VECTOR VALUED GROUPOIDS, SEMIGROUPS AND GROUPS

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§0. INTRODUCTION

The main aim of this work is an investigation of structures with one vector valued operation, i.e. vector valued groupoids, with a special attention on vector valued semigroups and groups. Almost all the results of this kind of structures known up to now are given here and many new results are obtained too (they are noted in §10).

This work is divided into ten section. In §1 we define the notion of "(n,m)-groupoid" as an ordered pair $\underline{Q}=(Q;f)$, where Q is a nonempty set, n and m are positive integers and $f:Q^n \to Q^m$ is a mapping. In §2 some classes of v.v. groupoids are considered (here and further on, "v.v." is an abbreviation for "vector valued"). Here we define the classes of commutative v.v. groupoids, v.v. quasigroups, v.v. semigroups and v.v. groups, and we investigate some elementary relations between these classes.

Although well-known, the necessary definitions and properties for presentations of semigroups are given in §3, the reason being clearness and completeness of the subsequent investigation. The problem of embedding of a v.v. groupoid into a semigroup, given with a corresponding presentation, is often placed in this work. To any v.v. groupoid $\underline{Q}$ one associates a semigroup $\underline{Q}^\wedge$ which is generated by the set Q and is defined by a set of defining relations. ($\underline{Q}^\wedge$ is called the universal semigroup for $\underline{Q}$.)

In §4 we consider the question for embedding of a v.v. groupoid into a semigroup and we give a complete answer to this question. We consider also the notion of "pure embedding" of (n,m)-groupoids into semigroups and we give a complete answer for $n \geq m$. For the case $n < m$, we show that every free (n,m)-groupoid has the above property; however, we have not a satisfactory description for the class of (n,m)-groupoids which are pure (n,m)-subgroupoids of semigroups when $1 < n < m$.

In §5 the general associative law for v.v. semigroups is proved and a number of characteristic properties of v.v. semigroups and v.v. groups are obtained. The notions "poly-(n,m)-groupoid" and "poly-(n,m)-semigroups" are introduced too, and it is shown that there is no essential difference between the *class* of (n,m)-semigroups and the class of poly-(n,m)-semigroups.

In §6 the free v.v. semigroups are described and it is shown that every free v.v. semigroup is cancellative. Also a description of the universal semigroups of the free v.v. semigroups is given. Further investigations of the universal semigroups for v.v. semigroups, called "universal coverings", are done in §7. Explicit descriptions of universal coverings of v.v. semigroups are obtained and it is shown that every cancellative v.v. semigroup is embedable into a cancellative semigroup.

In §8, v.v. groups are investigated by means of their universal coverings and corresponding v.v. variants of Post and Hosszu-Gluskin Theorems are proved. The main goal of §9 is the investigation of the (n,m)-groups in the cases $n=2m$ and $n=m+1$. It is shown that the theory of (2m,m)-groups in its great part is analogous to the theory of the (usual) groups and that every set is a carrier of a (2m,m)-group. The (m+1,m)-groups are also closely connected to the groups, but the situation is essentially different when existence is in question. Namely, if G is a finite set such that $|G| > 1$, then there is no (m+1,m)-group with a carrier G. By the results of [14] it follows that every infinite set G is a carrier of an (m+1,m)-group. However, we know only one kind of nontrivial (m+1,m)-groups, namely the free ones. Examples of (2m+1,m)-groups are given, and it is shown that there is no finite nontrivial (5,3)-group with an odd order.

Further discussions - the notes and the comments made in §10, are related to the preceding nine section and some results of other papers.

The work ends with an index, a list of notations, and also, according to our knowledge, a complete bibliography on vector valued algebraic structures.

§1. VECTOR VALUED GROUPOIDS

If m and n are positive integers, then an (n,m)-operation on a nonempty set Q is any mapping f from Q^n into Q^m, where Q^s denotes the s-th Cartesian power of Q: $Q^2 = Q\times Q$, $Q^3 = Q\times Q\times Q, \dots,$ and $Q^1 = Q$.

For example, a (3,2)-operation on Q is a mapping f: $Q^3 \longrightarrow Q^2$, a (3,3)-operation on Q is a mapping g: $Q^3 \longrightarrow Q^3$, and (1,3)-operation on Q is a mapping h: $Q \longrightarrow Q^3$. In this sense, a (2,1)-operation means a binary operation and an (n,1)-operation means an n-ary operation.

In some cases, when it will not be necessary to emphasize the integers n and m, we will say vector valued operation (v.v.o.) instead of (n,m)-operation.

Let f be an (n,m)-operation on a set Q. We can associate to f a sequence of n-ary operations $f_1, f_2, \dots, f_m$ by putting.

$$((\forall i\in\{1,2,\dots,m\})\ f_i(a_1,\dots,a_n) = b_i) \iff f(a_1,\dots,a_n) = (b_1,\dots,b_m). \qquad (1.1)$$

Then we call f_i the i-th component operation of f and we write

$$f = (f_1, f_2, \dots, f_m). \qquad (1.2)$$

Conversely, if $f_1, f_2, \dots, f_m$ is a sequence of n-ary operations on the set Q, then there exists a unique (n,m)-operation f on Q such that (1.2) is true.

Thus, every (n,m)-operation f: $Q^n \longrightarrow Q^m$ induces a sequence $f_1, f_2, \dots, f_m$ of n-ary operations on the set Q, and the converse is also true.

If f is an (n,m)-operation on a set Q, then by the analogy with the case $n \geq 2$, $m = 1$, we call the pair $\underline{Q} = (Q;f)$ an (n,m)-groupoid. In that case, if the equality (1.2) is true, we say that cp$\underline{Q}$ = $(Q;f_1,f_2,\dots,f_m)$ is the component algebra of $\underline{Q}$ and $(Q;f_i)$ is the i-th component n-groupoid of $\underline{Q}$.

We will call an (n,m)-groupoid also a vector valued groupoid (v.v.g.).

Here, we will introduce some short notations which will be used frequently further on.

