$|u_1^t| = \sum_{\nu=1}^{t} |u_\nu|$ for $u_1^t \in \bar{B}^+$ or $F(B)^+$.

Denote by $S(B)=S$ the set of all the reduced elements of $F(B)$.

Now we will define a mapping ψ: $S^+ \longrightarrow S^+$ as follows.

1) If $x \in S^\alpha$ and $1 \leq \alpha \leq m-1$, then $\psi(x)=x$.

Assume that $\psi(y) \in S$ is well defined for every $y \in S^+$ such that $|y| < |x|$ and $\psi(y)$ satisfies the following condition:

$$\psi(y) \neq y \implies m < d(y) < d(\psi(y)) \text{ and } |\psi(y)|<|y|. \quad (6.4)$$

Now, if x has a form $x=x'(1,z)\ldots(m,z)x''$, where $x',x'' \in S^*$, $(\nu,z) \in S$ and x' has the least possible dimension, then we define $\psi(x)$ by:

2) $\psi(x) = \psi(x'zx'')$.

And, if x has not such a form, then we put:

3) $\psi(x) = x$.

The assumption (6.4) implies that if $\psi(x)$ is defined by 2), then:

$$|\psi(x)| < |x| \text{ and } d(\psi(x)) > d(x),$$

and this implies that ψ: $S^+ \longrightarrow S^+$ is well defined mapping such that (6.4) holds for every $y \in S^+$.

By induction on length, the following statement can be easily shown.

<u>Proposition 6.5</u>. *If $x', x'', z \in S^*$, $(\nu,y) \in S$, $x \in S^+$, $i,\alpha,\beta \in N_m$, $\alpha \neq 1$, $\beta \neq m$, then:*

(i) $\psi(x'(1,y)\ldots(m,y)x'') = \psi(x'yx'')$,

(ii) $\psi(x'xx'') = \psi(x'\psi(x)x'')$,

(iii) $\psi^2 = \psi$,

(iv) $\psi(x) \neq x \implies m < d(x) < d(\psi(x))$ *and* $|\psi(x)| < |x|$,

(v) $d(\psi(x)) \equiv d(x) \pmod k$,

(vi) $\psi(yx) \neq (i,y)z$, $\psi(xy) \neq z(i,y)$,

(vii) $\psi((\alpha,y)x) = (\alpha,y)\psi(x)$, $\psi(x(\beta,y)) = \psi(x)(\beta,y)$,

(viii) $(i,x) \in S$ *iff* $x \in F(B)^{(n,m)}$ *and* $\psi(x)=x$. ∎

Now, we are ready to prove the main result of this section.

First, if $u_1^{m+sk} \in S^{(n,m)}$ and $i \in N_m$, then $v_i = (i, \psi(u_1^{m+sk})) \in S$, and thus we can define a poly-(n,m)-groupoid $\underline{S} = (S;g)$ by:

$$g(u_1^{m+sk}) = v_1^m \text{ iff } (\forall i \in N_m) \quad v_i = (i, \psi(u_1^{m+sk})). \tag{6.5}$$

Theorem 6.6. *$\underline{S}$ is a free poly-(n,m)-semigroup with a basis B.*

Proof. By P.6.5. (i), (iii), (viii) it can be easily seen that $\underline{S}$ is a poly-(n,m)-semigroup and it is clear that B is a generating subset of S.

Assume that $\underline{Q} = (Q;f)$ is a poly-(n,m)-semigroup and $\xi: b \mapsto \bar{b}$ a mapping from B into Q. Then, there is a unique homomorphism $\bar{\xi}: \underline{F}(B) \longrightarrow \underline{Q}$. Denote by ζ the restriction of $\bar{\xi}$ on S. We will show that ζ is a homomorphism from $\underline{S}$ in $\underline{Q}$, and this will complete the proof.

Let $g(u_1^{m+sk}) = v_1^m$, i.e. $v_i = (i, \psi(u_1^{m+sk})$, and $\zeta(u_\nu) = \bar{u}_\nu$, $\zeta(v_\lambda) = \bar{v}_\lambda$. If $(1, u_1^{m+sk}) \in S$, then we have $\psi(u_1^{m+sk}) = u_1^{m+sk}$ and thus

$$\bar{v}_i = \zeta(i, u_1^{m+sk}) = f_i(\bar{u}_1^{m+sk}), \text{ i.e. } f(\bar{u}_1^{m+sk}) = \bar{v}_1^m.$$

Assume that: $\quad u_1^{m+sk} = u_1^j(1, w_1^{m+rk}) \ldots (m, w_1^{m+rk}) u_{j+m+1}^{m+sk}$.

Then:

$$v_i = (i, \psi(u_1^j w_1^{m+rk} u_{j+m+1}^{m+sk})),$$

and by induction we have:

$$\begin{aligned} \bar{v}_i &= f_i(\bar{u}_1^j \bar{w}_1^{m+rk} \bar{u}_{j+m+1}^{m+sk}) \\ &= f_i(\bar{u}_1^j f(\bar{w}_1^{m+rk}) \bar{u}_{j+m+1}^{m+sk}) \\ &= f_i(\bar{u}_1^{m+sk}). \end{aligned}$$

Thus, we showed that

$$g(u_1^{m+sk}) = v_1^m \implies f(\bar{u}^{m+sk}) = \bar{v}_1^m. \ \square$$

As a corollary we obtain the following desired result:

Proposition 6.7. *If $u, v \in S \subset F(B)$ are such that $u \approx v$, then $u = v$.* $\square$

Note that we do not make difference between a free poly-(n,m)-semigroup and a free (n,m)-semigroup. Therefore, $\underline{S}(B)$ is a free (n,m)-semigroup.

Theorem 6.8. *Let $\underline{S}=(S;g)$ be a free (n,m)-semigroup, with a basis B of cardinality β, and $m \geq 2$. If α is a cardinal such that $\alpha \leq max\{\beta, \aleph_0\}$, then there exists an (n,m)-subsemigroup $\underline{T}$ of $\underline{S}$, which is a free (n,m)-semigroup with a basis C of cardinality α.*

Proof. Clearly, it is enough to show that if $B=\{b\}$ is a one-element set, then $\underline{S}$ has a free (n,m)-subsemigroup with an infinite basis.

Namely, if $\underline{S}$ is defined as in T.6.6., and if

$$a_r = (1, b^{m+rk}),$$

then the (n,m)-subsemigroup $\underline{T}$ of $\underline{S}$ generated by $A=\{a_r \mid r \geq 1\}$ is a free (n,m)-semigroup with a basis A. □

We note that the above result, in the case $m = 1$ holds only if $\beta \geq 2$.

Theorem 6.9. *Every free (n,m)-semigroup is cancellative.*

Proof. Let $\underline{S}=(S;g)$ be the free poly-(n,m)-semigroup with a basis B, defined as above. We will show that the following implication holds:

$$\psi(xy) = \psi(xz) \text{ or } \psi(yx) = \psi(zx) \Longrightarrow \psi(y) = \psi(z) \qquad (6.6)$$

for any x,y,z, and this will imply the desired result that $\underline{S}$ is cancellative.

