By induction on lengths (for $m \geq 2$) or on lengths and hierarchies (for $m=1$), it can be shown that ψ is a reduction for $\langle \overline{B};\Lambda\rangle$. Therefore, by P.3.5., if we define an operation $\bullet$ on R by

$$x\bullet y = \psi(xy),$$

we obtain that $\underline{R}=(R;\bullet)$ is the universal semigroup for $(\overline{B};f)$.

We note that in the case m=1, $\underline{R}=B^+$, i.e. B^+ is the universal semigroup for $(\overline{B};f)$.

§4. VECTOR VALUED SUBGROUPOIDS OF SEMIGROUPS

An (n,m)-groupoid $\underline{Q}=(Q;f)$ is called an (n,m)-subgroupoid of a semigroup $(S;\cdot)$ iff $Q\subseteq S$ and for every $a_1^n \in Q^n$, $b_1^m \in Q^m$

$$f(a_1^n) = b_1^m \text{ in } \underline{Q} \implies a_1\cdot a_2\cdot\ldots\cdot a_n = b_1\cdot b_2\cdot\ldots\cdot b_m \text{ in } \underline{S}. \quad (4.1)$$

If, in addition, for every $a_1^m, b_1^m \in Q^m$ the following implication is true

$$a_1\cdot a_2\cdot\ldots\cdot a_m = b_1\cdot b_2\cdot\ldots\cdot b_m \text{ in } \underline{S} \implies a_1^m=b_1^m, \quad (4.2)$$

then we say that $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{S}$.

It is clear that:

Proposition 4.1. *An (n,1)-groupoid $\underline{Q}$ is an (n,1)-subgroupoid of a semigroup $\underline{S}$ iff $\underline{Q}$ is a pure (n,1)-subgroupoid of $\underline{S}$.* □

Proposition 4.2. *An (n,n)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,n)-subgroupoid of a semigroup iff*

$$(\forall a_1^n \in Q^n) \quad f(a_1^n) = a_1^n,$$

i.e. f is the identity transformation on Q^n.

Proof. If f is the identity transformation on Q^n, then $(Q;f)$ is a pure (n,n)-subgroupoid of Q^+. □

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid, $\Lambda=\Lambda_{\underline{Q}}$ be the set of defining relations as in (3.5) and $\underline{Q}^\wedge$ be the corresponding semigroup $\langle Q;\Lambda\rangle$. The congruence $\overset{\Lambda}{=}$ is also defined in §3. Here we will write $\vdash$, $\sim$ instead of $\overset{\Lambda}{\vdash}$, $\overset{\Lambda}{\sim}$, respectively.

The definition of (n,m)-subgroupoids of semigroups can be restated by the following:

Proposition 4.3. *An (n,m)-groupoid $\underline{Q}=(Q;f)$ is an (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$ iff $Q\subseteq S$, and the inclusion $a\mapsto a$ from Q into S is a realization of (Q,Λ) in $\underline{S}$.* □

We recall that the presentation $\langle Q;\Lambda\rangle$ is proper iff

$$(\forall a,b\in Q)\ (a \overset{\Lambda}{=} b \Longrightarrow a = b), \tag{4.3}$$

and then we can assume that Q is a subset of $Q^{\wedge}=Q^{+}/\overset{\Lambda}{=}$.

Thus, we have the following description of the class of (n,m)-subgroupoids of semigroups.

Theorem 4.4. *An (n,m)-groupoid $\underline{Q}=(Q;f)$ is an (n,m)-subgroupoid of a semigroup iff the presentation $\langle Q;\Lambda\rangle$ is proper, and then $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^{\wedge}$. If this is satisfied, and if $\underline{Q}$ is an (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, then there exists a unique homomorphism $\zeta\colon \underline{Q}^{\wedge}\to\underline{S}$, such that $\zeta(a)=a$ for every $a\in Q$.* □

The following proposition is a consequence of P.3.3. b) and T.4.4.

Proposition 4.5. *If $\min\{n,m\}\geq 2$, then every (n,m)-groupoid is an (n,m)-subgroupoid of a semigroup.* □

For the class of pure (n,m)-subgroupoids of semigroups, we have the following result:

Theorem 4.6. *If $\underline{Q}=(Q;f)$ is an (n,m)-groupoid, then the following conditions are equivalent:*

(i) $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^{\wedge}$;

(ii) $\underline{Q}$ is a pure (n,m)-subgroupoid of a semigroup;

(iii) $$(\forall a_1^m,b_1^m\in Q^m)(a_1^m\overset{\Lambda}{=}b_1^m \Longrightarrow a_1^m=b_1^m). \tag{4.4}$$

Proof. Clearly, (i) ⟹ (ii).

Let $\underline{Q}$ be a pure (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, and let $a_1^m,b_1^m\in Q^m$ be such that $a_1^m\overset{\Lambda}{=}b_1^m$. By T.4.4., $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^{\wedge}$, and there exists a homomorphism $\zeta\colon \underline{Q}^{\wedge}\to\underline{S}$, such that $(\forall c\in Q)\zeta(c)=c$. Thus, we have $a_1a_2\cdots a_m=b_1\cdots b_m$ in $\underline{Q}^{\wedge}$, and this implies $a_1\bullet a_2\bullet\ldots\bullet a_m=b_1\bullet\ldots\bullet b_m$ in $\underline{S}$; hence we obtain that $a_1^m=b_1^m$. Thus, (ii) ⟹ (iii).

Assume now the condition (iii). If $a,b\in Q$ are such that $a\overset{\Lambda}{=}b$, then we have $a^m \overset{\Lambda}{=} b\,a^{m-1}$, and this, by (iii), implies $a^m = b\,a^{m-1}$, i.e. $a=b$. Therefore, by T.4.4, $\underline{Q}$ is an (n,m)-subgroupoid of $\underline{Q}^\wedge$. If $a_1^m, b_1^m \in Q^m$ are such that $a_1 a_2 \cdots a_m = b_1 b_2 \cdots b_m$ in $\underline{Q}^\wedge$, then $a_1^m \overset{\Lambda}{=} b_1^m$, and this by (iii) implies $a_1^m = b_1^m$ in Q^+. i.e. $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^\wedge$. □

In the case $n-m=k \geq 1$ we have the following:

Proposition 4.7. *Let $n-m=k\geq 1$. An (n,m)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,m)-subgroupoid of a semigroup iff $\underline{Q}$ is an (n,m)-semigroup.*

Proof. First suppose that $\underline{Q}$ is not an (n,m)-semigroup. Then, there exists $a_1^{n+k} \in Q^{n+k}$ and $i \in N_k$ such that

$$b_1^m = f(f(a_1^n)a_{n+1}^{n+k}) \neq f(a_1^i f(a_{i+1}^{i+n})a_{i+n+1}^{n+k}) = c_1^m.$$

Therefore, T.4.6. (iii) implies that $\underline{Q}$ is not a pure (n,m)-subgroupoid of a semigroup.

The converse follows from T.7.7.[1] □

The above discussion gives a complete description of all the (n,m)-subgroupoids of semigroups, except the (1,1+k)-subgroupoids of semigroups for $k \geq 1$. So, next we consider this case.

Suppose that $\underline{Q}=(Q;f)$ is a (1,1+k)-groupoid, where $k \geq 1$. For every $\alpha \in N$, define a set $\mathcal{P}_\alpha(f)$ of polynomial operations on Q with a degree α as follows.