1) The elements of Q^s, i.e. the sequences $(a_1,a_2,\dots,a_s)$ $(a_i \in Q)$ will be denoted by $a_1a_2\cdots a_s$ or a_1^s; in some cases when there will be no risk of misunderstanding the sequence $(a_1,a_2,\dots,a_s)$ will be denoted by one letter, often underlined: $\underline{a}$. Thus the symbols

$$(a_1,a_2,\dots,a_s);\ a_1a_2\cdots a_s;\ a_1^s;\ \underline{a}$$

will denote the very same object, namely an element of Q^s.

2) The symbol x_i^j will denote the sequence $x_ix_{i+1}\cdots x_j$ when $i \le j$, and the empty sequence when $i > j$.

3) If $x_1=x_2=\dots=x_p=x$ $(x_i \in Q)$, then the sequence x_1^p is denoted by the symbol $\overset{p}{x}$.

4) The set $\{1,2,\dots,s\}$ will be denoted by N_s and by N_0 sometimes will be denoted the empty set. The set of positive integers will be denoted by N.

Let us return to the v.v. groupoids. The fact that we can associate the algebra cp$\underline{Q}$ to a given v.v.groupoid $\underline{Q}$, allows us to carry over all the notions which make sense for universal algebras to v.v. groupoids without giving their explicit definitions. Such notions are: a subgroupoid, a direct product, a homomorphism, a congruence, a factor groupoid etc.

For example, if $(Q;f)$ is an (n,m)-groupoid, then a nonempty subset P of Q is an (n,m)-subgroupoid of $(Q;f)$ iff:

$$a_1^n \in P^n \ \&\ f(a_1^n) = b_1^m \Longrightarrow b_1^m \in P^m. \tag{1.3}$$

Another example: if $\underline{Q} = (Q;f)$ and $\underline{Q}' = (Q';f')$ are (n,m)-groupoids, then a mapping $\phi: x \mapsto \bar{x}$ from Q into Q' is a homomorphism from $\underline{Q}$ into $\underline{Q}'$ iff:

$$f(a_1^n) = b_1^m \Longrightarrow f'(\bar{a}_1^n) = \bar{b}_1^m. \tag{1.4}$$

One can define the other mentioned notions analogously.

Next we give a description of a free (n,m)-groupoid with a given basis B $(B \ne \emptyset)$, which ilustrates the use of homomorphisms as well. The description is simple enough, since it is the (absolute) free algebra with the basis B and signature $F=\{f_1,f_2,\dots,f_m\}$, where f_i is an n-ary operator and $f_i \neq f_j$ for $i \neq j$. This algebra can be described in the following way.

Put $H_o=B$ and suppose that the sequence $H_o,H_1,\dots,H_p$ of disjoint sets is built up. Let C_p be the set defined by:

$$C_p = \{u_1^n \in (H_o \cup H_1 \cup \dots \cup H_p)^n \mid (\exists i \in N_n) u_i \in H_p\}.$$

Then the set H_{p+1} is defined by:

$$H_{p+1} = N_m \times C_p.$$

Without loss of generality, we suppose that $H_{p+1} \cap H_i = \emptyset$ for every $i \in N$. Further on we will make such suppositions without an explicite explanation.

If $u \in H_p$, then we say that the element u has the hierarchy p and we write $\chi(u) = p$.

Now, put $\overline{B} = \bigcup_{p\geq 0} H_p$ and define n-ary operations $f_1,f_2,\dots,f_m$ in $\overline{B}$ by:

$$(\forall i \in N_m)\ f_i(u_1^n) = (i,u_1^n). \tag{1.5}$$

So we obtain an (n,m)-groupoid $(\overline{B};f)$, where $f=(f_1,f_2,\dots,f_m)$.

The following theorem shows that $(\overline{B};f)$ is a free (n,m)-groupoid with a basis B.

Theorem 1.1. *If $Q = (Q;g)$ is an (n,m)-groupoid, and ξ is a mapping from a nonempty set B into Q, then there exists a unique homomorphism ζ of the (n,m)-groupoid $(\overline{B};f)$ into the groupoid $(Q;g)$ which is an extension of ξ.*

Proof. Let D be an (n,m)-subgroupoid of $(\overline{B};f)$ such that $B=H_o \subseteq D$. Suppose that every element which has a hierarchy less than or equal to p is in D. If $\chi(u)=p+1$, then

$$u = (i,u_1^n) = f_i(u_1^n) \in D.$$

Thus, $H_{p+1} \subseteq D$ and so $\overline{B} = D$, which means that B is a generating set of $(\overline{B};f)$.

Let $(Q;g)$ be an (n,m)-groupoid and let $\xi: B \longrightarrow Q$ be a mapping. So we have a mapping ζ defined on H_0. Further on we use induction on hierarchy. If $\chi(u)=p+1$, then $u=(i,u_1^n)$, where $\chi(u_\nu) \leq p$, and so $\zeta(u_\nu)=\bar{u}_\nu \in Q$. Setting

$$\zeta(u) = g_i(\bar{u}_1^n),$$

we obtain

$$\zeta(f_i(u_1^n)) = g_i^*(\bar{u}_1^n).$$

Thus, ζ is a homomorphism which is an extension of ξ. The fact that B is a generating set implies the conclusion. □

Example 1.2. Let $A=\bigcup_{\alpha\geq 0} N_m^\alpha$ and define unary operations g_i $(i\in N_m)$ on A by:

$$u\in N_m^\alpha \Longrightarrow g_i(u) = iu\in N_m^{\alpha+1}$$

for $i\in N_m$. Let $g=(g_1,g_2,\dots,g_m)$. Then it is easily seen that $(A;g)$ is isomorphic to the free $(1,m)$-groupoid with a basis consisting of m elements. Thus, if $m=1$, then $(A;g)$ is isomorphic to $(N;s)$, where $s(x)=x+1$ for all $x\in N$.

§2. SOME TYPES OF VECTOR VALUED GROUPOIDS

Here we will consider the problem of classification of v.v. groupoids in general. A corresponding classification can be made according to assumed useful properties or by following the classifications of the binary groupoids. Here we will use the second possibility.