Assume that $\psi(xy)=\psi(xz)$. We will show that $\psi(y)=\psi(z)$, by induction on $|xyz|$. First if $\psi(xy)=xy$, $\psi(xz)=xz$, then $y=z$. By P.6.5. (ii) we have

$$\psi(\psi(x)y) = \psi(x\psi(y)) = \psi(\psi(x)z) = \psi(x(\psi(z))),$$

and thus we can assume:

$$\psi(x) = x, \quad \psi(y) = y, \quad \psi(z) = z, \text{ and } \quad \psi(xy) \neq xy.$$

Therefore, we have:

$$x = x'(1,x'')\ldots(\beta,x''), \quad y = (\beta+1,x'')\ldots(m,x'')y',$$

for some $\beta \in N_{m-1}$.

Then, one of the following conditions holds:

$$\text{a) } \psi(xz) = xz, \quad \text{b) } z = (\beta+1,x'')\dots(m,x'')z'.$$

In the case a) we would have:

$$x'(1,x'')\dots(\beta,x'')z = \psi(x'x''y')$$

and this would imply:

$$(1,x'')\dots(\beta,x'')z = \psi(x''y'),$$

which is impossible by P.6.5. (vi).

If b) holds, then we have:

$$\psi(x'x''y') = \psi(x'x''z'),$$

and this (by the induction) implies $\psi(y')=\psi(z')$, hence (by P.6.5. (vii)):

$$\begin{aligned}\psi(y) &= (\beta+1,x'')\dots(m,x'')\psi(y')\\ &= (\beta+1,x'')\dots(m,x'')\psi(z')\\ &= \psi(z).\end{aligned}$$

Thus, $\psi(xy)=\psi(xz) \Longrightarrow \psi(y)=\psi(z)$, and by symmetry: $\psi(yx)=\psi(zx) \Longrightarrow \psi(y)=\psi(z)$. This completes the proof of (6.6).

Assume that

$$g(u_1^{sk}v_1^m) = g(u_1^{sk}w_1^m),$$

where u_ν, v_λ, $w_\lambda \in S$, $s \geq 1$, i.e.

$$\psi(u_1^{sk}v_1^m) = \psi(u_1^{sk}w_1^m).$$

By (6.6), this implies $\psi(v_1^m)=\psi(w_1^m)$, hence by P.6.5. (iv), either $v_1^m=\psi(v_1^m)=\psi(w_1^m)=w_1^m$, or $\psi(v_1^m)=y=\psi(w_1^m)$, where $v_1^m=(1,y)(2,y)\cdots(m,y)=$ $=w_1^m$. □

Let $\underline{S}=(S;[\ ])$ be the free (n,m)-semigroup with a basis B as above. (Here we denote g by $[\ ]$). Denote by $\hat{S}$ the set $\psi(S^+)$, i.e.

$$\hat{S} = \{x \in S^+ \mid \psi(x) = x\}.$$

Define a (binary) operation $\bullet$ on $\hat{S}$ by:

$$x \bullet y = \psi(xy)$$

By P.6.5 (ii) and (6.6) it can be easily shown that the following statement is true:

Proposition 6.10. $\underline{\hat{S}} = (\hat{S};\bullet)$ *is a cancellative semigroup generated by* S. □

Theorem 6.11. $\underline{\hat{S}}$ *is the universal semigroup for the* (n,m)*-semigroup* $\underline{S}$.

Proof. We have to show that $\underline{\hat{S}}=\langle S;\Lambda\rangle$, where

$$\Lambda = \{(u_1^n, v_1^m) \mid [u_1^n] = v_1^m \text{ in } \underline{S}\}.$$

First, it is clear that the embedding from S in $\hat{S}$ is a realization of (S,Λ) in $\underline{\hat{S}}$.

Let ξ: $u \mapsto \bar{u}$ be a realization of (S,Λ) in a semigroup $\underline{H}=(H;\circ)$. We are looking for a homomorphism ζ: $\underline{\hat{S}} \to \underline{H}$, which is an extension of ξ.

Consider first the homomorphism ξ^+: $S^+ \to \underline{H}$, defined as in §3, i.e. by:

$$\xi^+(u_1^\alpha) = \bar{u}_1 \circ u_2 \circ \ldots \circ \bar{u}_\alpha,$$

for every $u_1^\alpha \in S^\alpha$. By induction on lengths and dimensions we will show that:

$$(\forall x \in S^+)\, \xi^+(x) = \xi^+(\psi(x)). \tag{6.7}$$

We have only to consider the case when $\psi(x) \neq x$.

If $x=(1,y)\ldots(m,y)$, where, $(\nu,y)\in S$, $y=u_1^{m+rk}$, then:

$$\xi^+(x) = \overline{(1,y)} \circ \overline{(2,y)} \circ \ldots \circ \overline{(m,y)}$$

and:

$$\xi^+(\psi(x)) = \xi^+(y) = \bar{u}_1 \circ \ldots \circ \bar{u}_{m+rk} = \xi^+(x),$$

for $[u_1^{m+rk}] = v_1^m$, where $v_\lambda = (\lambda, y)$.

Assume now that, $x=x'(1,y)\ldots(m,y)x''$, where $x',x''\in S^*$, $x'x''\in S^+$, $(\nu,y)\in S$. Then we have:

$$\begin{aligned}
\xi^+(\psi(x)) &= \xi^+(x'yx'') = \xi^*(x')\xi^+(y)\xi^*(x'') \\
&= \xi^*(x')\xi^+((1,y)(2,y)\ldots(m,y))\xi^*(x'') \\
&= \xi^+(x'(1,y)\ldots(m,y)x'') \\
&= \xi^+(x),
\end{aligned}$$

and this completes the proof of (6.7).

If $x,y\in\hat{S}$, then:

$$\begin{aligned}\zeta(x\bullet y) &= \zeta(\psi(xy)) = \xi^+(\psi(xy))\\ &= \xi^+(xy) = \xi^+(x)\circ\xi^+(y)\\ &= \zeta(x)\circ\zeta(y),\end{aligned}$$

and this implies that ζ: $\underline{\hat{S}} \longrightarrow \underline{H}$ is a homomorphism which is an extension of ξ. □

If we consider $\underline{S}$ as a poly-(n,m)-semigroup, then it is natural to consider the following presentation:

$$\Lambda' = \{(u_1^{m+sk}, v_1^m) \mid [u_1^{m+sk}] = v_1^m \text{ in } \underline{S}\}.$$

Then, we have $\langle S;\Lambda\rangle = \langle S;\Lambda'\rangle = \hat{S}$, and this statement is a corollary from the following more general

Proposition 6.12. *Let $\underline{P}=(P;g)$ be a poly-(n,m)-semigroup and Λ,Λ' sets of semigroup relations on P defined by:*

$$\Lambda = \{(a_1^n, b_1^m) \mid g(a_1^n) = b_1^m\}$$

$$\Lambda' = \{(a_1^{m+sk}, b_1^m) \mid g(a_1^{m+sk}) = b_1^m\}.$$

Then: $\langle P;\Lambda\rangle = \langle P;\Lambda'\rangle$. □

§7. UNIVERSAL COVERINGS OF VECTOR VALUED SEMIGROUPS

Here we will give a more precise description of the universal semigroup $\underline{Q}^{\wedge}$ of an (n,m)-semigroup $\underline{Q}=(Q;f)$, defined in §3. We recall that, as in §5, if $u=a_1^{sk+m}\in Q^{sk+m}$, then

$$[u] = f^{(s)}(a_1^{sk+m})\in Q^m$$

for every $s\geq 0$. The relations $\vdash$, ~ and $\overset{\Lambda}{=}$ are defined as in §3 with $\Lambda=\Lambda_Q$.