First, $\mathcal{P}_0(f)=\{1_Q\}$ (where $1_Q: x \mapsto x$ is the identity transformation on Q), and $\mathcal{P}_1(f)=\{f\}$. Suppose that for every α: $1 \leq \alpha < \beta$, $\mathcal{P}_\alpha(f)$ is a well-defined nonempty set of $(1,1+\alpha k)$-operations on Q. Then $h \in \mathcal{P}_\beta(f)$ iff there exist $g \in \mathcal{P}_{\beta-1}(f)$ and $i \in \{0,1,\ldots,(\beta-1)k\}$ such that

$$h(x)=x_1^{1+\beta k} \iff g(x)=x_1^i y x_{i+k+2}^{1+\beta k} \ \& \ f(y)=x_{i+1}^{i+k+1}. \tag{4.5}$$

Using the polynomial operations, we will describe the class of (1,1+k)-subgroupoids of semigroups and the class of pure (1,1+k)-subgroupoids of semigroups as well.

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[1] *P.4.7. is stated in this form only for completeness. It is not used in the proof of T.7.7.*

Theorem 4.8. *A (1,1+k)-groupoid $Q=(Q;f)$ is a (1,1+k)-subgroupoid of a semigroup iff for every positive integer β and polynomial operations $g,h\in\mathcal{P}_\beta(f)$, the following implication is satisfied:*

$$g(x) = h(y) \Longrightarrow x = y. \tag{4.6}$$

Proof. Let (Q;f) be a (1,1+k)-subgroupoid of a semigroup and let, for an arbitrary β, the polynomial operations $g,h\in\mathcal{P}_\beta(f)$ and $x,y\in Q$ be such that $g(x)=h(y)$. Then $x \overset{\Delta}{=} y$, which implies that $x = y$, by T.4.4.

For the converse we need the following two lemmas.

Lemma 4.9. *If $u,v,w\in Q^+$ are such that $v\vdash u$, $v\vdash w$, $u\neq w$, then there exists $v'\in Q^+$ such that $u\vdash v'$, $w\vdash v'$.*

Proof. By $v\vdash u$, $v\vdash w$ it follows that

$$v = a_1^{i-1}a_ia_{i+1}^p = a_1^{j-1}a_ja_{j+1}^p,$$

$$u = a_1^{i-1}b_1^{1+k}a_{i+1}^p, \quad w = a_1^{j-1}c_1^{1+k}a_{j+1}^p,$$

where $f(a_i)=b_1^{1+k}$, $f(a_j)=c_1^{1+k}$. Since $u\neq w$, it follows that $i\neq j$, for example $i < j$. Putting

$$v' = a_1^{i-1}b_1^{1+k}a_{i+1}^{j-1}c_1^{1+k}a_{j+1}^p,$$

we obtain that $u\vdash v'$, $w\vdash v'$. □

Lemma 4.10. *Let $u=a_1^s\in Q^s$, $v\in Q^+$. There exist $v_1,v_2,\dots,v_t\in Q^+$ such that*

$$u\vdash v_1\vdash v_2\vdash\cdots\vdash v_t = v$$

iff there exist integers $\alpha_1,\alpha_2,\dots,\alpha_s\geq 0$ and $h_\nu\in\mathcal{P}_{\alpha_\nu}(f)$ such that

$$v = h_1(a_1)h_2(a_2)\cdots h_s(a_s).$$

Proof. We will prove this by induction on t. For t=1, we have

$$v = a_1^{i-1}b_1^{1+k}a_{i+1}^s = 1(a_1)\cdots 1(a_{i-1})f(a_i)1(a_{i+1})\cdots 1(a_s)$$

Let $v_{t-1} = h_1(a_1)h_2(a_2)\cdots h_s(a_s)$

$= \cdots c_1^{1+\delta k}\cdots$

$$v_t = v = h_1(a_1)\cdots h_{j-1}(a_{j-1})c_1^{i-1}d_1^{1+k}c_{i+1}^{1+\delta k}h_{j+1}(a_{j+1})\cdots h_s(a_s),$$

where $h_j(a_j) = c_1^{1+\delta k}$, $f(c_i)=d_1^{1+k}$. Then

$$v = h_1(a_1)\cdots h_{j-1}(a_{j-1})h_j'(a_j)h_{j+1}(a_{j+1})\cdots h_s(a_s),$$

where $h_j'(a_j)=c_1^{i-1}d_1^{1+k}c_{i+1}^{1+\delta k}$. □

Now, to complete the proof of T.4.8., assume that for every $g,h\in\mathcal{P}_\beta(f)$, $x,y\in Q$,

$$g(x) = h(y) \implies x = y.$$

Let $x \overset{\Lambda}{=} y$, $x,y\in Q$. By T.4.4. it suffices to show that $x = y$. First, L.4.9. implies that there exist $u_1,\ldots,u_t$, $v_1,\ldots,v_t$ such that

$$x \vdash u_1 \vdash \cdots \vdash u_t = v_t \dashv v_{t-1} \dashv \cdots \dashv v_1 \dashv y.$$

Now, by L.4.10., there exist $g,h\in\mathcal{P}_t(f)$, such that

$$g(x) = u_t = v_t = h(y),$$

which implies $x = y$. □

The following theorem gives a description of the class of the pure (1,1+k)-subgroupoids of semigroups.

<u>Theorem 4.11</u>. *Let $\underline{Q}=(Q;f)$ be a $(1,1+k)$-groupoid, $k\geq 1$. Then, $\underline{Q}$ is a pure $(1,1+k)$-subgroupoid of a semigroup iff for every $\alpha_0,\ldots,\alpha_k$, $\beta_0,\ldots,\beta_k\geq 0$, such that $\alpha_0+\ldots+\alpha_k = \beta_0+\ldots+\beta_k$ and any $h_\nu\in\mathcal{P}_{\alpha_\nu}(f)$, $g_\nu\in\mathcal{P}_{\beta_\nu}(f)$, $\underline{Q}$ satisfies the following implication:*

$$h_0(x_0)\cdots h_k(x_k) = g_0(y_0)\cdots g_k(y_k) \implies x_0^k = y_0^k. \tag{4.7}$$

<u>Proof</u>. Let $\underline{Q}$ be a pure (1,1+k)-subgroupoid of a semigroup and let the assumption of the implication (4.7) holds. Then, by L.4.10., $x_0^k \overset{\Lambda}{=} y_0^k$ and this implies $x_0^k = y_0^k$. Thus, (4.7) is satisfied.

Conversely, suppose that (4.7) is satisfied in $\underline{Q}$, and $x_0^k,y_0^k\in Q^{k+1}$ be such that $x_0^k \overset{\Lambda}{=} y_0^k$. Then, the definition of $\overset{\Lambda}{=}$, L.4.9., and L.4.10. imply that there exist corresponding polynomial operations h_ν,g_ν such that

$$h_0(x_0)\cdots h_k(x_k) = g_0(y_0)\cdots g_k(y_k).$$ □

In the case $1 < n < m$, we do not know a satisfactory description of the class of pure (n,m)-subgroupoids of semigroups. The next theorem shows that such pure (n,m)-subgroupoids of semigroups do exist.