Suppose that $\mathcal{G}$ is a class of groupoids and consider the problem of defining a set $\{\mathcal{G}(n,m) \mid n,m \geq 1\}$ such that, for every $n,m \geq 1$, $\mathcal{G}(n,m)$ is a class of (n,m)-groupoids, where $\mathcal{G}(2,1)=\mathcal{G}$. Of course, this problem is not uniquely solvable because the only demand $\mathcal{G}(2,1)=\mathcal{G}$ has not some essential limitation.

One of the possibly „acceptable" solutions is the component definition. In that case, it is necessary first to define the set $\{\mathcal{G}(n) \mid n \geq 1\}$ such that for every $n \geq 1$, $\mathcal{G}(n)$ is a class of n-groupoids which satisfies the condition $\mathcal{G}(2)=\mathcal{G}$.

Then, $\mathcal{G}(n,m)$ will be defined by:

„$(Q;f) \in \mathcal{G}(n,m) \iff (\forall i \in N_m)(Q;f_i) \in \mathcal{G}(n)$",

where f_i is the i-th component operation of f, i.e. $f=(f_1,f_2,\ldots,f_m)$.

For example, if $\mathcal{G}$ is some of the classes

(a) commutative groupoids, (b) cancellative groupoids,

(c) quasigroups, (d) semigroups, (e) groups,

then the definition of $\mathcal{G}(n)$ for every $n \geq 1$[1] is well-known in every of the above five cases. Therefore, if we use the component method, we will obtain five classes of (n,m)-groupoids and it is not necessary to define them explicitly. However, these definitions, except in the case of commutativity, are not equivalent with the corresponding "direct" v.v. definitions.

(a) An (n,m)-groupoid (Q;f) is said to be commutative iff for every permutation $\sigma \in S_n$ the following identity holds

$$f(x_1^n) = f(\sigma(x_1^n)), \tag{2.1}$$

where $\sigma(x_1^n) = x_{\sigma(1)}x_{\sigma(2)}\ldots x_{\sigma(n)}$.

This definition makes sense for every $n,m \geq 1$. On the other hand, it is clear that an (n,m)-groupoid is commutative iff each one of its component n-groupoid is commutative. Therefore the given direct definition (2.1) of the concept of commutative (n,m)-groupoid coincides with the corresponding component definition.

Here we will define a congruence in the (n,m)-groupoid $(\overline{B};f)$ (see T.1.1) which will bring us to a free c o m m u t a t i v e (n,m)-groupoid.

Define a relation $\approx$ on the free (n,m)-groupoid $(\overline{B};f)$ with a basis B by induction on hierarchy, i.e. if $u,v \in \overline{B}$, then:

1) $\chi(u) = 0 \Longrightarrow (u \approx v$ iff $u = v)$,

2) $\chi(u) = p+1 \Longrightarrow (u \approx v$ iff $u = (i,u_1^n)$, $v = (i,v_1^n)$ for some

[1] *1-groupoid is in fact a unar and so the cancellation is equivalent with injectivity, the quasigroupness with bijectivity, while the commutativity and associativity are always satisfied.*

$i \in N_m$ and there exists a permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$ such that $v_\nu = u_{i_\nu}$ for every ν).

Then the following statement is true:

Proposition 2.1. *The relation $\approx$ is a congruence on $(\overline{B}; f)$ and the factor (n,m)-groupoid $(\overline{B}/\approx; f)$ is a free commutative (n,m)-groupoid with a basis B.* □

The direct definitions of the corresponding generalizations of the classes (b)-(e) differ essentially from the component ones.

Two different non-component definitions for cancellative v.v. groupoids and v.v. quasigroups can be immediate realized: one for arbitrary n,m and the other for $n-m \geq 1$.

Namely, an (n,m)-groupoid (Q;f) is said to be a partial (n,m)-quasigroup iff the following condition is satisfied:

(b.1) If $x_\nu, y_\nu \in Q$ are such that

$$f(x_1^n) = x_{n+1}^{n+m}, \quad f(y_1^n) = y_{n+1}^{n+m}$$

and if there exists a sequence of positive integers $i_1, i_2, \ldots, i_n$ such that $1 \leq i_1 < i_2 < \ldots < i_n \leq n+m$, $x_{i_\nu} = y_{i_\nu}$, for every $\nu \in N_n$, then

$$x_1^{n+m} = y_1^{n+m}.$$

Examples 2.2. 1) Let + denote the usual addition on N, and define an n-ary operation f on N by $f(x_1^n) = x_1 + x_2 + \ldots + x_n$. Then one obtains a partial (n,1)-quasigroup.

2) Define a (2,2)-operation f on Z by $f(x_1^2) = (x_1 + x_2,\ x_1 - x_2)$. Then one obtains a partial (2,2)-quasigroup.

3) A partial (n,m)-quasigroup (Z;f) for every $n, m \geq 1$ one obtains if f is defined by:

$$f(x_1^n) = y_1^m \iff y_i = \sum_{j=1}^{n} j^{i-1} x_j, \quad i \in N_m.$$

An (n,m)-groupoid (Q;f) is called an (n,m)-quasigroup iff the following condition is satisfied:

(c.1) For every element $y_1^n \in Q^n$ and for every sequence of positive integers i_1^n such that $i_\nu < i_{\nu+1}$ there exists a uniquelly determined element $x_1^{n+m} \in Q^{n+m}$ such that $f(x_1^n)=x_{n+1}^{n+m}$ and $y_\nu = x_{i_\nu}$ for every $\nu \in N_n$.

(Clearly, every quasigroup is a (2,1)-quasigroup.)

Obviously, the following statement is true:

Proposition 2.3. *Every (n,m)-quasigroup is a partial (n,m)-quasigroup.* □

E.2.2. shows that the converse of P.2.3. is not true, i.e. there are partial (n,m)-quasigroups which are not (n,m)-quasigroups.

Example 2.4. Let $Q=Z_7$ and define a (3,3)-operation f on Q by:

$$f(x_1^3) = y_1^3 \iff y_1=x_1+x_2+x_3,\ y_2=x_1+2x_2+3x_3,\ y_3=x_1+4x_2+2x_3$$

Then it is easy to check that (Q;f) is a (3,3)-quasigroup.

Now suppose that $n-m=k \geq 1$. An (n,m)-groupoid (Q;f) is said to be cancellative iff every implication of the following form is true in (Q;f):

(b.2) $f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k) \implies x_1^m = y_1^m$,

for every $i \in \{0,1,\dots,k\}$.