Proposition 7.1. *If $u\in Q^m$, $v\in Q^+$, then*

$$u \overset{\Lambda}{=} v \text{ iff } v\in Q^{sk+m} \text{ and } [v]=u, \text{ for some } s\geq 0.$$

Proof. Let $v\in Q^{sk+m}$ and $[v]=u\in Q^m$. If $s = 0$, then $u = v$, and clearly $u \overset{\Lambda}{=} v$; if $s = 1$, then the definition of Λ_Q implies $u \overset{\Lambda}{=} v$. Suppose that $v=a_1^{sk+m}$, $s \geq 2$ and $[v]=u$. Then $[v]=[wa_{k+m+1}^{sk+m}]$, where $w=[a_1^{k+m}]\in Q^m$ and $v \overset{\Lambda}{=} wa_{k+m+1}^{sk+m}$. Since $[wa_{k+m+1}^{sk+m}]=u$, by induction on s we have $wa_{k+m+1}^{sk+m} \overset{\Lambda}{=} u$, i.e. $u \overset{\Lambda}{=} v$.

Assume now that $u\in Q^m$, $v\in Q^+$ and $u \overset{\Lambda}{=} v$. Then there exist $u_o,\dots,u_t \in Q^+$, $t \geq 0$, such that $u=u_o$, $v=u_t$ and $u_{i-1} \sim u_i$ for $i\in N_t$. By P.3.3. c) we have that $d(u_{i-1}) \equiv d(u_i) \equiv m(\mathrm{mod}k)$ for every $i\in N_t$, and so it is enough to prove that $[u_{i-1}]=[u_i]$. But, the last equality is true by the definition of $\vdash$ and the GAL. □

Proposition 7.2. *If $u\in Q^\alpha$ and $\alpha \geq m$ then there exists a unique $\beta\in\{0,1,\dots,k-1\}$ such that $\alpha-m \equiv \beta(\mathrm{mod}k)$ and $u \overset{\Lambda}{=} v$ for some $v\in Q^{m+\beta}$.*

Proof. Let $\alpha=m+\gamma k+\beta$, $0 \leq \beta < k$, and suppose that $u=u'u''$, where $u'\in Q^{m+\gamma k}$, $u''\in Q^\beta$. Then, by P.7.1., $u' \overset{\Lambda}{=} [u']$, which implies $u \overset{\Lambda}{=} v$, where $v=[u']u''\in Q^{m+\beta}$. □

As a consequence of P.7.1. and P.7.2., we have that:

$$u,v\in Q^m \text{ and } u \overset{\Lambda}{=} v \text{ imply } u = v.$$

i.e.

$$Q^m \subseteq Q^\wedge.$$

Thus, by P.3.3. and the above remark, we have the following description of the universal semigroup $\underline{Q}^\wedge$.

Theorem 7.3. *The universal semigroup $\underline{Q}^\wedge$ of an (n,m)-semigroup $\underline{Q}$ has a carrier $Q^\wedge$ represented as a disjoint union of the form*

$$Q \cup Q^2 \cup \dots \cup Q^m \cup Q_{m+1} \cup \dots \cup Q_{n-1} \tag{7.1}$$

where $Q_{m+\beta}=Q^{m+\beta}/\overset{\beta}{=}$ and $\overset{\beta}{=}$ is the restriction of $\overset{\Lambda}{=}$ on $Q^{m+\beta}$ for every $\beta\in N_{k-1}$. □

Note that, by using the multiplicative notation $\bullet$ for the operation on $Q^\wedge$, we have that

$$a_1\dots a_i\bullet b_1\dots b_j = \begin{cases} a_1\dots a_i b_1\dots b_j & \text{if } i+j<n \\ [a_1^i b_1^{n-i}]\bullet b_{n-i+1}\dots b_j & \text{if } i+j\geq n \end{cases} \tag{7.2}$$

We will denote by $Q^\vee$ the subset

$$Q^m \cup Q_{m+1} \cup \dots \cup Q_{n-1}$$

of $Q^\wedge$ and we say that $Q^\vee$ is the universal envelope of $\underline{Q}$. It is clear that:

Proposition 7.4. *$Q^{\vee}$ is an ideal in $\underline{Q}^{\wedge}$.* □

Note that the set $N_{n-1}=N_{k+m-1}=\{1,\dots,m,m+1,\dots,n-1\}$ is a cyclic semigroup generated by 1, of an index m and a period k, with respect to the operation ⊕ defined by:

$$\alpha\oplus\beta = \begin{cases} \alpha+\beta & \text{if } \alpha+\beta \leq n-1 \\ \alpha+\beta-tk & \text{if } m+tk \leq \alpha+\beta < m+(t+1)k \end{cases} \tag{7.3}$$

The following proposition follows directly from (7.2) and (7.3).

Proposition 7.5. *The map $\|\ \|: Q^{\wedge} \longrightarrow N_{n-1}$ defined by $\|u\| = \alpha$ iff $u\in Q^{\alpha}$ or $u\in Q_{\alpha}$, is a homomorphism from $\underline{Q}^{\wedge}$ onto $(N_{n-1};\oplus)$.* □

If $m \leq \ell k < m+k$, then ℓk is the neutral element in the subgroup $Z_k'=\{m,m+1,\dots,m+k-1\}$ of $(N_{n-1};\oplus)$. This, and P.7.5. imply:

Corollary 7.6. *$Q_{\ell k}$ is a subsemigroup of $\underline{Q}^{\vee}$ and so of $\underline{Q}^{\wedge}$.* □

T.7.3. implies that the following is true:

Theorem 7.7. *Every (n,m)-semigroup $\underline{Q}$ is a pure (n,m)-subgroupoid of its universal semigroup $\underline{Q}^{\wedge}$.*

(In this case we say that $\underline{Q}$ is a pure (n,m)-subsemigroup of $\underline{Q}^{\wedge}$.) □

This result is a generalization of Post's theorem for polyadic groups ([4], [6], [41]) and that is why we reffer to it as Post Theorem.

Further on, according to T.7.7., the semigroup $\underline{Q}^{\wedge}$ will be called a universal covering of the (n,m)-semigroup $\underline{Q}$.

We note that, if $\underline{P}=(P;g)$ is a poly-(n,m)-semigroup, then the semigroup $\langle P;\ \Gamma(\underline{P})\rangle$, where

$$\Gamma(\underline{P}) = \{(a_1^{m+sk}, b_1^m) \mid g(a_1^{m+sk}) = b_1^m,\ s\geq 1,\ a_\nu, b_\lambda \in P\},$$

coincides with the universal covering $\underline{Q}^{\wedge}$ of the restriction $\underline{Q}=\underline{P}_{\#}$ of $\underline{P}$.

