^2 a_{k-2}+\ldots+\gamma^k a_0$$

is the corresponding polynomial mapping.

Example 5. The mapping $\psi: P_3 \to P_2$ which is defined by

$$\psi: t^3+a_2t^2+a_1t+a_0 \to t^2+(a_2+\delta)\, t+a_1+\delta a_2+\delta^2$$

induces a $(3, 2)$-group on $\mathbf{C}$.

## §3. *The problem of isomorphism of the obtained groups*

**Proposition** 3.1. All of the $(m+k, m)$-groups of the example 2 are isomorphic.

**Proof**. For the sake of simplicity we shall prove this proposition for $m=2$ and $k=1$.

Let us consider two groups for $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. Then

$$[z_1 z_2 z_3]_{\alpha_1} = (w_1, w_2) \Leftrightarrow$$

$$\text{(3.1)} \qquad \begin{cases} z_1+z_2+z_3=w_1w_2 \\ \alpha_1+z_1z_2+z_1z_3+z_2z_3=w_1w_2 \end{cases}$$

and

$$[z_1 z_2 z_3]_{\alpha_2} = (w_1, w_2) \Leftrightarrow$$

$$\begin{cases} z_1 z_2 z_3 = w_1 + w_2 \\ \alpha_2 + z_1 z_2 + z_1 z_3 + z_2 z_3 = w_1 w_2. \end{cases}$$

We shall find a bijection $\phi : \mathbf{C} \to \mathbf{C}$ such that

$$[z_1 z_2 z_3]_{\alpha_1} = (w_1, w_,) \Rightarrow [\phi(z_1)\, \phi(z_2)\, \phi(z_3)]_{\alpha_2} = (\phi(w_1), \phi(w_2))$$

i.e. (3.1) $\Rightarrow$ (3.2), where

$$\text{(3.2)} \qquad \begin{cases} \phi(z_1) + \phi(z_2) + \phi(z_3) = \phi(w_1) + \phi(w_2) \\ \alpha_2 + \phi(z_1)\, \phi(z_2) + \phi(z_1)\, \phi(z_3) + \phi(z_2)\, \phi(z_3) = \phi(w_1)\, \phi(w_2). \end{cases}$$

The implication (3.1) $\Rightarrow$ (3.2) is satisfied for $\phi(z) = z\sqrt{\alpha_2/\alpha_1}$ . ||

**Proposition 3.2**. All of the $(m+k, m)$-groups of the example 3 are isomorphic.

**Proof**. The mapping $\phi : \mathbf{C}\setminus\{0\} \to \mathbf{C}\setminus\{0\}$ which is defined by $\phi(z) = z\beta_1/\beta_2$ is a bijection, and it yields to the required isomorphism between the groups which are parametrized by $\beta_1$ and $\beta_2$. ||

**Proposition 3.3**. All of the $(m+k, m)$-groups of the example 4 are isomorphic.

**Proof**. The mapping $\phi : \mathbf{C}\setminus\{1/\gamma_1\} \to \mathbf{C}\setminus(1/\gamma_2\}$ which is defined by $\phi(z) = z\gamma_1/\gamma_2$ is a bijection, and it is an isomorphism between the groups which are parametrized by $\gamma_1$ and $\gamma_2$. ||

If $\phi$ is a homomorphism of $(m+k, m)$-groups, then for each positive integer $n$, $\phi$ induces a mapping $\xi_n : P_n \to P_n$ such that

$$\xi_n [(t - z_1) \ldots (t - z_n)] = (t - \phi(z_1)) \ldots (t - \phi(z_n)),$$

and such that the following diagram

$$\begin{array}{ccc} P_{m+k} & \xrightarrow{\ \psi\ } & P_m \\ \downarrow{\scriptstyle \xi_{m+k}} & & \downarrow{\scriptstyle \xi_m} \\ P_{m+k} & \xrightarrow{\ \psi'\ } & P_m \end{array}$$

is commutative, where $\psi$ and $\psi'$ are the corresponding polynomial mappings for the given $(m+k, m)$-groups. Inversely, if the mappings $\xi_{m+k} : P_{m+k} \to P_{m+k}$ and $\xi_m : P_m \to P_m$ are induced from a mapping $\phi$ and the above diagram is commutative, then $\phi$ induces a homomorphism of $(m+k, m)$-groups which are given by $\psi$ and $\psi'$. This will be used in the proof of the following proposition.