Using a shortened notation, we can write this implication in the following way:

(b.2´) $f(\underline{a}\underline{x}\underline{b}) = f(\underline{a}\underline{y}\underline{b}) \implies \underline{x} = \underline{y}$,

where $\underline{a}\underline{b} \in Q^k$ and $\underline{x},\underline{y} \in Q^m$.

An (n,m)-groupoid (Q;f) is called a weak (n,m)-quasigroup ($n-m=k \geq 1$), iff the following condition is satisfied:

(c.2) For every $\underline{a}\underline{b} \in Q^k$ and $\underline{c} \in Q^m$, there exists a unique element $\underline{x} \in Q^m$ such that

$$f(\underline{a}\underline{x}\underline{b}) = \underline{c}.$$

The following proposition follows directly from the preceding definitions:

Proposition 2.5. *If $n-m=k \geq 1$, then:*

(i) Every (n,m)-quasigroup is a weak (n,m)-quasigroup.

(ii) Every weak (n,m)-quasigroup is a cancellative (n,m)-groupoid.

(iii) Every partial (n,m)-quasigroup is a cancellative (n,m)-groupoid. □

Let $n-m=k \geq 1$. An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-semigroup iff the following equality

(d.1) $f(f(x_1^n)x_{n+1}^{n+k}) = f(x_1^j f(x_{j+1}^{j+n})x_{j+n+1}^{n+k})$

is an identity in $(Q;f)$, for every $j \in N_k$.

Examples 2.6. Here we suppose that Q is a nonempty set and that $n-m=k \geq 1$.

1) Fix an element $a_1^m \in Q^m$ and put $f(x_1^n)=a_1^m$ for every $x_1^n \in Q^n$. Then $(Q;f)$ is an (n,m)-semigroup, called a constant (n,m)-semigroup on Q.

2) Define an (n,m)-operation f on Q by $f(x_1^n)=x_1^m$ for every $x_1^n \in Q^n$. Then $(Q;f)$ is an (n,m)-semigroup, called a left zero (n,m)-semigroup. Dually, a right zero (n,m)-semigroup $(Q;g)$ can be defined by $g(x_1^n)=x_{k+1}^n$.

3) Let $Q=A \times B=\{(a,b) \mid a \in A,\ b \in B\}$ and define an (n,m)-operation f on Q by

$$f(c_1^n) = d_1^m \iff (c_i=(a_i,b_i),\ d_j=(a_j,b_{j+k}),\ i \in N_n,\ j \in N_m).$$

Then $(Q;f)$ is an (n,m)-semigroup, and it is a direct product of a left-zero (n,m)-semigroup on A and a right zero (n,m)-semigroup on B. We say that $(Q;f)$ is an (n,m)-rectangular band.

4) Let $(Q;g)$ be a $t+1$-semigroup, $t \geq 1$, $m \geq 1$ and $n=(t+1)m$. Define an (n,m)-operation f on Q, by:

$$f(x_1^n) = y_1^n \iff y_i = g(x_i x_{i+m} \cdots x_{i+tm}),\quad i \in N_m.$$

Then $(Q;f)$ is an (n,m)-semigroup.

Note that a given semigroup $(Q;\cdot)$ induces a $t+1$-semigroup $(Q;g)$ by $g(x_1^{t+1})=x_1 \cdot x_2 \cdots x_{t+1}$. Together with the above, this gives new examples of $((t+1)m,m)$-semigroups.

5) Given a lattice $(Q;\wedge,\vee)$ one can define a (3,2)-semigroup $(Q;f)$ by

$$f(x_1^3) = (x_1\wedge x_2\wedge x_3,\ x_1\vee x_2\vee x_3).$$

Similarly, if $(Q;\wedge)$ is a semilattice and if f is defined by

$$f(x_1^n) = y_1^m \iff y_i = x_1\wedge x_2\wedge\ldots\wedge x_n,\ i\in N_m,$$

then one obtains an (n,m)-semigroup $(Q;f)$.

6) Let Q=N and define f by

$$f(x_1^n) = (1,\ldots,1,\ x_1\cdot x_2\cdots x_n).$$

Then $(Q;f)$ is an (n,m)-semigroup.

An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-<u>group</u> iff:

(e.1) $\underline{Q}$ is an (n,m)-semigroup; and

(e.2) $(\forall \underline{a}\in Q^k)(\forall \underline{b}\in Q^m)(\exists \underline{x},\underline{y}\in Q^m)(f(\underline{a}\underline{x})=\underline{b}=f(\underline{y}\underline{a}))$.

<u>Examples 2.7</u>. 1) Let $(Q;g)$ be a t+1-group and k=tm, $t\geq 1$. Then $(Q;f)$, where f is defined in E.2.6. 4), is an (n,m)-group, $n=(t+1)m$.

2) If $(G;+)$ is an abelian group, then $(G;f)$, where $f_i(a_1^m b c_1^m) = a_i - b + c_i$, is a (2m+1,m)-group.

3) Let $G=Z_4=\{0,1,2,3\}$ and $\overline{0}=\overline{2}=0$, $\overline{1}=\overline{3}=1$. Then $(G;f)$, where

$$f(xyzt) = (x+z-\overline{y}-\overline{t}+\overline{y+t},\ y+t-\overline{x}-\overline{z}+\overline{x+z})$$

is a (4,2)-group.

4) Let $\phi\colon R^2 \longrightarrow R^2$ be defined by

$$\phi(s,u) = (s+\tfrac{1}{2}\sin u,\ u+\tfrac{1}{2}\sin s),$$

where R is the set of real numbers. Then ϕ is a bijection and $(R;f)$, where

$$f(stuv) = \phi^{-1}(s+u+\tfrac{1}{2}(\sin t+\sin v),\ t+v+\tfrac{1}{2}(\sin s+\sin u)),$$

is a (4,2)-group.

5) If $f\colon C^5 \longrightarrow C^2$, where C is the set of complex numbers, is defined by:

$$f(z_1^5) = (z_1+z_4 - \frac{1+i\sqrt{3}}{2} z_3,\ z_2+z_5 - \frac{1-i\sqrt{3}}{2} z_3),$$

then (C;f) is a (5,2)-group.