A semigroup $\underline{S}=(S;\cdot)$ is said to be a covering of an (n,m)-groupoid $\underline{Q}$ iff $\underline{Q}$ is a pure (n,m)-subgroupoid of $\underline{S}$ and $\underline{S}$ is generated by Q. Every covering of an (n,m)-semigroup $\underline{Q}$ is a homomorphic image of the universal covering $\underline{Q}^{\wedge}$, i.e.

Proposition 7.8. *If a semigroup $\underline{S}$ is a covering of an (n,m)-semigroup $\underline{Q}$, then the inclusion $a \mapsto a$ of Q into S can be uniquely extended to a homomorphism of $\underline{Q}^{\wedge}$ into $\underline{S}$.* □

Proposition 7.9. *If the universal envelope $\underline{Q}^{\vee}$ is a cancellative semigroup, then $\underline{Q}$ is a cancellative (n,m)-semigroup. In this case, if a_1^{m+i}, $b_1^{m+i} \in Q^{m+i}$, $0 \le i < k$, then the following conditions are equivalent:*

(i) $a_1 \ldots a_{m+i} = b_1 \ldots b_{m+i}$ *in* Q_{m+i};

(ii) the equality

$$[c_1^{sk-i} a_1^{m+i}] = [c_1^{sk-i} b_1^{m+i}] \tag{7.4}$$

holds in $\underline{Q}$ for every $s \ge 1$ and every $c_\nu \in Q$;

(iii) there exist $s \ge 1$ and $c_\nu \in Q$ such that the equality (7.4) holds in $\underline{Q}$.

Proof. Let $u \in Q^k$, $v,w \in Q^m$ and suppose that $[uv]=[uw]$. Then in $\underline{Q}^{\vee}=(Q^{\vee};\bullet)$ we have $u\bullet v=[uv]=[uw]=u\bullet w$, which implies $v=w$. Similarly, $[vu]=[wu]$ implies $v=w$.

It is clear that (i) ⟹ (ii), (ii) ⟹ (iii).

Suppose that for some $s \ge 1$ and some $c_\nu \in Q$ the equality (7.4) holds in $\underline{Q}$. Then we have in $\underline{Q}^{\vee}$

$$c_1\bullet \ldots \bullet c_{sk-i}\bullet a_1 \ldots a_{m+i} = c_1\bullet \ldots \bullet c_{sk-i}\bullet b_1 \ldots b_{m+i}$$

and multiplying by $d_1 \ldots d_{m+i}$,

$$[d_1^{m+i} c_1^{sk-i}]\bullet a_1 \ldots a_{m+i} = [d_1^{m+i} c_1^{sk-i}]\bullet b_1 \ldots b_{m+i}.$$

The last equality implies $a_1 \ldots a_{m+i}=b_1 \ldots b_{m+i}$ in $\underline{Q}^{\vee}$, i.e. in $\underline{Q}^{\wedge}$. □

Next we will show that every cancellative (n,m)-semigroup admits a cancellative covering.

Theorem 7.10. *Let $\underline{Q}$ be a cancellative (n,m)-semigroup and define a relation $\approx$ on Q^+ by*

$$u \approx v \iff (\exists w \in Q^+)[uw] = [vw]. \tag{7.5}$$

Then $\approx$ is a congruence on Q^+ and $\underline{Q}^{\sim} = Q^+/\approx$ is a cancellative covering of $\underline{Q}$.

Proof. If $u\in Q^{\alpha}$, $v\in Q^{\beta}$ and $u \approx v$ then $\alpha \equiv \beta\,(\mathrm{mod}\,k)$; namely, if $w\in Q^{\gamma}$ and $[uw]=[vw]$, then $\alpha+\gamma \equiv \beta+\gamma\ (\mathrm{mod}\,k)$.

From P.5.12., it follows that $u \approx v$ iff $d(u) \equiv d(v)\,(\mathrm{mod}\,k)$ and $[w_1uw_1]=[w_1vw_2]$ for every $w_1,w_2\in Q^+$ such that $d(w_1uw_2)\equiv m\ (\mathrm{mod}\,k)$. Now, it is easy to see that $\approx$ is a congruence on Q^+ and that

$$wu \approx wv \text{ or } uw \approx vw \text{ implies } u \approx v.$$

Thus, the factor semigroup $\underline{Q}^{\sim} = Q^+/\approx$ is cancellative.

We can assume that $Q\subseteq Q^{\sim} = Q^+/\approx$, since

$$a,b\in Q \text{ and } a \approx b \text{ implies } [\overset{n}{a}]=[ba^{n-1}] \text{ in } \underline{Q},$$

i.e. $\overset{m}{a}=\overset{m}{b}\,\overset{m-1}{a}$, after cancelling. Hence, $a=b$.

Let $[a_1^n]=b_1^m$ in $\underline{Q}$. Then $a_1^n \approx b_1^m$, i.e. $a_1a_2\ldots a_n=b_1\ldots b_m$ in $\underline{Q}^{\sim}$. This means that $\underline{Q}$ is an (n,m)-subsemigroup of $\underline{Q}^{\sim}$. In fact, $\underline{Q}$ is a pure (n,m)-subsemigroup of $\underline{Q}^{\sim}$, since

$$a_1^m \approx b_1^m \text{ implies } [c_1^k a_1^m] = [c_1^k b_1^m] \text{ in } \underline{Q},$$

and the cancellativity of $\underline{Q}$ implies $a_1^m = b_1^m$.

It is clear that $\underline{Q}^{\sim}$ is generated by Q. □

Example 7.11. Let $(Q;[\ ])$ be a constant (n,m)-semigroup defined as in E.2.6.1), i.e. there is an $a_1^m\in Q^m$ such that

$$[x_1^n] = a_1^m$$

for all $x_\nu\in Q$. Then:

$$u \overset{\Lambda}{=} v \iff u = v \text{ or } (u=u'a_1^m u'',\ v=v'a_1^m v''), \tag{7.6}$$

where $u,v\in Q^+$, $u'u''$, $v'v''\in Q^+$ (and $\Lambda=\Lambda_{\underline{Q}}$). Namely, let $u=b_1^i a_1^m b_{i+1}^j$, $v=c_1^p a_1^m c_{p+1}^j$ for some $b_\nu, c_\lambda\in Q$, $i,p \geq 0$. Then, the fact that $x_1^n \overset{\Lambda}{=} y_1^n \overset{\Lambda}{=} a_1^m$ for every $x_\nu, y_\lambda\in Q$ implies

$$\begin{aligned} u &\overset{\Lambda}{=} b_1^i x_1^n b_{i+1}^j \overset{\Lambda}{=} b_1^i x_1^{n-i} x_{n-i+1}^n b_{i+1}^j \\ &\overset{\Lambda}{=} c_1^p y_1^{n-p} x_{n-i+1}^n b_{i+1}^j \overset{\Lambda}{=} c_1^p y_1^n c_{p+1}^j \\ &\overset{\Lambda}{=} c_1^p a_1^m c_{p+1}^j \overset{\Lambda}{=} v. \end{aligned}$$

Conversely, let $u \overset{\Lambda}{=} v$ and $u \neq v$. Then we can apply some of the defining relations from Λ iff u and v contain a_1^m as a subword.