Theorem 4.12. *If $n < m$, then every free (n,m)-groupoid is a pure (n,m)-subgroupoid of a semigroup.*

Proof. Let $\underline{Q}=(\overline{B};f)$ be the free (n,m)-groupoid with a basis B. By T.4.6. we have to show that $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{Q}^{\wedge}$. We use the description of $\underline{Q}^{\wedge}$ given in §3. Thus $\underline{Q}^{\wedge}=(R;\bullet)$, where $R=\psi(\overline{B}^{+})$, and $x\bullet y=\psi(xy)$. Suppose that $u_1^m, v_1^m \in \overline{B}^m$ are such that $u_1\bullet u_2\bullet\ldots\bullet u_m = v_1\bullet v_2\bullet\ldots\bullet v_m$ in $(R;\bullet)$, i.e. $\psi(u_1^m)=\psi(v_1^m)$. Since $n < m$, $\psi(u_1^m)\neq u_1^m$ iff $u_\lambda=(\lambda,w_1^n)$, for $\lambda\in N_m$, and then $\psi(u_1^m) = w_1^n$. □

It is natural to ask the question about the existence of pure (n,m)-subgroupoids $\underline{Q}$ of semigroups, under the assumption that $\underline{Q}$ has a corresponding property $\mathcal{C}$. In the case when $n-m \geq 0$, the answer to this question give P.4.2. and P.4.7.

Here we will show that, for $2 \leq n \leq m$, there are no nontrivial commutative (n,m)-groupoids which are pure (n,m)-subgroupoids of semigroups.

Let $\underline{Q}=(Q;f)$ be a commutative pure (n,m)-subgroupoid of a semigroup $\underline{S}=(S;\cdot)$, where $2 \leq n \leq m$. In the case $n=m$, by P.4.2., we have $f(a_1^n)=a_1^n$ and so $f(a_2a_1a_3^n)=a_2a_1a_3^n$, which implies that $a_1^n=a_2a_1a_3^n$, i.e. $a_1=a_2$. Since a_1 and a_2 are arbitrary, the equality $a_1=a_2$ implies that $|Q|=1$.

Now suppose that $2 \leq n < m$. If $b_1,\ldots,b_m\in Q$, then

$$b_1b_2\ldots b_m = (b_1b_2\ldots b_n)b_{n+1}\ldots b_m =$$
$$= (b_2b_1\ldots b_n)b_{n+1}\ldots b_m = b_2b_1\ldots b_m$$

implies that $b_1=b_2$ for arbitrary $b_1,b_2\in Q$. Hence $|Q|=1$.

Thus we have

Proposition 4.13. *If $2 \leq n \leq m$, then a commutative (n,m)-groupoid $\underline{Q}=(Q;f)$ is a pure (n,m)-subgroupoid of a semigroup iff $|Q|=1$.* □

(Note that every (1,1+k)-groupoid is commutative, so that in this case, P.4.8. and P.4.11. can be applied.)

§5. THE GENERAL ASSOCIATIVE LAW

The general associative law is true for v.v. semigroups too. We will prove this fact in details. First we will introduce the notion of polynomial operation.

Throughout this section we will assume that $\underline{Q}=(Q;f)$ is a given (n,m)-groupoid, where $n-m=k \geq 1$.

Let $g, g_1, g_2, \ldots, g_t$ be v.v. operations on a set Q, such that g is an (n,m)-operation, and g_ν is an (n_ν, m_ν)-operation for every $\nu \in N_t$, and let the following equalities be satisfied:

$$p = n_1+n_2+\ldots+n_t, \qquad n = m_1+m_2+\ldots+m_t.$$

Define a (p,m)-operation $h=g(g_1^t)$ by:

$$h(x_1^p) = g(g_1(\underline{x}_1)g_2(\underline{x}_2)\cdots g_t(\underline{x}_t)), \tag{5.1}$$

where $x_1^p = \underline{x}_1\underline{x}_2\ldots\underline{x}_t$, $\underline{x}_i \in Q^{n_i}$.

For every positive integer α define a set $\mathcal{P}_\alpha(\underline{Q}) = \mathcal{P}_\alpha(f)$ of polynomial operations[1], with a degree α, inductively, in the following way. First put

$$\mathcal{P}_0(f) = \{1_Q\}, \quad \mathcal{P}_1(f) = \{f\}. \tag{5.2}$$

(As before, 1_Q: $x \mapsto x$ is the identity transformation on Q; further on, we will write 1 instead of 1_Q.)

Suppose that the number $\beta \geq 2$ is such that for every α: $0 \leq \alpha < \beta$ the set $\mathcal{P}_\alpha(f)$ of v.v. operations is well-defined. Then the set $\mathcal{P}_\beta(f)$ of v.v. operations on Q will be defined in the following way: $h \in \mathcal{P}_\beta(f)$ iff there exist v.v. operations g, g_ν on Q such that $g \in \mathcal{P}_\alpha(f)$, $g_\nu \in \mathcal{P}_{\alpha_\nu}(f)$, $\beta = \alpha + \sum_{\nu=1}^{t} \alpha_\nu$, $0 < \alpha < \beta$ and

$$h = g(g_1^t). \tag{5.3}$$

---

[1] *The definition of $\mathcal{P}_\alpha(f)$ in this section differs from the definition of $\mathcal{P}_\alpha(f)$ in §4, because here we have $n > m$.*

Proposition 5.1. *For every integer $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ is nonempty. If $\alpha > 0$ and $h \in \mathcal{P}_\alpha(f)$, then h is an $(m+\alpha k,m)$-operation.*

Proof. Let $\alpha \geq 0$. Define a set $\{f^{(s)} \mid s \geq 1\}$, of v.v. operations on Q, such that $f^{(s)} \in \mathcal{P}_s(f)$, by induction on s. First, for $s=1$, put $f^{(1)}=f$. Suppose that $f^{(\alpha)} \in \mathcal{P}_\alpha(f)$ is a well-defined $(m+\alpha k,m)$-operation on Q and define a v.v.o. $f^{(\alpha+1)}$ by:

$$f^{(\alpha+1)} = f(f^{(\alpha)}{}_1^k). \tag{5.4}$$

Then $f \in \mathcal{P}_1(f)$, $1 \in \mathcal{P}_0(f)$ and $f^{(\alpha)} \in \mathcal{P}_\alpha(f)$ imply that $f^{(\alpha+1)} \in \mathcal{P}_{\alpha+1}(f)$ and that $f^{(\alpha+1)}$ is an $(m+(\alpha+1)k,m)$-operation on Q. Thus, for every $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ is nonempty.

Next we show the second part of the proposition, using the induction again. Suppose that

$g \in \mathcal{P}_\alpha(f)$, $g_\nu \in \mathcal{P}_{\alpha_\nu}(f)$, $h=g(g_1^t) \in \mathcal{P}_\beta(f)$, $\beta=\alpha+\alpha_1+\ldots+\alpha_t$, where $0<\alpha<\beta$, g is an $(m+\alpha k,m)$-operation, and g_ν is an (n_ν,m_ν)-operation on Q.

If $\alpha_\nu=0$, then $g_\nu=1$ which means that $n_\nu=1=m_\nu$, and for $\alpha_\nu>0$ we have $n_\nu=m+\alpha_\nu k$, $m_\nu=m$. Therefore, h is a (p,m)-operation, where

$$p = n_1+n_2+\ldots+n_t, \quad m+\alpha k = m_1+m_2+\ldots+m_t.$$

Let i integers from the sequence $\alpha_1,\alpha_2,\ldots,\alpha_t$ be positive, and the other t-i be zeros. Without loss of generality, we may assume that $\alpha_1,\alpha_2,\ldots,\alpha_i>0$, $\alpha_{i+1}=\ldots=\alpha_t=0$. Then we have:

$$p = (m+\alpha_1k)+\ldots+(m+\alpha_ik)+(t-i) = mi+(\alpha_1+\ldots+\alpha_i)k+(t-i)$$

and $m+\alpha k=mi+(t-i)$, which implies that

$$p = m+\alpha k+(\alpha_1+\ldots+\alpha_i)k = m+\beta k \quad (\beta=\alpha+\alpha_1+\ldots+\alpha_i).$$

Thus, h is an $(m+\beta k,m)$-operation. □

Proposition 5.2. *If $g \in \mathcal{P}_\alpha(f)$, then $\mathcal{P}_\beta(g) \subseteq \mathcal{P}_{\alpha\beta}(f)$.*

Proof. The statement is obviously true for $\beta \in \{0,1\}$ and so we will assume that $\beta \geq 2$.