**Proposition 3.4.** The (3, 2)-group which was defined in example 5, is isomorphic to the additive (3,2)-group on **C**. The required isomorphism is induced by the mapping $\phi : \mathbf{C} \to \mathbf{C}$, $\phi(z) = z + \delta$.

**Proof.** The mappings $\xi_3 : P_3 \to P_3$ and $\xi_2 : P_2 \to P_2$ which are induced by the bijection $\phi(z) = z + \delta$ are

$$\xi_3(t^3 + a_2 t^2 + a_1 t + a_0) = t^3 + (a_2 - 3\delta)t^2 + (a_1 - 2\delta a_2 + 3\delta^2)t + a_0 - \delta a_1 + \delta^2 a_2 - \delta^3,$$

$$\xi_2(t^2 + a_1 t + a_0) = t^2 + (a_1 - 2\delta)\, t + a_0 - \delta a_1 + \delta^2.$$

It is easy to verify that the following diagram

$$\begin{array}{ccc} P_2 & \xrightarrow{\psi} & P_2 \\ \downarrow{\scriptstyle \xi_3} & & \downarrow{\scriptstyle \xi_2} \\ P_3 & \xrightarrow{\psi'} & P_2 \end{array}$$

is commutative, where

$$\psi(t^3 + a_2 t^2 + a_1 t + a_0) = t^2 + a_2 t + a_1$$

and

$$\psi'(t^3 + a_2 t^2 + a_1 t + a_0) = t^2 + (a_2 + \delta)\, t + a_1 + \delta a_2 + \delta^2. \quad ||$$

As a summary of the above considerations we obtain that each of the above (3, 2)-groups is isomorphic to one of the four groups, which are given by the following polynomial mappings:

(i) $\psi_1 : t^3 + a_2 t^2 + a_1 t + a_0 \to t^2 + a_2 t + a_1$, on **C**

(ii) $\psi_2 : t^3 + a_2 t^2 + a_1 t + a_0 \to t^2 + a_2 t + a_1 + 1$, on **C**

(iii) $\psi_3 : t^3 + a_2 t^2 + a_1 t + a_0 \to t^2 - a_1 t - a_0$, on $\mathbf{C} \setminus \{0\}$

(iv) $\psi_4 : t^3 + a_2 t^2 + a_1 t + a_0 \to t^2 + a_2 t + a_1 + a_0$, on $\mathbf{C} \setminus \{1\}$.

**Theorem 3.5.** The four (3, 2)-groups induced by the polynomial mappings $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ are pairwise non-isomorphic.

**Proof.** We remind ourselves [3] that if $(Q, [\ ])$ is a (3, 2)-group then there exists a unique $e_1^2 \in Q^{(2)}$ (called a unit of $(Q; [\ ])$) such that $[x_1^2 e_1^2] = x_1^2$.

Denote by $[\ ]_i$ the (3, 2)-group operation induced by $\psi_i$. It can be easily seen that:

a) (0,0) is the unit of $[\ ]_1$,

b) $(\sqrt{2}, -\sqrt{2})$ is the unit of $[\ ]_2$,

c) $(i, -i)$ is the unit of $[\ ]_3$, and

d) (0, 0) is the unit of $[\ ]_4$.