As a consequence of (7.6) we have that

$$Q_\alpha = \{u\in Q^+ \mid |u|=\alpha,\ a_1^m \text{ is not a subword of } u\} \cup \{a_1^m\, b^{\alpha-m}\}$$

for every α: $m < \alpha < n$, where b is a fixed element of Q. Hence, the multiplication in Q^ can be given by

$$c_1^s \bullet d_1^t = \begin{cases} c_1^s d_1^t, & \text{if } s+t < n \text{ and } a_1^m \text{ is not a subword of } c_1^s d_1^t \\ a_1^m\, b^r, & \text{otherwise,} \end{cases}$$

where $0 \le r < k$, $r \equiv s+t-m \pmod{k}$.

It can be easily seen that if $|Q|=q < \infty$, then $|Q^\wedge| < \infty$ as well, and moreover

$$|Q^\wedge| = k-2 + \frac{q^{m+1}-1}{q-1} + \frac{kq^{k+1}-(k+1)q^k+1}{(q-1)^2}$$

for $q \ge 2$, and $|Q^\wedge| = n-1$ for $q = 1$.

Example 7.12. Let $(Q;[\ ])$ be a left zero (n,m)-semigroup defined as in E.2.6. 2). Then we have

$$u \overset{\wedge}{=} v \iff u = v \text{ or } (d(u)=d(v)) \text{ and } (\exists c_1^m \in Q^m)(u=c_1^m u',\ v=c_1^m v') \tag{7.7}$$

where $u,v\in Q^+$, $u',v'\in Q^*$. Namely, the defining relations imply that if $u \overset{\wedge}{=} v$, then the first m elements of u and v are the same. On the other hand, since $u \overset{\wedge}{=} v$ imply $|u| \equiv |v| \pmod{k}$, if $u=c_1^m b_1^j$, $v=c_1^m d_1^j$ $(j \ge 0)$ then

$$u \overset{\wedge}{=} c_1^m b_1^j \overset{\wedge}{=} c_1^m x_1^n b_1^j \overset{\wedge}{=} c_1^m y_1^m d_1^j$$
$$\overset{\wedge}{=} c_1^m d_1^j \overset{\wedge}{=} v.$$

Now we have the following description of $Q^\wedge$:

$$Q_\alpha = \{u\in Q^+ \mid u = v\, b^{\alpha-m},\ |v| = m\},$$

where $m < \alpha < n$ and b is a fixed element of Q. The multiplication on Q^ is given by

$$c_1^s \bullet d_1^t = \begin{cases} c_1^s d_1^t, & \text{if } s+t \le m \\ c_1^m\, b^r, & \text{if } s \ge m \\ c_1^s d_1^{m-s}\, b^r, & \text{otherwise} \end{cases}$$

where $0 \le r < k$, $r \equiv s+t-m \pmod{k}$.

If $|Q|=q < \infty$ then $|Q^\wedge| < \infty$ as well, and

$$|Q^\wedge| = kq^{m-1} + \frac{q^m-1}{q-1}$$

for $q \geq 2$, and $|Q^\wedge| = n-1$ for $q = 1$.

By duality, we have corresponding results for right zero (n,m)-semigroups too.

Example 7.13. Let $(Q;[\ ])$ be an (n,m)-rectangular band, defined in E.2.6. 3), where $Q=A\times B$, $\underline{A}$ is a left zero and $\underline{B}$ is a right zero (n,m)-semigroup. One can show that if $u,v\in Q^+$, $m<|u| = |v| <n$, then

$$u \stackrel{\wedge}{=} v \iff u = (a_1,b_1)\dots(a_i,b_i),$$
$$v = (c_1,d_1)\dots(c_i,d_i),\ a_1^m = c_1^m,\ b_{i+1-m}^i = d_{i+1-m}^i,$$

where $a_\nu,c_\nu\in A$, $b_\lambda,d_\lambda\in B$. Also, if $|A| = \alpha < \infty$, $|B| = \beta < \infty$, then

$$|Q^\wedge| = |(A\times B)^\wedge| = k\alpha^m\beta^m-1 + \frac{\alpha^m\beta^m-1}{\alpha\beta-1}$$

for $\alpha\beta \geq 2$, and $|Q^\wedge| = n-1$ for $\alpha\beta = 1$.

Example 7.14. A universal covering semigroup of a free (n,m)-semigroup $\underline{S}=(S;[\ ])$ is the semigroup $\underline{\hat{S}}=(\hat{S};\bullet)$ defined in §6. But $\underline{\hat{S}}$ has not the usual form:

$$\hat{S} = S \cup S^2 \cup \dots \cup S^m \cup S_{m+1} \cup \dots \cup S_{n-1}. \tag{7.8}$$

To get such a form we have to make a modification.

Define first a mapping $x \mapsto \bar{x}$ from $\hat{S}$ in S^+, in the following way. If $x\in\hat{S}$, then:

$$\bar{x} = \begin{cases} x & \text{if } 1 \leq d(x) < n \\ (1,y)\dots(m,y)z, & \text{where } x=yz,\ y\in S^{(n,m)},\ 0 \leq d(z) < k. \end{cases}$$

Denote by $S^\wedge$ the set $\{\bar{x} \mid x\in\hat{S}\}$, and define a (binary) operation $\bullet$ on $S^\wedge$ by:

$$\bar{x}\bullet\bar{y} = \overline{\psi(xy)}.$$

Then we get a semigroup $\underline{S}^\wedge=(S^\wedge;\bullet)$ isomorphic with $\underline{\hat{S}}$, and morover (7.8) holds.

§8. VECTOR VALUED GROUPS AND THEIR COVERINGS

In this section we will give some characterizations of v.v. groups, mainly using their coverings.

Let n,m be given integers, $n-m=k\geq 1$. Recall that an (n,m)-semigroup (Q;[ ]) is an (n,m)-group if for each $\underline{a}\in Q^k$, $\underline{b}\in Q^m$, there exist $\underline{x},\underline{y}\in Q^m$ such that $[\underline{a}\underline{x}]=\underline{b}=[\underline{y}\underline{a}]$. The question about the existence of v.v. groups will be considered later, but we know that (n,m)-groups do exist; see E.2.7. Since an (n,m)-group (Q;[ ]) is also an (n,m)-semigroup, we have the universal covering semigroup $Q^{\wedge}=\langle Q;\wedge_Q\rangle$, (see §3 and §7), and by T.7.3:

$$Q^{\wedge} = Q\cup Q^2\cup \ldots \cup Q^{m-1}\cup Q^{\vee},$$

where

$$Q^{\vee} = Q^m\cup Q_{m+1}\cup \ldots \cup Q_{n-1}$$

is the universal envelope of Q.

Proposition 8.1. *Let (Q;[ ]) be an (n,m)-semigroup. Then the following conditions are equivalent:*

(i) (Q;[ ]) is an (n,m)-group;

(ii) $\underline{Q}^{\vee}$ is a group.