The proof of the following proposition will be given in § 8.

Proposition 2.8. *If $\underline{Q}=(Q;f)$ is an (n,m)-semigroup, then the following statements are equivalent:*

(i) $\underline{Q}$ is an (n,m)-group;

(ii) $(\forall \underline{a}\in Q^k)(\forall \underline{b}\in Q^m)(\exists! \underline{x},\underline{y}\in Q^m)(f(\underline{a}\underline{x})=\underline{b}=f(\underline{y}\underline{a}))$;

(iii) $\underline{Q}$ is a weak (n,m)-quasigroup. □

As consequences of the above proposition one can obtain the following two descriptions of v.v. groups.

Corollary 2.9. *An (n,m)-semigroup $(Q;f)$ is an (n,m)-group iff there exist (n,m)-groupoids $(Q;{}^{-}f)$ and $(Q;f^{-})$ such that for every $\underline{a}\in Q^k$, $\underline{b}\in Q^m$*

$$f(\underline{a}\,{}^{-}f(\underline{a}\underline{b})) = \underline{b},\quad f(f^{-}(\underline{b}\underline{a})\underline{a}) = \underline{b}. \ □$$

Corollary 2.10. *Let $n-m=k\geq 2$. An (n,m)-semigroup $(Q;f)$ is an (n,m)-group iff there exist a positive integer $i\in N_{k-1}$ and an (n,m)-groupoid $(Q;f^{-i})$ such that for every $\underline{a}\in Q^i$, $\underline{b}\in Q^{k-i}$, $\underline{c}\in Q^m$*

$$f(\underline{a}f^{-i}(\underline{a}\underline{c}\underline{b})\underline{b}) = \underline{c}. \ □$$

The following proposition is obviously true:

Proposition 2.11. *Let $n\geq 2$. If $\underline{Q}=(Q;f)$ is an $(n,1)$-groupoid, then:*

(i) $\underline{Q}$ is an n-quasigroup $\iff$ $\underline{Q}$ is an $(n,1)$-quasigroup $\iff$ $\underline{Q}$ is a weak $(n,1)$-quasigroup.

(ii) $\underline{Q}$ is an n-semigroup $\iff$ $\underline{Q}$ is an $(n,1)$-semigroup.

(iii) $\underline{Q}$ is an n-group $\iff$ $\underline{Q}$ is an $(n,1)$-group. □

Further on we will say also: v.v. semigroup, v.v. group,... instead of (n,m)-semigroup, (n,m)-group,... .

If $\mathscr{C}$ is a property of v.v. groupoids, then we say that a v.v. semigroup has the property $\mathscr{C}$ iff it has that property as a v.v. groupoid. Therefore, the meanings of the following expessions are clear: "a commutative v.v. semigroup", "a cancellative v.v. semigroup", etc.

In this sense, it could be said that an (n,m)-group is commutative iff it is commutative as an (n,m)-groupoid. However, if $n,m \geq 2$, there are no nontrivial commutative (n,m)-groups as we can see by the following

Proposition 2.12. *Let $n,m \geq 2$. If $\underline{Q}=(Q;f)$ is a commutative (n,m)-groupoid, then the following statements are equivalent:*

(i) $|Q|=1$, i.e. Q is a one-element set.

(ii) $\underline{Q}$ is a cancellative (n,m)-groupoid.

(iii) $\underline{Q}$ is an (n,m)-group.

Proof (ii) $\Longrightarrow$ (i). Let (ii) be true and $a,b \in Q$. Since $\underline{Q}$ is commutative, we have

$$f(\overset{k+1}{a}\ \overset{m-1}{b}) = f(\overset{k}{a}\, b\, a\ \overset{m-2}{b})$$

and because of (ii), i.e. (b.2), it follows that

$$a\overset{m-1}{b} = ba\overset{m-2}{b} \quad \text{i.e. } a = b.$$

(iii) $\Longrightarrow$ (ii). Follows from P.2.5. and P.2.8. □

Each of the defined classes of v.v. groupoids can be described by the component operations too. So, if $cp\underline{Q}=(Q;f_1,f_2,\dots,f_m)$ is the component algebra which corresponds to the (n,m)-groupoid $\underline{Q}=(Q;f)$, then $\underline{Q}$ is an (n,m)-semigroup iff the following equality

$$\begin{aligned} &f_i(f_1(x_1^n)f_2(x_1^n)\cdots f_m(x_1^n)x_{n+1}^{n+k}) = \\ &= f_i(x_1^j f_1(x_{j+1}^{j+n})\cdots f_m(x_{j+1}^{j+n})x_{j+n+1}^{n+k}) \end{aligned} \qquad (ij)$$

is an identity in $cp\underline{Q}$, for every $j \in N_k$ and $i \in N_m$.

Note that the identity (ij) can be written in the following „component-vector" form:

$$f_i(f(x_1^n)x_{n+1}^{n+k}) = f_i(x_1^j f(x_{j+1}^{j+n})x_{j+n+1}^{n+k}) \qquad (ij^*)$$

for every $j \in N_k$, $i \in N_m$.

If $\mathcal{C}(n,m)$ is a class of (n,m)-groupoids, then the question for a suitable description of the free objects in the class $\mathcal{C}(n,m)$ (if such objects do exist) occurs naturally.

By the construction of the free (n,m)-groupoid in §1 and the definition of partial (n,m)-quasigroup, it is clear that:

Proposition 2.13. *Every free (n,m)-groupoid ($\overline{B}$;f), with a basis B, is a partial (n,m)-quasigroup.*

Proof. Let the conditions of (b.1) be satisfied. If there exists a positive integer ν such that $i_\nu > n$, i.e. $x_{n+i} = y_{n+i}$ for some $i \in N_m$, then

$$f_i(x_1^n) = f_i(y_1^n), \text{ i.e. } (i,x_1^n) = (i,y_1^n), \text{ i.e. } x_1^n = y_1^n,$$

which implies $x_1^{n+m} = y_1^{n+m}$.