Thus the units of the first and the fourth groups have the form $(x, x)$, but the units of the second and the third groups have the form $(x,y)$ where $x \neq y$. So the first group is not isomorphic with the second and the third group, and the fourth group is not isomorphic with the second and third groups either.

It can easily be seen that

$$[xxx]_1 = (y,y) \Rightarrow x = y \quad (=0)$$

and

$$\left[\frac{3}{4}\,\frac{3}{4}\,\frac{3}{4}\right]_4 = \left(\frac{9}{8}, \frac{9}{8}\right)$$

which implies that $[\ ]_1$ and $[\ ]_4$ are not isomorphic.

Now we shall prove that the third and the second groups are not isomorphic. Indeed, the second group satisfies the following implication

$$[\sqrt{2}\ -\sqrt{2}\ x]_2 = (y, y) \Rightarrow x=2i \vee x=-2i,$$

but the third group satisfies the following implication

$$[i\ -i\ \ x]_3 = (y,y) \Rightarrow x=1/4,$$

and hence there does not exist an isomorphism between these two groups. ||

If we consider (2, 1)-groups instead of (3, 2)-groups, then we obtain the following groups

(i) $[x\ y] = x+y$, on **C**

(ii) $[x\ y] = x+y-1$, on **C**

(iii) $[xy] = x \cdot y$, on $\mathbf{C}\setminus\{0\}$

[iv) $[xy] = x+y-x \cdot y$, on $\mathbf{C}\setminus\{1\}$

It is well known that the second group is isomorphic with the first group, the fourth group is isomorphic with the third group, and the first and the third groups are not isomorphic. The same result also holds for the arbitrary $(k+1, 1)$-group.

In a general case, ecah of the above $(m+k, m)$-groups from examples 1—4 is isomorphic to one of four $(m+k, m)$-groups. Since all of them are induced from $(m+1, m)$-groups, we can consider the units of the corresponding $(2m, m)$-groups. The unit for the first and the unit for the fourth group is $\underbrace{(0,0,\ldots,0)}_{m}$. The unit for the third group is $(z_1, \ldots, z_m)$ where $z_1, \ldots, z_m$ are roots of the polynomial $z^m+(-1)^m$, and the unit for the second group is $(z_1, \ldots, z_m)$, where $-z_1, \ldots, -z_m$ are roots of the polynomial $z^m-m$. Hence we obtain that if $m \geqslant 2$, then the first group is not isomorphic with the second and the third group, and the fourth group is not isomorphic with the second and the third group, either.

## § 4. *Some properties*

**Proposition 4.1**. Let $n$ be an arbitrary positive integer. Then:

(i) in the additive (3, 2)-group $[\ \ ]_+$ we have

$[x^n y^n z^n]_+ = [u^n v^n]_+$ iff $[xyz]_+ = (u,v)$, and

(ii) the equality $[x^n y^n z^n]_\cdot = [u^n v^n]_\cdot$ holds in the multiplicative (3,2)-group $[\ \ ]_\cdot$ iff at least one of the equalities $[xyz]_\cdot^{(\beta_i)} = (u,v)$, $i=0, 1, \ldots, n-1$ is satisfied, where $\beta_0, \beta_1, \ldots, \beta_{n-1}$ are $n$-th roots of unit, and $[\ \ ]_\cdot^{(\beta_i)}$ is the corresponding parametric group of the example 3.

**Proof**. We shall prove this proposition for $n=2$.

(i) $[xxyyzz]_+ = [uuvv]_+$

$$\Leftrightarrow \begin{cases} 2x+2y+2z=2u+2v \\ x^2+y^2+z^2+4xy+4yz+4zx=u^2+v^2+4uv \end{cases}$$

$$\Leftrightarrow \begin{cases} x+y+z=u+v \\ xy+yz+zx=uv \end{cases}$$

$\Leftrightarrow [xyz]_+ = (u,v)$.