Proof. (i) ⟹ (ii): Let $a_1\ldots a_{m+p}$, $b_1\ldots b_{m+q}\in Q$, and let $m+p=sk+r$ for $0\leq r<k$. Let $a\in Q$. Then for $a_1^{m+p}\overset{k-r}{a}\in Q^{(s+1)k}$ there exists $c_1^m\in Q^m$ such that $[a_1^{m+p}\overset{k-r}{a}c_1^m]=b_1^m$ (see T.5.8). This implies that

$$a_1a_2\ldots a_{m+p}\cdot \overset{k-r}{a}\cdot c_1\ldots c_m\cdot b_{m+1}\ldots b_{m+q} = b_1b_2\ldots b_{m+q} \text{ in } \underline{Q}^{\vee}.$$

Hence, the equations $\underline{a}\cdot\underline{x}=\underline{b}$ have solutions in $\underline{Q}^{\vee}$, and the proof that the equations $x\cdot a=b$ have solutions in $\underline{Q}^{\vee}$, is symmetrical. So, $\underline{Q}^{\vee}$ is a group.

(ii) ⟹ (i): Let $a_1^{rk}\in Q^{rk}$, $b_1^m\in Q^m$ be given, where $m\leq rk<n$. Then, there exists $c_1\ldots c_{m+p}\in\underline{Q}^{\vee}$, such that $a_1\ldots a_{rk}\cdot c_1\ldots c_{m+p}=b_1\ldots b_m$ in $\underline{Q}^{\vee}$. P.3.3 implies that $m\equiv m+p+rk \pmod k$, i.e. p=0. P.7.1 implies that $[a_1^{rk}c^m]=b_1^m$. Hence the equations $[a_1^{rk}x_1^m]=b_1^m$, have solutions in Q, and symmetrically, the equations $[x_1^m a_1^{rk}]=b_1^m$ have solutions in Q. This, together with P.5.5., implies that $(Q;[\ ]^{(r)})$ is v.v. group, which together with T.5.8. implies that $\underline{Q}$ is a v.v. group. □

The proofs of the following two corollaries follow directly from P.8.1, P.3.3, C.7.6, T.5.8 and P.7.9.

Corollary 8.2. *Let $(Q;[\ ])$ be an (n,m)-group, and $m \le \ell k < m+k$. Then $(Q_{\ell k};\cdot)$ is a normal subgroup of $\underline{Q}^{\vee}$. Moreover, the factor group $\underline{Q}^{\vee}/\underline{Q}_{\ell k}$ is a cyclic group of order k.* □

Corollary 8.3. *If $\underline{Q}$ is a v.v. group, then $\underline{Q}$ is a cancellative v.v. semigroup.* □

Corollary 8.4. *Let $(Q;[\ ])$ be an (n,m)-semigroup, $m \le \ell k < m+k$, and $t=\ell k-m$. Then $(Q;[\ ])$ is an (n,m)-group iff for each $a_1^t \in Q^t$, $(Q^m;*)$ is a group, where $\underline{x}*\underline{y}=[\underline{x}a_1^t\underline{y}]$. Moreover, each $(Q^m;*)$ is isomorphic to $(Q_{\ell k};\cdot)$.* □

Now we give the proof of P.2.8.

Proof of P.2.8. The implication (i) ⟹ (ii) follows from the definition of v.v. group and C.8.3. The implications (ii) ⟹ (i) and (iii) ⟹ (i) are obvious. The implication (i) ⟹ (iii) follows from T.5.8 and C.8.3.

We note that P.7.9 is applicable for v.v. groups, because the universal envelope of a v.v. group is a cancellative semigroup.

Next we have the following propositions.

Proposition 8.5. *If $\underline{H}$ is an (n,m)-subgroup[1] of an (n,m)-group $\underline{Q}$, then $\underline{H}^{\wedge}$ is a subsemigroup of $\underline{Q}^{\wedge}$, and $\underline{H}^{\vee}$ is a subgroup of $\underline{Q}^{\vee}$.*

Proof. If $a_1 \ldots a_{m+i}=b_1 \ldots b_{m+i}$ in $\underline{Q}^{\wedge}$, for $a_\nu, b_\nu \in H$, then P.7.9. implies that $a_1 \ldots a_{m+i}=b_1 \ldots b_{m+i}$ in $\underline{H}^{\wedge}$. □

Proposition 8.6. *If $\underline{Q}$ is an (n,m)-group and $a_1^i \in Q^i$, $b_1^{m+i} \in Q^{m+i}$, $1 \le i < k$, then for each $0 \le j \le i$, there exists a unique $x_1^m \in Q^m$ such that*

$$a_1 \ldots a_j \cdot x_1 \ldots x_m \cdot a_{j+1} \ldots a_i = b_1 \ldots b_{m+i} \text{ in } \underline{Q}^{\wedge}.$$

Proof. Let $c_1^{k-i} \in Q^{k-i}$ be an arbitrary element. Then the equation

[1] *i.e. $H \subseteq Q$ and $\underline{H}$ is an (n,m)-group with respect to the (n,m)-operation of $\underline{Q}$.*

$$[c_1^{k-i}a_1^j x_1^m a_{j+1}^i] = [c_1^{k-i}b_1^{m+i}]$$

has a unique solution $x_1^m \in Q^m$. Now, the conclusion follows from P.7.9. □

As a consequence of the above propositions we have the following:

Corollary 8.7. *Let* $(Q;[\ ])$ *be an* (n,m)*-group. Then, for each* $a \in Q$

$$Q^{\vee} = Q^m \cup aQ^m \cup \ldots \cup \overset{k-1}{a} Q^m,$$

where $\overset{i}{a}Q^m = \{\overset{i}{a}x_1^m \mid x_1^m \in Q^m\}$. *Moreover, the operation* $\bullet$ *on* $Q^{\wedge}$ *is given by:*

$$(x_1 \ldots x_r)\bullet(y_1 \ldots y_s) = \begin{cases} x_1 \ldots x_r y_1 \ldots y_s & \text{if } r+s \le m \\ \overset{r+s-pk-m}{a} z_1^m & \text{if } m+pk \le r+s < m+(p+1)k, \end{cases}$$

where z_1^m *is the unique element from* Q^m *such that*

$$x_1 \ldots x_r y_1 \ldots y_s = \overset{r+s-pk-m}{a} z_1 \ldots z_m$$

(see P.8.6.), i.e.

$$[\overset{(p+1)k+m-r-s}{a} x_1^r y_1^s] = [\overset{k}{a} z_1^m].\ \square$$

Using P.7.5 and C.8.7 we obtain:

Corollary 8.8. *Let* $(Q;[\ ])$ *be an* (n,m)*-group. Then* $\underline{Q}^{\vee}$ *is isomorphic to* $(Z_k \times Q^m;*)$ *where*

$$(i,x_1^m)*(j,y_1^m) = (i+j,z_1^m),$$

z_1^m *is defined as in C.8.7, and* $i+j$ *is in the group* $(Z_k;+)$.