Let $\ell \in \mathcal{P}_\beta(g)$. Then, there exist integers $\gamma,\gamma_1,\ldots,\gamma_t$ such that $0<\gamma<\beta$, $\beta=\gamma+\gamma_1+\ldots+\gamma_t$, $\ell=h(h_1^t)$, $h \in \mathcal{P}_\gamma(g)$, $h_\nu \in \mathcal{P}_{\gamma_\nu}(g)$. By the inductive hypothesis it follows that $h \in \mathcal{P}_{\alpha\gamma}(f)$, $h_\nu \in \mathcal{P}_{\alpha\gamma_\nu}(f)$ and

$$\alpha\beta = \alpha\gamma + \alpha\gamma_1 + \ldots + \alpha\gamma_t,$$

and thus $\ell \in \mathcal{P}_{\alpha\beta}(f)$. □

For the proof of the general associative law (GAL) we need the following lemma:

Lemma 5.3. *Let $(Q;f)$ be an (n,m)-semigroup. Then*

$$f^{(\gamma+\delta)} = f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}\, \overset{\gamma k-r}{1}) \qquad (5.5)$$

for every $\gamma, \delta \geq 1$, and $0 \leq r \leq \gamma k$.

Proof. The proof will be done by induction. Since $(Q;f)$ is an (n,m)-semigroup, it follows that $\mathcal{P}_2(f)=\{f^{(2)}\}$, i.e. $f^{(1+1)}=f^{(2)}=f(\overset{r}{1}\, f\, \overset{k-r}{1})$.

1) Let (5.5) be satisfied for $1,\delta$. Then

$$f^{(1+\delta+1)} = f(f^{(1+\delta)}\overset{k}{1}) = f(f(\overset{r}{1}\, f^{(\delta)}\overset{k-r}{1})\overset{k}{1})$$
$$= f^{(2)}(\overset{r}{1} f^{(\delta)}\overset{2k-r}{1}) = f(\overset{r}{1} f(f^{(\delta)}\overset{k-r}{1})) = f(\overset{r}{1} f^{(\delta+1)}\overset{k-r}{1}).$$

2) Let (5.5) be satisfed for $\gamma,1$. Then by 1):

$$f^{(\gamma+1+1)} = f(\overset{k}{1} f^{(\gamma+1)}) = f(\overset{k}{1} f^{(\gamma)}(\overset{r-k}{1}\, f\, \overset{\gamma k-r+k}{1}))$$
$$= f(\overset{k}{1} f^{(\gamma)})(\overset{r}{1} f \overset{(\gamma+1)k-r}{1}) = f^{(\gamma+1)}(\overset{r}{1} f \overset{(\gamma+1)k-r}{1}),$$

for $r \geq \gamma k$; and

$$f^{(\gamma+1+1)} = f(f^{(\gamma+1)}\overset{k}{1}) = f(f^{(\gamma)}(\overset{r}{1} f \overset{\gamma k-r}{1})\overset{k}{1})$$
$$= f(f^{(\gamma)}\overset{k}{1})(\overset{r}{1}\, f \overset{(\gamma+1)k-r}{1}) = f^{(\gamma+1)}(\overset{r}{1}\, f \overset{(\gamma+1)k-r}{1}),$$

for $r < \gamma k$.

3) Let (5.5) be satisfied for γ,δ. Then

$$f^{(\gamma+\delta+1)} = f^{(\gamma+\delta)}(\overset{r}{1}\, f \overset{(\gamma+\delta)k-r}{1}) =$$
$$= f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}\overset{\gamma k-r}{1})(\overset{r}{1}\, f \overset{(\gamma+\delta)k-r}{1}) =$$
$$= f^{(\gamma)}(\overset{r}{1}\, f^{(\delta)}(f\overset{\delta k}{1})\overset{\gamma k-r}{1}) = f^{(\gamma)}(\overset{r}{1}\, f^{(\delta+1)}\overset{\gamma k-r}{1}). \; □$$

Theorem 5.4. (GAL). *An (n,m)-groupoid $Q=(Q;f)$ is an (n,m)-semigroup iff, for every integer $\alpha \geq 0$, the set $\mathcal{P}_\alpha(f)$ has exactly one element.*

Proof. The definition of (n,m)-semigroups implies that $\underline{Q}$ is an (n,m)-semigroup iff $\mathcal{P}_2(f)=\{f^{(2)}\}$, where $f^{(s)}$ is defined in the proof of P.5.1. Therefore, if for every α, $\mathcal{P}_\alpha(f)$ is a one-element set, then $\underline{Q}$ is an (n,m)-semigroup.

Conversely, let $\underline{Q}$ be an (n,m)-semigroup. Obviously, $|\mathcal{P}_\alpha(f)|=1$ for $\alpha \le 2$. We will show, by induction, that $\mathcal{P}_\beta(f)=\{f^{(\beta)}\}$ for every $\beta \ge 1$, which will complete the proof.

Let $h \in \mathcal{P}_\beta(f)$, i.e. $h=g(g_1^t)$, where $g \in \mathcal{P}_\alpha(f)$, $g_\lambda \in \mathcal{P}_{\alpha_\lambda}(f)$, $\beta=\alpha+\alpha_1+\ldots+\alpha_t$, $0<\alpha<\beta$. The inductive hypothesis implies that $g=f^{(\alpha)}$ and: $\alpha_\lambda=0 \Longrightarrow g_\lambda = 1$, $\alpha_\lambda > 0 \Longrightarrow g_\lambda = f^{(\alpha_\lambda)}$. Since $0<\alpha<\beta$, there exists an integer λ, such that $\alpha_\lambda>0$. Let r+1 be the least positive integer, such that $\alpha_{r+1}=\tau\neq 0$. Then $g_{r+1}=f^{(\tau)}$ and $g_1=g_2=\ldots=g_r=1$. Using L.5.3., we have:

$$\begin{aligned} h &= g(g_1^t) = f^{(\alpha)}(\overset{r}{1} f^{(\tau)} g_{r+2}^t) = \\ &= f^{(\alpha)}(\overset{r}{1} f^{(\tau)}(\overset{\alpha k-r}{1})(\overset{r+m+\tau k}{1} g_{r+2}^t) = \\ &= f^{(\alpha+\tau)}(\overset{r+m+\tau k}{1} g_{r+2}^t). \end{aligned}$$

Applying the same procedure a finite number of times, we obtain that $h=f^{(\beta)}(\overset{\beta k+m}{1})=f^{(\beta)}$. □

As a consequence of GAL and P.5.2. we have

Proposition 5.5. *If $(Q;f)$ is an (n,m)-semigroup, then $(Q;f^{(s)})$ is an $(sk+m,m)$-semigroup, for every $s \ge 1$.* □

Using GAL and the usual induction, we obtain

Proposition 5.6. *If $(Q;f)$ is a commutative (n,m)-semigroup, then $(Q;f^{(s)})$ is a commutative $(m+sk,m)$-semigroup, for every $s \ge 1$.* □

Now we give the following characterization for the cancellative v.v. semigroups.