If all $i_\nu \leq n$, then we have again $x_1^n = y_1^n$, which implies the same conclusion. □

§3. PRESENTATIONS OF SEMIGROUPS

If $\underline{Q}=(Q;f)$ is a given v.v. groupoid, then we can associate to $\underline{Q}$ a semigroup $\underline{Q}^\wedge$, obtained by a corresponding presentation induced by $\underline{Q}$. Therefore we will make a brief discussion of presentations in the class of semigroups.

Let B be a nonempty set and denote by B^+ the set of all non empty finite sequences on B, i.e.

$$B^+ = \bigcup_{p\geq 1} B^p.$$

Sometimes the elements of B^+ will be called words in B. Thus,

$$B^+ = \{b_1 b_2 \cdots b_t \mid b_\nu \in B,\ t \geq 1\},$$

and if $a_\nu, b_\lambda \in B$, $p,q \geq 1$, then

$$a_1 a_2 \cdots a_p = b_1 b_2 \cdots b_q \iff p = q \text{ and } (\forall \nu \in N_p)\ a_\nu = b_\nu.$$

We denote by B^* the set $B^+ \cup \{1\}$, where 1 denotes the empty word and $1 \notin B^+$.

If $u \in B^p \subseteq B^*$, then p is called the dimension of u and we write $p=d(u)$. (Thus, $d(1)=0$).

The set B^+ with the operation of concatenation of sequences is a semigroup, and moreover it is a free semigroup with a basis B.

In other words, if $\underline{S}=(S;\cdot)$ is a semigroup and $\xi: B \longrightarrow S$ is a mapping, then there exists a unique homomorphism ξ^+ from B^+ into $\underline{S}$ which is an extension of ξ. So, if $u=b_1b_2\cdots b_t \in B^+$, then

$$\xi^+(u) = \xi(b_1)\cdot\xi(b_2)\cdot\dots\cdot\xi(b_t).$$

The set B^* with the operation of concatenation is a free monoid with a basis B. So, if $\underline{S}=(S;\cdot)$ is a monoid with the identity e and $\xi: B \longrightarrow S$ is a mapping, then ξ can be extended, in a unique way, to a homomorphism ξ^* from B^* into $\underline{S}$, where

$$\xi^*(1)=e \quad \& \quad (\forall u \in B^+)\,\xi^*(u)=\xi^+(u).$$

Further on we will often write $\xi(u)$ instead of $\xi^+(u)$ and $\xi^*(u)$.

Any subset Λ of the set $B^+\times B^+$[1] is called a set of defining relations on B, and the pair $\langle B;\Lambda\rangle$ is called a (semigroup) presentation.

Let $\underline{S}=(S;\cdot)$ be a semigroup and $\xi: B \longrightarrow S$ be a mapping such that $\xi(u)=\xi(v)$[2], for every pair $(u,v)\in\Lambda$. Then we say that ξ is a realization of the pair (B,Λ) in $\underline{S}$. If, moreover, for any realization ξ' of (B,Λ) in a semigroup $\underline{S}'=(S';\cdot)$ there exists a unique homomorphism $\zeta: \underline{S} \longrightarrow S'$ such that $\xi'=\zeta\xi$, then we say that the semigroup S is determined by the presentation $\langle B;\Lambda\rangle$.

Proposition 3.1. *If $\underline{S}$ and $\underline{T}$ are two semigroups determined by a presentation $\langle B;\Lambda\rangle$, then $\underline{S}$ and $\underline{T}$ are isomorphic.* □

Further on we will usually write „$\underline{S} = \langle B;\Lambda\rangle$" instead of „$\underline{S} = (S;\cdot)$ is a semigroup determined by the presentation $\langle B;\Lambda\rangle$". We will also write $S=\langle B;\Lambda\rangle$. Thus, $\langle B;\Lambda\rangle$ will have three meanings.

Proposition 3.2. *Let $\overset{\Lambda}{=}$ be the least congruence on the semigroup B^+ containing Λ. Then $B^+/\overset{\Lambda}{=} \;= \langle B;\Lambda\rangle$.* □

Here we give a more explicit description of the congruence $\overset{\Lambda}{=}$.

1) *To avoid any confusion, the elements of $B^+\times B^+$ will be denoted by (u,v), where $u,v\in B^+$.*

2) *We write ξ instead of ξ^+ (as we said above).*

First, if $u,v \in B^+$ are such that $u=u_1u'u_2$, $v=u_1v'u_2$, where $(u',v') \in \Lambda$, and $u_1,u_2 \in B^*$, then we write $u \overset{\Lambda}{\vdash} v$. Let $\overset{\Lambda}{\sim}$ be the symmetric extension of $\overset{\Lambda}{\vdash}$, i.e.

$$u \overset{\Lambda}{\sim} v \iff u \overset{\Lambda}{\vdash} v \text{ or } v \overset{\Lambda}{\vdash} u.$$

Finally, let $\overset{\Lambda}{=}$ be the reflexive and transitive extension of $\overset{\Lambda}{\sim}$, i.e. $u \overset{\Lambda}{=} v$ iff there exist $u_0,u_1,\ldots,u_t \in B^+$, such that $t \geq 0$, $u=u_0$, $v=u_t$, and $u_{i-1} \overset{\Lambda}{\sim} u_i$ for any $i \in N_t$.

If $u \in B^+$, then we denote by u^Λ the element of $B^+/\overset{\Lambda}{=}$ containing u, i.e.

$$u^\Lambda = \{v \in B^+ \mid u \overset{\Lambda}{=} v\}. \tag{3.1}$$

The <u>dimension</u> $\overline{d}(u^\Lambda)$ of u^Λ is defined by:

$$\overline{d}(u^\Lambda) = \{d(v) \mid v \in u^\Lambda\}. \tag{3.2}$$

A presentation $\langle B;\Lambda\rangle$ is said to be <u>proper</u> iff

$$(\forall a,b \in B)(a \overset{\Lambda}{=} b \implies a=b). \tag{3.3}$$

In this case we may assume that B is a subset of $B^+/\overset{\Lambda}{=}$. (We note that the assertion „$\langle B;\Lambda\rangle$ is proper", does not mean „$(\forall b \in B)\, b^\Lambda = \{b\}$".)