Proof. C.8.7 implies that $\underline{Q}^{\vee}$ is isomorphic to $(Z_k' \times Q^m, *')$ where

$$Z_k' = \{m,m+1,\ldots,m+k-1\},\quad (i,x_1^m)*'(j,y_1^m) = (i\oplus j,z_1^m).$$

The conclusion follows from the fact that $(Z_k';\oplus)$ is isomorphic to $(Z_k;+)$. □

Now we consider the question about the existence of a covering semigroup $\underline{S}$ for a given v.v. group $\underline{Q}$, such that $\underline{S}$ is a group. We note that the universal covering semigroup $\underline{Q}^{\wedge}$ is not a group for $m \ge 2$, but the answer to the above question is positive.

Proposition 8.9. *The universal cancellative covering semigroup $\underline{Q}^{\sim}$ (defined in §7.) for a v.v. group $\underline{Q}$ is isomorphic to the universal envelope $\underline{Q}^{\vee}$.*

Proof. Suppose that $\underline{S}=(S;\bullet)$ is an arbitrary covering semigroup of $\underline{Q}$. Let $S^{\vee}=\{a_1\bullet\ldots\bullet a_t \mid t\geq m,\ a_{\nu}\in Q\}$. Then $S^{\vee}$ is an ideal of $\underline{S}$, and moreover, $\underline{S}^{\vee}$ is a homomorphic image of $\underline{Q}^{\vee}$. Hence $\underline{S}^{\vee}$ is a group. Now let $\underline{S}=\underline{Q}^{\sim}$ be the universal cancellative covering semigroup for $\underline{Q}$. Let $a\in Q$ be a fixed element. Then by C.8.5., $\underline{Q}^{\vee}=Q^m\cup aQ^m\cup\ldots\cup \overset{k-1}{a}Q^m$. Let $b\in Q$ and let $j\in\{1,\ldots,k-1\}$ such that k is a divisor of n+j-1. For $[\overset{n-1}{a}b]$ there exists a unique $b_1^m\in Q^m$ such that $[\overset{n-1}{a}b]=[\overset{n+j-1}{a}b_1^m]$. Hence

$$\overset{n-1}{a}b = \overset{n+j-1}{a}b_1\ldots b_m \text{ in } \underline{Q}^{\sim},\text{ i.e. } b=a^jb_1\ldots b_m.$$

If $\overset{i}{a}x_1^m=\overset{j}{a}y_1^m$ in $\underline{Q}^{\sim}$, then i=j and $x_1^m=y_1^m$ in $\underline{Q}^{\sim}$, and since $\underline{Q}$ is a group, $x_1^m=y_1^m$ in Q^m. Now, since Q generates $\underline{Q}^{\sim}$ and each element of Q is an image of an element of $\underline{Q}^{\vee}$, it follows that $\underline{Q}^{\sim}$ is a group. □

Corollary 8.10. *If $\underline{S}$ is a cancellative covering semigroup of a v.v. group, then $\underline{S}$ is a group.* □

Next we give the following:

Corollary 8.11. *If $\underline{Q}$ is a v.v. group, such that $|Q|=q<\infty$, then*

(a) $|Q_{m+i}| = |a^iQ^m|=q^m,\ i\in\{0,1,2,\ldots,k-1\}$;

(b) $|Q^{\vee}| = k\cdot q^m$; *and*

(c) $|Q^{\wedge}| = q+q^2+\ldots+q^{m-1}+k\cdot q^m$.

Proof. Follows from C.8.7. □

Corollary 8.12. *(Lagrange Theorem). Let $\underline{H}$ be an (n,m)-subgroupoid of an (n,m)-group $\underline{Q}$, and let $|Q|=q<\infty$. Then $|H|=p$ is a divisor of q.*

Proof. P.8.5 implies that $\underline{H}^{\vee}$ is a subgroup of $\underline{Q}^{\vee}$. Using C.8.11 and Lagrange Theorem for groups, we have that $k\cdot p^m$ is a divisor of $k\cdot q^m$, which implies that p is a divisor of q. □

Now, we give a description of the universal covering semigroup for a special kind of v.v. groups, by considering a few examples.

Example 8.13. Let $(Q;[\ ])$ be an (n,m)-group and $m \le \ell k < m+k$. Suppose that the subgroup $\underline{Q}_{\ell k}$ of $\underline{Q}^{\vee}$ has a neutral element equal to $\overset{\ell k}{e}$ for $e \in Q$. Then

$$[x_1^m \overset{\ell k}{e}\ ] = [\ \overset{\ell k}{e} x_1^m] = x_1^m$$

for each $x_1^m \in Q^m$. C.8.7 implies that the universal envelope $Q^{\vee}$ has the form

$$Q^{\vee} = Q^m \cup eQ^m \cup \ldots \cup \overset{k-1}{e} Q^m,$$

and moreover, the multiplication on $Q^{\wedge}$ is given by:

$$x_1^r \bullet y_1^s = \begin{cases} x_1^r y_1^s & \text{if } r+s \le m \\ \overset{t}{e}\left[\overset{\ell k-t}{e} x_1^r y_1^s\right] & \text{if } r+s \ge m, \end{cases}$$

where $0 \le t = r+s-m-pk < k$.

Example 8.14. Let $\underline{G}$ be a group with a neutral element $e \in G$. Then $(G;[\ ])$, where $[x_1^m y_1^m] = (x_1 y_1, \ldots, x_m y_m)$ is a $(2m,m)$-group; see E.2.7. 1). Such $(2m,m)$-groups are called „trivial" $(2m,m)$-groups. In this case $\ell k = m$, and the multiplication in

$$G^{\wedge} = G \cup \ldots \cup G^m \cup eG^m \cup \ldots \cup \overset{m-1}{e} G^m$$

is given by:

$$x_1^r \bullet y_1^s = \begin{cases} x_1^r y_1^s & \text{if } r+s \le m \\ \overset{r+s-m}{e}\left[\overset{2m-r-s}{e} x_1^r y_1^s\right] & \text{if } r+s > m, \end{cases}$$

$$e^r x_1^m \bullet y_1^s = \begin{cases} \overset{r+s}{e}\left[\overset{m-s}{e} x_1^m y_1^s\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{m-s}{e} x_1^m y_1^s\right] & \text{if } r+s \ge m, \end{cases}$$

$$x_1^r \bullet \overset{s}{e} y_1^m = \begin{cases} \overset{r+s}{e}\left[\overset{m-r-s}{e} x_1^r \overset{s}{e} y_1^m\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{2m-r-s}{e} x_1^r \overset{s}{e} y_1^m\right] & \text{if } r+s \ge m, \end{cases}$$

$$e^r x_1^m \bullet \overset{s}{e} y_1^m = \begin{cases} \overset{r+s}{e}\left[\overset{m-s}{e} x_1^m \overset{s}{e} y_1^m\right] & \text{if } r+s < m \\ \overset{r+s-m}{e}\left[\overset{2m-s}{e} x_1^m \overset{s}{e} y_1^m\right] & \text{if } r+s \ge m, \end{cases}$$

where $0 \le r,\ s < m$.