Theorem 5.7. *If $(Q;f)$ is an (n,m)-semigroup, then the following conditions are equivalent:*

(i) $(Q;f)$ is cancellative;

(ii) $(Q;f^{(s)})$ *is a cancellative* $(m+sk,m)$*-semigroup, for every* $s \geq 1$*;*

(iii) $(Q;f^{(s)})$ *is a cancellative* $(m+sk,m)$*-semigroup, for some* $s \geq 1$*;*

(iv) $(Q;f)$ *satisfies the following implication:*

$$f(a_1^k\ x_1^m) = f(a_1^k\ y_1^m) \text{ or } f(x_1^m\ b_1^k) = f(y_1^m\ b_1^k) \Longrightarrow x_1^m = y_1^m;$$

(v) there exist $i,s \geq 1$, $i \in N_{sk-1}$, $sk \geq 2$, *such that the following implication in* $(Q;f)$ *is true:*

$$f^{(s)}(a_1^i x_1^m a_{i+1}^{sk}) = f^{(s)}(a_1^i y_1^m a_{i+1}^{sk}) \Longrightarrow x_1^m = y_1^m.$$

Proof. (i) $\Longrightarrow$ (ii). Let $(Q;f)$ be cancellative and let $t \geq 0$ be an integer such that, for every s: $1 \leq s < t$, the v.v. semigroup $(Q;f^{(s)})$ is cancellative. Suppose that the equality

$$f^{(t)}(a_1^i x_1^m a_{i+1}^{tk}) = f^{(t)}(a_1^i y_1^m a_{i+1}^{tk}) \tag{5.6}$$

is true, where $0 \leq i \leq tk$, and let $\alpha \leq \min\{i,k\}$. By the equality (5.6),

$$f(a_1^\alpha f^{(t-1)}(a_{\alpha+1}^i x_1^m a_{i+1}^{(t-1)k-i+\alpha}) a_{(t-1)k-i+\alpha+1}^{tk}) =$$
$$= f(a_1^\alpha f^{(t-1)}(a_{\alpha+1}^i y_1^m a_{i+1}^{(t-1)k-i+\alpha}) a_{(t-1)k-i+\alpha+1}^{tk});$$

now, using the facts that $(Q;f)$ is cancellative and that $(Q;f^{(t-1)})$ is cancellative, we obtain that $x_1^m = y_1^m$, i.e. the equality (5.6) implies that $x_1^m = y_1^m$. Therefore the $(m+tk,m)$-semigroup $(Q;f^{(t)})$ is cancellative.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i). Suppose that, for some $s \geq 2$, the $(m+sk,m)$-semigroup $(Q;f^{(s)})$ is cancellative and let

$$f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k), \tag{5.7}$$

where $i \in \{0,1,\ldots,k\}$. Then, for arbitrary $b_1,\ldots,b_{(s-1)k} \in Q$,

$$f^{(s)}(b_1^{(s-1)k} a_1^i x_1^m a_{i+1}^k) = f^{(s)}(b_1^{(s-1)k} a_1^i y_1^m a_{i+1}^k),$$

by which, since $(Q;f^{(s)})$ is cancellative, we obtain that $x_1^m = y_1^m$. Thus (5.7) implies that $x_1^m = y_1^m$, i.e. $(Q;f)$ is cancellative.

It is clear that (i) $\Longrightarrow$ (iv). We will show that (iv) $\Longrightarrow$ (i).

We can assume that $k \geq 2$, for if $k=1$, then (iv) is the condition (b.2) of §2, i.e. $(Q;f)$ is cancellative.

Let (iv) be true and let

$$f(\underline{a}\ \underline{x}\ \underline{b}) = f(\underline{a}\ \underline{y}\ \underline{b}),$$

where $\underline{a}\underline{b} \in Q^k$, $\underline{x},\underline{y} \in Q^m$. Then:

$$f^{(2)}(\underline{b}\ \underline{a}\ \underline{x}\ \underline{b}\ \underline{a}) = f^{(2)}(\underline{b}\ \underline{a}\ \underline{y}\ \underline{b}\ \underline{a}),\ \text{or}$$

$$f(f(\underline{b}\underline{a}\underline{x})\underline{b}\underline{a}) = f(f(\underline{b}\underline{a}\underline{y})\underline{b}\underline{a}),$$

by which, after two applications of (iv), one obtains $\underline{x}=\underline{y}$. Thus (iv) $\Longrightarrow$ (i).

Since the implication (ii) $\Longrightarrow$ (v) is obviously true, it remains to show that (v) implies some of the conditions (i)-(iv).

Suppose first that the following condition is true:

(v′) $k \geq 2$ and there exists $i \in N_{k-1}$ such that the following implication in $(Q;f)$ is true:

$$f(a_1^i x_1^m a_{i+1}^k) = f(a_1^i y_1^m a_{i+1}^k) \Longrightarrow x_1^m = y_1^m.$$

We will show that (v′) $\Longrightarrow$ (iv). First, one can show by induction on s that, by (v′), the following implication is true:

$$f^{(s)}(a_1^{si} x_1^m a_{si+1}^{sk}) = f^{(s)}(a_1^{si} y_1^m a_{si+1}^{sk}) \Longrightarrow x_1^m = y_1^m. \qquad (5.8)$$

Let $f(\underline{x}\underline{a})=f(\underline{y}\underline{a})$, $d(\underline{a})=k$, $d(\underline{x})=d(\underline{y})=m$, and choose $\underline{b}$ and $\underline{c}$ such that:

$$d(\underline{b}\underline{x}\underline{a}\underline{c}) = sk+m, \quad d(\underline{b})=si, \quad d(\underline{a}\underline{c})=sk-si.$$

Then one can obtain the following equality:

$$f^{(s)}(\underline{b}\underline{x}\underline{a}\underline{c}) = f^{(s)}(\underline{b}\underline{y}\underline{a}\underline{c}).$$

Applying (5.8), we have $\underline{x}=\underline{y}$. By symmetry, we have also $f(\underline{a}\underline{x})=f(\underline{a}\underline{y}) \Longrightarrow \underline{x} = \underline{y}$. Thus we showed that (v′) $\Longrightarrow$ (iv), i.e. (v′) $\Longrightarrow$ (i).

Now, since $f^{(s)}$ satisfies the condition (v′), (when k is replaced by sk), we conclude that $(Q;f^{(s)})$ is cancellative, i.e. (v) $\Longrightarrow$ (iii). □

The next theorem gives a characterization of v.v. groups, using the operations $f^{(s)}$.

Theorem 5.8. *If $(Q;f)$ is an (n,m)-semigroup, then the following conditions are equivalent:*

(i) $(Q;f)$ is a v.v. group.

(ii) $(Q;f^{(s)})$ is a v.v. group, for each $s \geq 1$.

(iii) $(Q;f^{(s)})$ is a v.v. group, for some $s \geq 1$.

(iv) There exist $s \geq 1$ satisfying $sk \geq 2$, and $i \in N_{sk-1}$, such that the equation

$$f^{(s)}(\underline{a}\ \underline{x}\ \underline{b}) = \underline{c} \qquad (5.9)$$

is solvable for given $\underline{a} \in Q^i$, $\underline{b} \in Q^{sk-i}$, $\underline{c} \in Q^m$.

(v) There exists $s \geq 1$, such that $sk \geq 2$ and for each $i \in N_{sk-1}$, the equation (5.9) is solvable.