(ii) First we notice that, for the multiplicative (3, 2)-group, we have:

$$[z_1^p]_\cdot = [w_1^q]_\cdot$$

$$\Leftrightarrow \begin{cases} z_1 z_2 \ldots z_p = w_1 w_2 \ldots w_q \\ z_1 \ldots z_{p-1} + z_1 \ldots z_{p-2} z_p + \ldots + z_2 z_3 \ldots z_p = w_1 w_2 \ldots w_{q-1} + \ldots + \\ \qquad + \ldots + w_2 \ldots w_q \end{cases}$$

$$\Leftrightarrow \begin{cases} z_1 z_2 \ldots z_p = w_1 w_2 \ldots w_q \\ \dfrac{1}{z_1} + \ldots + \dfrac{1}{z_p} = \dfrac{1}{w_1} + \ldots + \dfrac{1}{w_q}. \end{cases}$$

Using this fact, we obtain

$$[xxyyzz]_{.} = [uuvv]$$

$$\Leftrightarrow \begin{cases} x^2y^2z^2 = u^2v^2 \\ \dfrac{2}{x}+\dfrac{2}{y}+\dfrac{2}{z}=\dfrac{2}{u}+\dfrac{2}{v} \end{cases}$$

$$\Leftrightarrow \begin{cases} xyz=uv \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{u}+\dfrac{1}{v} \end{cases} \quad \text{or} \quad \begin{cases} xyz=-uv \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{u}+\dfrac{1}{v} \end{cases}$$

$$\Leftrightarrow \quad [xyz]_{.} = (u,v) \qquad \text{or} \quad [xyz]_{.}^{(-1)} = (u,v). \ ||$$

**Remark**. The implication $[xyz] = (u,v) \Rightarrow [x^n y^n z^n] = [u^n v^n]$ holds for the arbitrary (3,2)-group. The proposition 4.1. can be generalized for arbitrary additive and multiplicative $(m+k,\ m)$-groups.

**Proposition** 4.2. For the additive and the multiplicative (3, 2)-groups the following implication is satisfied

$$[xyz]_{+} = [x'y'z']_{+} \Rightarrow$$

(4.1) $$[[pqx]\ .\ [pqy]\ .\ [pqz]\ .]_{+} = [[pqx']\ .\ [pqy']\ .\ [pqz']\ .\ ]_{+}.$$

**Proof**. First we remark that

$$[xyz]_{+} = [x'y'z']_{+} \Leftrightarrow$$

(4.2) $$x+y+z = x'+y'+z' \quad \text{and}$$

(4.3) $$xy+yz+zx = x'y'+y'z'+z'x'.$$

Let us suppose that

$$[pqx]_{.} = (u_1,\ u_2),\ [pqy]_{.} = (u_3,\ u_4),\ [pqz]_{.} = (u_5,\ u_6)$$

and $[pqx']_{.} = (u_1',u_2')$, $[pqy']_{.} = (u_3',u_4')$, $[pqz']_{.} = (u_5',u_6')$. We should prove that

(4.4) $$u_1+u_2+u_3+u_4+u_5+u_6=u_1'+u_2'+u_3'+u_4'+u_5'+u_6', \quad \text{and}$$

(4.5) $$\sum_{1\leqslant i<j\leqslant 6} u_i u_j = \sum_{1\leqslant i<j\leqslant 6} u_i'\ u_j' .$$

Since $(u_1+u_2)+(u_3+u_4)+(u_5+u_6)=(pq+px+qx)+(pq+py+qy)+(pq+pz+qz)=$ $=3pq+(p+q)(x+y+z)$, and analogously $u_1'+u_2'+u_3'+u_4'+u_5'+u_6'=3pq+$ $+(p+q)\ (x'+y'+z')$, the proof of (4.4) follows from (4.2).