We will denote by $\langle B;\Lambda\rangle$ the semigroup $B^+/\overset{\Lambda}{=}$. In order to simplify the notations, we will write u instead of u^Λ, i.e. we will use the elements of B^+ as „names" for the elements of $\langle B;\Lambda\rangle$. Thus, „$u=v$ in $\langle B;\Lambda\rangle$" means „$u \overset{\Lambda}{=} v$", and $\overline{d}(u)=\{d(v) \mid u \overset{\Lambda}{=} v\}$. Of course, if $\langle B;\Lambda\rangle$ is not proper, it may happen $a=b$ in $\langle B;\Lambda\rangle$, but $a \neq b$ in B.

<u>Proposition 3.3</u>. *Let m and k be two positive integers such that for every $(u,v) \in \Lambda$, $d(u) \geq m$, $d(v) \geq m$ and $d(u) \equiv d(v) \equiv m \pmod{k}$.*

a) If $w_1 \in B^+$ and $d(w_1) < m$, then for all $w_2 \in B^+$, $w_1 \overset{\Lambda}{=} w_2$ iff $w_1 = w_2$.

b) If $m > 1$, then the presentation $\langle B;\Lambda\rangle$ is proper and moreover we may assume that

$$B \cup B^2 \cup \cdots \cup B^{m-1} \subset \langle B;\Lambda\rangle.$$

c) If $u,v \in B^+$ and $u \overset{\Lambda}{=} v$, then $d(u) \equiv d(v) \pmod{k}$. □

It is usually desirable to have a procedure for answering the question whether or not two given words $u,v\in B^+$ are equal in $\langle B;\Lambda\rangle$. For this aim one uses very often a special mapping ψ of B^+ into B^+ which associate, to every equivalent class u^Λ ($u\in B^+$), a uniquely determined element $\psi(u)\in u^\Lambda$.

Namely, a mapping $\psi: B^+ \longrightarrow B^+$ is called a reduction for $\langle B;\Lambda\rangle$ iff the following conditions are satisfied:

(i) $\psi(uvw) = \psi(u\psi(v)w)$ for every $u,w\in B^*$, $v\in B^+$.

(ii) $(u,v)\in\Lambda \implies \psi(u)=\psi(v)$.

(iii) $\psi(u) \overset{\Lambda}{=} u$ for every $u\in B^+$.

Proposition 3.4. *If ψ is a reduction for $\langle B;\Lambda\rangle$, then $\psi(u)=\psi(v)$ iff $u \overset{\Lambda}{=} v$, for all $u,v\in B^+$, i.e. $\ker\psi = \overset{\Lambda}{=}$.* □

Proposition 3.5. *Let $\psi: B^+ \longrightarrow B^+$ be a reduction for $\langle B;\Lambda\rangle$ and let $S=\psi(B^+)$. If the operation $\bullet$ is defined on S by:*

$$(\forall u,v\in S)\ u\bullet v = \psi(uv), \tag{3.4}$$

then $(S;\bullet) = \langle B;\Lambda\rangle$.

Proof. First, (i) implies that $\psi^2=\psi$, i.e. ψ is a retract and, moreover, ψ is a surjective homomorphism from B^+ onto $(S;\bullet)$. The conditions (ii) and (iii) imply that $\overset{\Lambda}{=}$ is a kernel of ψ, and therefore $B^+/\overset{\Lambda}{=}$ is isomorphic to $(S;\bullet)$. □

Proposition 3.6. *A presentation $\langle B;\Lambda\rangle$ is proper iff there exists a reduction $\psi: B^+ \longrightarrow B^+$ for $\langle B;\Lambda\rangle$ such that $(\forall b\in B)\ \psi(b)=b$.* □

We note that any presentation admits a reduction. Namely, let S be a subset of B^+ such that for every $u\in B^+$ there exists exactly one element $v\in S$ which satisfies the relation $u \overset{\Lambda}{=} v$, i.e. S contains one and only one element from each class of the equivalence $\overset{\Lambda}{=}$. Then a reduction ψ can be defined by:

$$(\forall u\in B^+)\big(\psi(u)\in S \ \ \&\ \ u \overset{\Lambda}{=} \psi(u)\big).$$

Now we will apply the notion of reduction for a presentation in order to prove a v.v. variant of Cohn-Rebane's Theorem. First we will introduce two more notions.

A partial (n,m)-operation (p.v.v.o.) on a set A is a mapping $f: \mathcal{D} \to A^m$, where $\mathcal{D} = \mathcal{D}_f$ („the domain of f") is a subset of A^n. If F is a set of p.v.v.o. on the set A, then $\underline{A}=(A;F)$ is called a partial vector valued algebra (p.v.v.a.).

Let $\underline{A}=(A;F)$ be a p.v.v.a., and let F' be a set such that $F' \cap (A \cup F)=\emptyset$ and $f \mapsto f'$ is a bijection from F to F'. Denote by B the set $A \cup F'$, and consider the presentation $\langle B;\Lambda\rangle$, where

$$\Lambda = \{(f'a_1^n, b_1^m) \mid f(a_1^n) = b_1^m \text{ in } \underline{A}\}.$$

If $u \in B^+$, then we denote by dg(u) (= degree of u) the number of occurences of elements of F' in the word u. (Namely, dg(a)=0, dg(f')=1, dg(uv)=dg(u)+dg(v), for any $a \in A$, $f \in F$, $u,v \in B^+$.)

If $u \in B^+$ has a form $u=u'f'a_1^n u''$, where $u',u'' \in B^*$, $f \in F$, $a_1^n \in \mathcal{D}_f$ (= the domain of f), then u is said to be reducible. We say that $u \in B^+$ is reduced if it is not reducible. We define a reduction ψ: $B^+ \to B^+$ by induction on degree. Denote by S the set of reduced words. Then

(0) $(\forall u \in S)\ \psi(u) = u.$

Let $u=u'f'a_1^n u''$ be a reducible word, where $f \in F$, $a_1^n \in \mathcal{D}_f$ and u' has the least possible dimension. If $b_1^m=f(a_1^n)$ in $\underline{A}$, and $v=u'b_1^m u''$, then dg(v)=dg(u)-1. Therefore, by the inductive hypothesis, $\psi(v) \in S$ is well defined. Now we define $\psi(u)$ by:

(1) $\psi(u) = \psi(v)$

So, we have defined a mapping ψ: $B^+ \to S$, and by induction on dimensions and degrees it can be shown that the conditions (i), (ii), (iii) are satisfied, i.e. ψ is a reduction for the presentation $\langle B;\Lambda\rangle$. Therefore $(S;\bullet)=\langle B;\Lambda\rangle$, where „$\bullet$" is defined by (3.4).