(vi) For each $s \geq 1$, satisfying $sk \geq 2$, there exists $i \in N_{sk-1}$, such that the equation (5.9) is solvable.

(vii) For each $s \geq 1$, satisfying $sk \geq 2$, and each $i \in N_{sk-1}$, the equation (5.9) is solvable.

Proof. It is obvious that: (ii) $\Longrightarrow$ (iii), (vii) $\Longrightarrow$ (v), (vi) $\Longrightarrow$ (iv), (vii) $\Longrightarrow$ (vi), and (v) $\Longrightarrow$ (iv).

(i) $\Longrightarrow$ (ii): Let $s \geq 1$, $\underline{a}_1, \ldots, \underline{a}_s \in Q^k$, $\underline{c} \in Q^m$. Then there exist $\underline{x}_1, \ldots, \underline{x}_s \in Q^m$ such that $f(\underline{x}_s \underline{a}_s) = \underline{c}$, $f(\underline{x}_{s-1} \underline{a}_{s-1}) = \underline{x}_s, \ldots, \ldots, f(\underline{x}_1 \underline{a}_1) = \underline{x}_2$. This implies that $f^{(s)}(\underline{x}_1 \underline{a}_1 \ldots \underline{a}_s) = \underline{c}$. Symmetrically, there exists $\underline{y} \in Q^m$ such that $f^{(s)}(\underline{a}_1 \ldots \underline{a}_s \underline{y}) = \underline{c}$. This, together with P.5.5., implies that $(Q;f^{(s)})$ is a v.v. group.

(iii) $\Longrightarrow$ (i): Let $\underline{a} \in Q^k$, $\underline{c} \in Q^m$, $a \in Q$, and let $\underline{b} = \overset{(s-1)k}{a}$. Since $(Q;f^{(s)})$ is a v.v. group, there exists $\underline{d} \in Q^m$, such that $f^{(s)}(\underline{a}\underline{b}\underline{d}) = \underline{c}$. This implies that the equation $f(\underline{a}\underline{x}) = \underline{c}$ has a solution. Symmetrically, the equation $f(\underline{y}\underline{a}) = \underline{c}$ has a solution. This, together with the assumption that $(Q;f)$ is a v.v. semigroup, implies that $(Q;f)$ is a v.v. group.

(ii) $\Longrightarrow$ (vii): Let $s \geq 1$, $sk \geq 2$ and $i \in N_{sk-1}$. Let $\underline{a} \in Q^i$, $\underline{b} \in Q^{sk-i}$, $\underline{c} \in Q^m$, $a \in Q$, $\underline{u} = \overset{sk-i}{a}$, $\underline{v} = \overset{i}{a}$. Then there exists $\underline{w} \in Q^m$, such that $f^{(s)}(\underline{a}\underline{u}\underline{w}) = \underline{c}$, and there exists $\underline{z} \in Q^m$, such that $f^{(s)}(\underline{z}\underline{v}\underline{b}) = \underline{w}$. Now, $f^{(s)}(\underline{a} f^{(s)}(\underline{u}\underline{z}\underline{v})\underline{b}) = \underline{c}$, i.e. the equation $f^{(s)}(\underline{a}\underline{x}\underline{b}) = \underline{c}$ is solvable.

(iv) $\Longrightarrow$ (v). Let s and i be the numbers which exist by (iv). The proof of (v) will be devided into three parts. Let $j\in N_{sk-1}$, $a_1^j\in Q^i$, $b_1^{sk-j}\in Q^{sk-j}$, $\underline{c}\in Q^m$, $a\in Q$.

(a) If $j<i$, then (iv) implies that there exist $\underline{d},\underline{g}\in Q^m$ such that $f^{(s)}(a_1^j\,\overset{i-j}{a}\,\underline{d}\,b_{i-j+1}^{sk-j})=\underline{c}$ and $f^{(s)}(\overset{i}{a}\,\underline{g}\,\overset{sk-2i+j}{a}\,b_1^{i-j})=\underline{d}$. Then $f^{(s)}(a_1^j h_1^m b_1^{sk-j})=\underline{c}$ for $h_1^m=f^{(s)}(\overset{2i-j}{a}\,\underline{g}\,\overset{sk-2i+j}{a})$.

(b) If $0<t=j-i\le i$, then (a) and (iv) imply that there exist $\underline{d},\underline{g}\in Q^m$ such that $f^{(s)}(a_1^t\underline{d}\,\overset{i}{a}\,b_1^{sk-j})=\underline{c}$ and $f^{(s)}(a_{t+1}^j\underline{g}\,\overset{sk-i}{a})=\underline{d}$. Then $f^{(s)}(a_1^j h_1^m b_1^{sk-j})=\underline{c}$, for $h_1^m=f^{(s)}(\underline{g}\,\overset{sk}{a})$.

(c) The general case follows from (a) and (b).

(v) $\Longrightarrow$ (iii). Let s be the number which exists by (v). Let $\underline{a}\in Q^{sk}$, $\underline{c}\in Q^m$. Then $\underline{a}=b\underline{d}$ for some $b\in Q$, $\underline{d}\in Q^{sk-1}$. Let $u\in Q$ and $\underline{v}=\overset{sk-1}{u}$. Then (v) implies that there exist $\underline{\alpha},\underline{\beta}\in Q^m$ such that $f^{(s)}(b\underline{\alpha}\underline{v})=\underline{c}$ and $f^{(s)}(\underline{d}\underline{\beta}u)=\underline{\alpha}$. So, $f^{(s)}(\underline{\beta}\,\overset{sk}{u})$ is a solution of the equation $f^{(s)}(\underline{a}\underline{x})=\underline{c}$. Symmetrically, the equation $f^{(s)}(\underline{x}\underline{a})=\underline{c}$ is solvable. □

By using GAL, we can consider v.v. semigroups as v.v. algebras with infinitely many v.v. operations and these algebras we call „poly-(n,m)-semigroups". The notion of poly-(n,m)-semigroup will be introduced here, and it will be used in the construction of free v.v. semigroups (§6).

Let P be a non-empty set and let n,m,k be as above, i.e. $n-m=k\ge 1$. If

$$g:\ P^{(n,m)}\longrightarrow P^m,\ \text{where}\ P^{(n,m)}=\bigcup_{s\ge 1}P^{m+sk},$$

is a mapping, then $\mathbf{P}=(P;g)$ is called a poly-(n,m)-groupoid.

From a given (n,m)-groupoid we can obtain a poly-(n,m)-groupoid as follows.

Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and let π be a choice of one and only one polynomial operation $\pi_s\in\mathcal{P}_s(f)$. For this π, define a mapping

$$f^\pi:\ Q^{(n,m)}\longrightarrow Q^m\ \text{by:}\ f^\pi(a_1^{m+sk})=\pi_s(a_1^{m+sk}).$$

Then we obtain a poly-(n,m)-groupoid $\underline{Q}^{\pi}=(Q;f^{\pi})$ which is said to be induced by the (n,m)-groupoid $\underline{Q}$.

In general, since the set $\mathcal{P}_s(f)$ may have more than one element, it is possible a given (n,m)-groupoid to induce more than one poly-(n,m)-groupoid.

For the choice $\pi_s=f^{(s)}$, we use the notations $f^{\#}$ and $\underline{Q}^{\#}$ instead of f^{π} and $\underline{Q}^{\pi}$ respectively. (Here, $f^{(s)}$ is as in (5.4).)

Note that there are poly-(n,m)-groupoids that are not induced by an (n,m)-groupoid.