3*

$$\sum_{1\leqslant i<j\leqslant 6} u_i u_j = u_1u_2+u_3u_4+u_5u_6+(u_1+u_2)\ (u_3+u_4)\ +\ (u_1+u_2)\ (u_5+u_6)\ + +(u_3+u_4)\,(u_5+u_6)=pqx+pqy+pqz+(pq+px+qx)\,(pq+py+qy)+ +\ (pq+px+qx)\ (pq+pz+qz)+(pq+py+qy)\ (pq+pz+qz)= =pq(1+2p+2q)\ (x+y+z)+3p^2q^2+(p+q)^2\ (xy+xz+yz).$$

Analogously one obtains

$$\sum_{1\leqslant i<j\leqslant 6} u_i'\ u_j' = pq\ (1+2p+2q)\ (x'+y'+z')+3p^2q^2+(p+q)^2(x'y'+y'z'+z'x')$$

and the proof of (4.5) follows from (4.2) and (4.3). ||

**Proposition 4.3**. The usual multiplication is distributive with respect to the additive $(m+k,\ m)$-group operation, i.e.

$$z\ .\ [z_1^{m+k}]_+ = [(zz_1)\ (zz_2)\ \ldots\ (zz_{m+k})\ ]_+,$$

where a multiplication of complex number and an $m$-type is defined in the usual way.

**Proof**. Let us denote by $\psi$ the polynomial mapping for the additive $(m+k,\ m)$-group. Then

$$\psi\ ((t-zz_1)\ \ \ldots\ \ (t-zz_{m+k})) =$$

$$=\psi(t^{m+k}-z(\sum_{i=1}^{m+k} z_i)\ t^{m+k-1}+\ldots+(-z)^{m+k}z_1\ldots z_{m+k}) =$$

$$= t^m-(\sum_{i=1}^{m+k} zz_i)t^{m-1}+\ldots+(-1)^m\left(\sum_{\substack{i_1,\ldots,i_m=1\\ i_1<\ldots<i_m}}^{m+k} (zz_{i_1})\ \ldots\ (zz_{i_m})\right)$$

$$= (t-zw_1)\ldots(t-zw_m)$$

where $w_1, \ldots, w_m$ are roots of the following polynomial

$$t^m-(\sum_{i=1}^{m+k} z_i)\ t^{m-1}+\ldots+(-1)^m \sum_{\substack{i_1,\ldots,i_m=1\\ i_1<\ldots<i_m}}^{m+k} z_{i_1}\ldots z_{i_m}\,.$$

Hence we obtain

$$[(zz_1)\ldots(zz_{m+k})\ ]_+ = (zw_1,\ldots,zw_m) = z\ .\ (w_1,\ldots,w_m) = = z\ .\ [z_1^{m+k}]_+\,.\ \ ||$$

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*КОСТАДИН ТРЕНЧЕВСКИ*

## НЕКОЛКУ ПРИМЕРИ НА ПОТПОЛНО КОМУТАТИВНИ ВЕКТОРСКО ВРЕДНОСНИ ГРУПИ

(Резиме)

Потполно комутативните $(n, m)$-групоиди (квазигрупи, полугрупи, групи) се разгледувани во [2] и [3]. Следните резултати се добиени во [3]. Ако $k \geq 1$, $m \geq 2$, тогаш: (i) на непразното конечно множество $Q$ постои потполно комутативна $(m+k,m)$. група акко $Q$ содржи најмногу два елемента; (ii) Секое бесконечно множество $Q$ е носач на потполно комутативна $(m+k,m)$-група. Во [4] е даден погоден опис на слободната потполно комутативна $(m+k,m)$-група. Да уочиме дека слободните потполно комутативни $(m+k.m)$ групи се единствените познати примери на такви структури. Во овој труд се дадени неколку природни примери на потполно комутативни $(m+k,m)$-групи над полето од комплексни броеви $\mathbf{C}$, и испитувани се некои нивни особини. Да забележиме дека $\mathbf{C}$ може да биде заменето со произволно алгебарски затворено поле.

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