Example 8.15. Let (G;[ ]) be the (4,2)-group given in E.2.7. 3). Then $G^{\wedge}=G\cup G^2\cup 0\,G^2$, and $x\cdot y=xy$, $x\cdot(yz)=(xy)\cdot z=0[0xyz]$, $(xy)\cdot(zt)=[xyzt]$, $x\cdot 0yz=[x0yz]$, $0xy\cdot z=[0xyz]$, $xy\cdot 0zt=0[yxzt]$, $0xy\cdot zt=0[xyzt]$ and $0xy.0zt=[yxzt]$.

Example 8.16. Let (G;[ ]) be an (n+1,n)-group. Then $G^{\wedge}=G\cup G^2\cup\ldots\cup G^n$ and the multiplication on $G^{\wedge}$ is given by:

$$x_1^r\cdot y_1^s=\begin{cases}x_1^r y_1^s & \text{if } r+s\le n\\ [x_1^r y_1^s] & \text{if } r+s>n.\end{cases}$$

At the end of this section, we give a corresponding generalization of Hosszú-Gluskin's theorem for some types of v.v. groups.

Theorem 8.17. *Let (G;[ ]) be an (sm,m)-group. Then there exist: a binary group* $(G^m;\bullet)$, *an element* $c\in G^m$, *and an automorphism* θ *of this group, such that for each* $a_1,\ldots,a_s\in G^m$,

$$[a_1\ldots a_s]=a_1\bullet\theta(a_2)\bullet\ldots\bullet\theta^{s-2}(a_{s-1})\bullet\theta^{s-1}(a_s)\bullet c, \tag{8.1}$$

where

$$\theta(c)=c \text{ and } \theta^{s-1}(b)=c\bullet b\bullet c^{-1}, \text{ for } b\in G^m. \tag{8.2}$$

Proof. Since (G;[ ]) is an (sm,m)-group, $(G^m;g)$, where

$$g(x_1^m x_{m+1}^{2m}\ldots x_{(s-1)m+1}^{sm})=[x_1^{sm}], \tag{8.3}$$

is an (s,1)-group. Then, Hosszú-Gluskin's theorem implies that there exist: a binary group $(G^m;\bullet)$, an element $c\in G^m$ and an automorphism θ of $(G^m;\bullet)$, satisfying (8.2) and

$$g(u_1^s)=u_1\bullet\theta(u_2)\bullet\ldots\bullet\theta^{s-1}(u_s)\bullet c \tag{8.4}$$

for each $u_\nu\in G^m$. Now, (8.1) follows from (8.4) and (8.3). □

§9. SOME CLASSES OF VECTOR VALUED GROUPS

In this section we consider some classes of v.v. groups, touching upon existence problems for them also.

The investigation of (n,m)-groups pushes forward naturally the cases n=2m and n=m+1, because for m=1 one obtains the class of groups in both cases. Further on, we will assume that m is a given positive integer such that $m \geq 2$.

Recall that any (2m,m)-group defined as in E.2.7. 1), is said to be trivial (see E.8.14). By [5], there exist also non-trivial (2m,m)-groups.

Note that if $\underline{G}=(G;[\ ])$ is a (2m,m)-semigroup and if we define a (binary) operation $\bullet$ on G^m by:

$$x_1^m \bullet y_1^m = [x_1^m y_1^m],$$

then we obtain a semigroup $(G^m;\bullet)$, called associated semigroup to $\underline{G}$.

Proposition 9.1. *A (2m,m)-semigroup $\underline{G}=(G;[\ ])$ is a (2m,m)-group iff its associated semigroup $(G^m;\bullet)$ is a group.*

In that case, the identity of the group $(G^m;\bullet)$ has a form $\overset{m}{e}$, where $e \in G$, and moreover the following equality in $\underline{G}$

$$[x_1^i \overset{m}{e} x_{i+1}^m] = x_1^m$$

holds for every $i \in \{0,1,\dots,m\}$ and $x_1^m \in G^m$.

(We say that e is the identity of $\underline{G}$ and that $(G^m;\bullet)$ is the associated group to $\underline{G}$.)

Proof. The first part of the proposition follows easily as a consequence of C.8.4.

Let $\underline{G}=(G;[\ ])$ be a (2m,m)-group and let e_1^m be the identity of the associated group. If $x_1^m \in G^m$ and $0 \leq i \leq m$, then:

$$\begin{aligned}[x_1^i e_1^m x_{i+1}^m] &= e_1^m \bullet [x_1^i e_1^m x_{i+1}^m] = \\ &= [e_1^i [e_{i+1}^m x_1^i e_1^m] x_{i+1}^m] \\ &= [e_1^m x_1^m] = x_1^m;\end{aligned}$$

therefore

$$[e_1 e_1^m e_2^m] = e_1^m = [e_1^m e_1^m],$$

which implies $e_1=e_2=\ldots=e_m$ (=e). □

As a direct consequence of P.9.1 we obtain:

Corollary 9.2. *Let $\underline{G}=(G;[\ ])$ be a $(2m,m)$-group with the identity e and let $H\subseteq G$. Then H is a subgroup of $\underline{G}$ iff H^m is a subgroup of the associated group, and in that case $e\in H$.* □

(Here, the notion "a subgroup of a v.v. group" means "a v.v. subgroup of a v.v. group".) □

The existence of the identity e enables us to introduce the notion of a normal subgroup of a (2m,m)-group as a kernel of a homomorphism.

Proposition 9.3. *If $\xi: \underline{G} \to \underline{G}'$ is a homomorphism from the $(2m,m)$-group $\underline{G}=(G;[\ ])$ to the $(2m,m)$-group $\underline{G}'=(G';[\ ]')$, then*

$$H = Ker\xi = \{x\in G \mid \xi(x) = e'\} = \xi^{-1}(e')$$

is a subgroup of $\underline{G}$ with the following properties:

$$[x_1^{i-1} H^m x_i^m] = [x_1^m H^m], \tag{9.1}$$

$$[x_1^m H^m] = [y_1^m H^m] \iff (\forall j\in N_m)[\overset{m}{x}_j H^m] = [\overset{m}{y}_j H^m] \tag{9.2}$$

for every $x_\nu, y_\nu \in G$, $i\in N_m$.

(Here, e′ is the identity of $\underline{G}'$, and $[x_1^{i-1}H^m x_i^m]$ has the usual meaning, i.e.

$$[x_1^{i-1} H^m x_i^m] = \{[x_1^{i-1} h_1^m x_i^m] \mid h_1^m \in H^m\}.)$$

Proof. Denote by $\bar\xi$ the homomorphism from $(G^m;\bullet)$ to $(G'^m;\bullet)$ induced by ξ, i.e.

$$\xi(x_1^m) = y_1^m \iff (\forall i\in N_m)\ y_i = \xi(x_i).$$

Then

$$Ker\bar\xi = H^m,$$

which implies that H^m is a normal subgroup of the group $(G^m;\bullet)$ and by P.9.2, H is a subgroup of $\underline{G}$; now, by the fact that H^m is a normal subgroup of G^m, one obtains that (9.1) is true.

Suppose that $x_1^m, y_1^m \in G^m$ are such that $[x_1^m H^m]=[y_1^m H^m]$. Then $\bar\xi(x_1^m)=\bar\xi(y_1^m)$, i.e. $\xi(x_j)=\xi(y_j)$ for every $j\in N_m$, which implies that