On the other hand, if $\underline{P}=(P;g)$ is a poly-(n,m)-groupoid, then one can obtain an (n,m)-groupoid $\underline{P}_{\#}=(P;g_{\#})$, where

$$g_{\#}(a_1^n) = g(a_1^n),$$

i.e. $g_{\#}$ is the restriction of g on P^n. Therefore, the (n,m)-groupoid $\underline{P}_{\#}$ is called a restriction of the poly-(n,m)-groupoid $\underline{P}$.

Obviously, if $\underline{Q}=(Q;f)$ is an (n,m)-groupoid, then

$$(f^{\pi})_{\#}(a_1^n) = f^{\pi}(a_1^n) = f(a_1^n),$$

i.e.

$$(\underline{Q}^{\pi})_{\#} = \underline{Q},$$

for any choice π.

However, if $\underline{P}$ is a poly-(n,m)-groupoid, then it may happen $(\underline{P}_{\#})^{\pi}\neq\underline{P}$. We will give below (P.5.10) a sufficient condition for the equality $(P_{\#})^{\pi}=\underline{P}$.

A poly-(n,m)-groupoid $\underline{P}=(P;g)$ is called a poly-(n,m)-semigroup iff the following equality

$$g(a_1^j g(b_1^{m+rk})a_{j+1}^{sk}) = g(a_1^j b_1^{m+rk} a_{j+1}^{sk}) \qquad (5.10)$$

is an identity in $\underline{P}$, for every $a_\nu, b_\lambda \in P$, $r,s\geq 1$ and $j\in\{0,1,\ldots,sk\}$.

By GAL we obtain the following:

Proposition 5.9. *Let $\underline{Q}=(Q;f)$ be an (n,m)-semigroup. Then:*

(i) $\underline{Q}^{\#}$ is the unique poly-(n,m)-groupoid induced by $\underline{Q}$;

(ii) $(\underline{Q}^{\#})_{\#} = \underline{Q}$;

(iii) $\underline{Q}^{\#}$ is a poly-(n,m)-semigroup. □

The following proposition is also true.

Proposition 5.10. *Let $\underline{P}=(P;g)$ be a poly-(n,m)-semigroup. Then:*

(i) the restriction $\underline{P}_{\#}$ is an (n,m)-semigroup;

(ii) $(\underline{P}_{\#})^{\#} = \underline{P}$. □

The notion of a cancellative (n,m)-groupoid and of a poly-(n,m)-group as well can be introduced in an obvious way. Namely, a poly-(n,m)-groupoid $\underline{P}=(P;g)$ is said to be cancellative iff the following quasi-identity

$$g(a_1^j x_1^m a_{j+1}^{sk}) = g(a_1^j y_1^m a_{j+1}^{sk}) \Longrightarrow x_1^m = y_1^m, \tag{5.11}$$

holds in $\underline{P}$ for every $s \geq 1$ and every $j \in \{0,1,\dots,sk\}$.

A poly-(n,m)-groupoid $\underline{P}=(P;g)$ is called a poly-(n,m)-group iff $\underline{P}$ is a poly-(n,m)-semigroup and any equations on x_1^m, y_1^m of the form

$$g(a_1^{sk} x_1^m) = b_1^m = g(y_1^m a_1^{sk}), \tag{5.12}$$

are solvable in $\underline{P}$ for every $s \geq 1$.

The following proposition is true:

Proposition 5.11. *Let $\underline{Q}$ be an (n,m)-groupoid.*

(i) $\underline{Q}$ is a cancellative (n,m)-semigroup iff $\underline{Q}^{\#}$ is a cancellative poly-(n,m)-semigroup;

(ii) $\underline{Q}$ is an (n,m)-group iff $\underline{Q}^{\#}$ is a poly-(n,m)-group. □

By P.5.9. and P.5.10., it is not necessary to make any distinction between an (n,m)-semigroup and its induced poly-(n,m)-semigroup because there is no essential difference between them. However there is a reasonable motivation for introducing poly-(n,m)-semigroups, as we will see in §6 in the construction of free v.v. semigroups.

The above discussion allows us to write $[a_1^{m+sk}]$ instead of $g(a_1^{m+sk})$ in both cases: for (n,m)-semigroups (s=1) and for poly-(n,m)-semigroups ($s \geq 1$, arbitrary). In this notation we will admit also s=0, setting $[a_1^m]=a_1^m$.

We will use this notation in the following:

Proposition 5.12. *Let $(Q;f)$ be a cancellative (n,m)-semigroup and let $\underline{a}\underline{x}\underline{b}$, $\underline{a}\underline{y}\underline{b}$, $\underline{a}'\underline{x}\underline{b}'$, $\underline{a}'\underline{y}\underline{b}' \in Q^{(n,m)}$. Then:*

$$[\underline{a}\underline{x}\underline{b}] = [\underline{a}\underline{y}\underline{b}] \Longrightarrow [\underline{a}'\underline{x}\underline{b}'] = [\underline{a}'\underline{y}\underline{b}'].$$

Proof. Let $[\underline{a}\underline{x}\underline{b}]=[\underline{a}\underline{y}\underline{b}]$ and choose an arbitrary element $e \in Q$. Let $\underline{a} \in Q^{(s-1)k-r}$ for $0 \leq r < k$. Then $[\underline{a}\underline{x}\underline{b}\overset{sk}{e}] = [\underline{a}\underline{y}\underline{b}\overset{sk}{e}]$ implies that $[\underline{x}\underline{b}\overset{k-r}{e}] = [\underline{y}\underline{b}\overset{k-r}{e}]$. Let $\underline{a}' \in Q^{(t-1)k+\ell}$ for $0 \leq \ell < k$. Then $[\underline{a}'\underline{x}\underline{b}\overset{k-r}{e}\overset{k-\ell}{e}] = [\underline{a}'\underline{y}\underline{b}\overset{k-r}{e}\overset{k-\ell}{e}]$, and $[\overset{k-r-\ell}{e}\underline{a}'\underline{x}\underline{b}\overset{2k}{e}] = [\overset{2k-r-\ell}{e}\underline{a}'\underline{y}\underline{b}\overset{2k}{e}]$. This implies that $[\overset{2k-r-\ell}{e}\underline{a}'\underline{x}\underline{b}] = [\overset{2k-r-\ell}{e}\underline{a}'\underline{y}\underline{b}]$. Because $\underline{a}\underline{x}\underline{b}$, $\underline{a}\underline{y}\underline{b}$, $\underline{a}'\underline{x}\underline{b}'$, $\underline{a}'\underline{y}\underline{b}' \in Q^{(n,m)}$, the same procedure leads to $[\underline{a}'\underline{x}\underline{b}']=[\underline{a}'\underline{y}\underline{b}']$. □

§6. FREE VECTOR VALUED SEMIGROUPS

In this section we will give a construction of free (n,m)-semigroups. Although the construction will make sense for $m=1$ too, we will assume that $m \geq 2$, because the description of free n-semigroups (i.e. $(n,1)$-semigroups) with a basis B is well known.

Namely, if $m=1$ and $n \geq 3$, then the subset D of B^+ defined by

$$D = \{b_1^{\alpha k+1} \mid \alpha \geq 1, \quad b_\nu \in B\} \tag{6.1}$$

is a free n-semigroup with a basis B, the operation in D being the usual concatenation of sequences.