Clearly, $A \cup F' \subseteq S$, i.e. ψ is a proper reduction.

If $f(a_1^n)=b_1^m$ in $\underline{A}$, then we have:

$$f' \bullet a_1 \bullet \ldots \bullet a_n = \psi(f'a_1 \ldots a_n) = \psi(b_1^m) = b_1 \bullet b_2 \bullet \ldots \bullet b_m.$$

Conversely, if $f \in F$ is an (n,m)-operation and $a_1^n \in A^n$, $b_1^m \in A^m$ are such that

$$f'\bullet a_1\bullet\ldots\bullet a_n = b_1\bullet b_2\bullet\ldots\bullet b_m$$

in $(S;\bullet)$, then we have

$$\psi(f'a_1^n)=f'\bullet a_1\bullet\ldots\bullet a_n=b_1\bullet\ldots\bullet b_m=\psi(b_1^m)=b_1^m,$$

and therefore $f(a_1^n)=b_1^m$ in $\underline{A}$.

This proves the following v.v. variant of Cohn-Rebane's Theorem:

Theorem 3.7. *If $\underline{A}=(A;F)$ is a p.v.v.a., then there exists a semigroup $\underline{S}=(S;\bullet)$ and a mapping $f\mapsto f'$ from F into S such that $A\subseteq S$ and*

$$f(a_1^n) = b_1^m \text{ in } \underline{A} \iff f'\bullet a_1\bullet a_2\bullet\ldots\bullet a_n = b_1\bullet b_2\bullet\ldots\bullet b_m \text{ in } \underline{S},$$

for any $a_\nu, b_\lambda\in A$ and any (n,m)-operation $f\in F$. □

Now we will consider the question of associating a semigroup to a given v.v. groupoid by using presentations of semigroups.

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and let $\Lambda=\Lambda_{\underline{Q}}$ where $\Lambda_{\underline{Q}}$ is the following set of defining relations:

$$\Lambda_{\underline{Q}} = \{(a_1^n,b_1^m) \mid f(a_1^n) = b_1^m \text{ in } \underline{Q}\}. \tag{3.5}$$

Then $\underline{Q}^\wedge=\langle Q;\Lambda\rangle$ is called a universal semigroup for the given (n,m)-groupoid $\underline{Q}$. (Note that $\Lambda_{\underline{Q}}$ is the graph of f.)

Having in mind P.3.2., we have the following:

Proposition 3.8. *Let $\underline{Q}$ be an (n,m)-groupoid and $\Lambda=\Lambda_{\underline{Q}}$. Then $\underline{Q}^\wedge=\underline{Q}^+/\overset{\Lambda}{=}$ and the natural mapping*

$$nat^\wedge\colon a\mapsto a^\Lambda$$

is a realization of (Q,Λ) in $\underline{Q}^\wedge$.

For every realization $\eta\colon Q\to S$ of (Q,Λ) in a semigroup $\underline{S}=(S;\cdot)$, there is a unique homomorphism $\zeta\colon \underline{Q}^\wedge\to\underline{S}$, such that

$$(\forall b\in Q)\ \eta(b) = \zeta(b^\Lambda). \ \square$$

Next we will give a description of the universal semigroup for the free (n,m)-groupoid $(\overline{B};f)$ with a basis B, defined in §1.

We say that a word $x \in \overline{B}^+$ is reducible iff it has a form

$$x = x'(1,y)(2,y)\cdots(m,y)x'',$$

where $x',x'' \in \overline{B}^*$ and $y=u_1^n \in \overline{B}^n$. And, if this is not satisfied, then we say that x is reduced. The set of all reduced elements of $\overline{B}^+$ will be denoted by R.

We note that in this case $\Lambda=\Lambda_{\overline{B}}$ has the following form:

$$\Lambda = \{(u_1^n,(1,u_1^n)(2,u_1^n)\cdots(m,u_1^n)) \mid u_\nu \in \overline{B}\}.$$

The length $|x|$ of an element x of $\overline{B}^+$ is defined in the usual manner. Namely

$$|b| = 1, \quad |(i,u^n)| = \sum_\nu |u_\nu|, \quad |yz|=|y|+|z|,$$

for every $b \in B$, $u_1^n \in \overline{B}^n$, $y,z \in \overline{B}^+$.

By induction on lengths we define a reduction ψ in $\langle\overline{B};\Lambda\rangle$ as follows:

(0) $\psi(x) = x$, for every $x \in R$.

Let x be a reducible element of $\overline{B}^+$ and assume that $\psi(y)$ is a well defined element of R for every $y \in \overline{B}^+$ such that $|y| < |x|$.

Consider, first, the case $m \geq 2$.

If $x=x'(1,u^n)(2,u_1^n)\cdots(m,u_1^n)x''$, where x' has the least possible dimension, then we put $y=x'u_1^n x''$. Clearly, $|y|<|x|$, and thus, $\psi(y) \in R$ is well defined. Then $\psi(x)$ is defined by:

(1) $\psi(x) = \psi(y)$,

and therefore $\psi: \overline{B}^+ \to R$ is a well defined mapping in the case $m \geq 2$.

In the case when $m = 1$, a reduction $\psi: \overline{B}^+ \to \overline{B}^+$ can be defined by induction on hierarchy. If $u=(1,u_1^n) \in \overline{B}$ then $\psi(u)$ is defined by:

(1') $\psi(u) = \psi(u_1)\psi(u_2)\cdots\psi(u_n)$.

And, if $x=u_1^p \in \overline{B}^p$, then $\psi(x)$ is defined by

(1") $\psi(x) = \psi(u_1)\cdots\psi(u_p)$.