First we will state the following:

Proposition 6.1. *Let B be a non-empty set and $(\overline{B};f)$ be the free (n,m)-groupoid with a basis B. If $\approx$ is the least congruence on $(\overline{B};f)$ such that the corresponding quotient (n,m)-groupoid $(\overline{B}/\approx;f)$ is an (n,m)-semigroup, then $(\overline{B}/\approx;f)$ is a free (n,m)-semigroup with a basis B.*

Proof. Let $(Q;g)$ be an (n,m)-semigroup, $\xi\colon B \longrightarrow Q$ be a mapping and $\overline{\xi}\colon (\overline{B};f) \longrightarrow (Q;g)$ be the unique homomorphism which is an extension of ξ. Then $(\overline{B}/\ker\overline{\xi};f)$ is an (n,m)-semigroup.

This implies that $\approx\subseteq \ker\overline{\xi}$. If we define $\eta\colon \overline{B}/\approx \longrightarrow Q$ by $\eta(u^{\approx}) = \overline{\xi}(u)$, then we obtain that η is an extension of ξ and a homomorphism of (n,m)-semigroups. Moreover, η is unique with respect to these properties. □

Using the characterisation of (n,m)-semigroups as poly-(n,m)-semigroups (P.5.9., P.5.10.), we can give a description of the free (n,m)-semigroups by a corresponding factorization of free poly-(n,m)-groupoids. Namely:

Proposition 6.2. *Let $\underline{F}(B)$ be a free poly-(n,m)-groupoid with a basis B. If $\approx$ is the least congruence on $\underline{F}(B)$ such that $\underline{F}(B)/\approx$ is a poly-(n,m)-semigroup, then $\underline{F}(B)/\approx$ is a free (n,m)-semigroup with a basis B.* □

(Here, the notion of: a congruence, a homomorphism and a free poly-(n,m)-groupoid, we will not define explicitely. However, we will give a description of a free poly (n,m)-groupoid as well.)

In the first section (§1) we gave a description of a free (n,m)-groupoid $(\overline{B};f)$ with a given basis B. By a similar discussion one obtains the following description of a free poly-(n,m)-groupoid with a basis B.

Proposition 6.3. *Let B be a nonempty set and let*

$$C_0 = B,\ C_{p+1} = C_p \cup N_m \times C_p^{(n,m)}\,^{1)},\ F(B) = \bigcup_{p\geq 0} C_p$$

Define a mapping $f\colon F(B)^{(n,m)} \longrightarrow F(B)^m$ by:

$$f(u_1^{m+sk}) = v^m \iff (\forall i\in N_m)\ \ v_i = (i, u_1^{m+sk}).$$

Then $\underline{F}(B) = (F(B);f)$ is a free poly-(n,m)-groupoid with a basis B. □

Below we will give a more explicit description of the congruences in $(\overline{B};f)$ and $\underline{F}(B)$, which are denoted by a same symbol $\approx$.

First we define a relation $\overset{o}{\longmapsto}$ in $\overline{B}$ and a relation in F(B) with the same notation $\overset{o}{\longmapsto}$. Namely, the relation $\overset{o}{\longmapsto}$ in $\overline{B}$ is defined by:

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1) See §5 for the definition of $C_p^{(n,m)}$, p. 34: $P^{(n,m)}$.

$$u \overset{o}{\longmapsto} v \iff u = (i, u_1^j(1, u_{j+1}^{j+n}) \dots (m, u_{j+1}^{j+n}) u_{j+n+1}^{n+k}),$$
$$v = (i, (1, u_1^n), \dots, (m, u_1^n) u_{n+1}^{n+k})$$

for some $j \in N_k$.

If $u, v \in F(B)$, then:

$$u \overset{o}{\longmapsto} v \iff [u = (i, x'(1,y) \dots (m,y) x''), \quad v = (i, x'yy'')]$$

where $i \in N_m$, $x', x'' \in F(B)^*$, $x'x'' \in F(B^+)$ and $y \in F(B)^{(n,m)}$. Suppose that $\overset{\alpha}{\longmapsto}$ is defined in $\overline{B}'$ and in F(B) as well. Then in each one of these cases we set

$$u \overset{\alpha+1}{\longmapsto} v \iff u = (i, x'u'x''), \quad v = (i, x'v'x''),$$

where $u' \overset{\alpha}{\longmapsto} v'$. Now,

$$u \longmapsto v \iff (\exists \alpha) \quad u \overset{\alpha}{\longmapsto} v. \tag{6.2}$$

If ~ is the symetric extension of the relation $\longmapsto$, then the relation ≈ is the reflexive and transitive extension of .

In other words,

$$u \sim v \iff u \longmapsto v \text{ or } v \longmapsto u,$$

and

$u \approx v$ iff there exist $t \geq 0$ and $u_1, u_2, \dots, u_t$ such that $u_o = u$, $u_t = v$, $u_{i-1} \sim u_i$ for $i \in \{1, 2, \dots, t\}$.

The following problem rises naturally in the both cases: given two elements $u, v \in \overline{B}$ or $u, v \in F(B)$, find an effective procedure for answering the question whether or not $u \approx v$. One of the possibilities to solve this problem is the notion of reduced elements.

It is natural to say (in both cases) that an element u is reduced iff there is no element v such that $u \longmapsto v$. Directly from the definition of $\longmapsto$ in the two cases it follows that every sequence of elements $u_o, u_1, \dots$, such that $u_i \longmapsto u_{i+1}$ is finite. This implies that, for every w, after a finite number of steps, one can get an element u, such that $w \approx u$ and u is reduced. If, in addition, the following proposition is true:

„u,v are reduced and $u \approx v \Longrightarrow u = v$",

then the following proposition would be true:

„$(\forall w)(\exists! u, \text{reduced})\ w \approx u$".

Therefore, to answer the question whether or not $w \approx v$, one would find the reduced representatives of w and v, and $w \approx v$ iff these representatives are equal.

In the first case (in P.6.1.) it is possible two distinct reduced elements to be equivalent (and thus this procedure can not be used), as the following example shows.

Example 6.4. Let n=3 and m=2. If $a_\nu \in B$, and

$$w = (1,a_1(1,a_2(1,a_3^5)(2,a_3^5))(2,a_2(1,a_3^5)(2,a_3^5)))\in\overline{B},$$

then:

$$w \approx (1,a_1(1,(1,a_2^4)(2,a_2^4)a_5)(2,(1,a_2^4)(2,a_2^4)a_5)) \approx$$

$$\approx (1,(1,a_1(1,a_2^4)(2,a_2^4))(2,a_1(1,a_2^4)(2,a_2^4))a_5) \approx$$

$$\approx (1,(1,(1,a_1^3)(2,a_1^3)a_4)(2,(1,a_1^3)(2,a_1^3)a_4)a_5) = u.$$

We also have:

$$w \approx (1,(1,a_1^2(1,a_3^5))(2,a_1^2(1,a_3^5))(2,a_3^5)) = v$$

and this implies that $u \approx v$. It is clear that u and v are reduced. Thus two different reduced elements in $\overline{B}$ are equivalent. We note that such examples we have for any n,m of $m \geq 2$.

Fortunately, this is not possible in F(B), i.e. two distinct reduced elements are not equivalent. Thus, to obtain the free poly-(n,m)-semigroup with a basis B, it is necessary to give a description of the reduced elements and the procedure for obtaining reduced representatives of given elements as well.

First we define a notion of a length of an element $x \in X$, in notation: $|x|$, where $X \in \{\overline{B}, F(B), \overline{B}^+, F(B)^+\}$, by induction, as follows (see also p. 20):

$|b| = 1$ for $b \in B$;

$|(i,u_1^{m+rk})| = \sum_{\nu=1}^{m+rk} |u_\nu|$ for $(i,u_1^{m+rk}) \in C_{p+1}$; and