$\overline{\xi}(\overset{m}{x_j}) = \overline{\xi}(\overset{m}{y_j})$ for every $j\in N_m$. This implies that (9.2) is also true. □

Now suppose that H is a subgroup of the (2m,m)-group $\underline{G}=(G;[\ ])$ with the properties (9.1) and (9.2). Then H^m is a normal subgroup of the associated group $(G^m;\bullet)$. Consider the quotient group G^m/H^m and the subset G/H of G^m/H^m defined by:

$$G/H = \{\overset{m}{x}H^m \mid x\in G\}. \tag{9.3}$$

To shorten the notation we will write $\overline{x}$ instead of $\overset{m}{x}H^m$. Therefore:

$$\overline{x} = \overline{y} \ \text{ iff } \ \overset{m}{x}H^m = \overset{m}{y}H^m.$$

Consider the canonical mapping nat_H: $x \mapsto \overline{x}$ from G onto G/H. We will show that:

Proposition 9.4. *There exists a unique (n,m)-operation* $[\ ]'$ *on G/H such that* nat_H *is a homomorphism from* $(G;[\ ])$ *to* $(G/H;[\ ]')$.

Proof. By the fact that nat_H is a surjective mapping from G onto G/H it follows that there exists at most one operation $[\ ]'$ with the demanded properties.

To prove that such an operation exists it is necessary only to see that (9.2) implies directly that the following implication is true:

$$(\forall i\in N_{2m})\, \overline{x}_i=\overline{y}_i \text{ and } [x_1^{2m}]=u_1^m,\ [y_1^{2m}]=v_1^m \Longrightarrow (\forall j\in N_m)\ \overline{u}_j=\overline{v}_j. \tag{9.4}$$

By (9.4) one obtains that a (2m,m)-operation $[\ ]'$ on G/H is well-defined by:

$$[x_1^{2m}] = u_1^m \Longrightarrow [\overline{x}_1^{2m}]' = \overline{u}_1^m, \tag{9.5}$$

and also that nat_H: $(G;[\ ]) \longrightarrow (G/H;[\ ]')$ is a homomorphism. □

By P.9.4 one obtains the following:

Proposition 9.5. *If* $\underline{G}=(G;[\ ])$ *is a (2m,m)-group with the identity e, H is a subgroup of* $\underline{G}$ *with the properties (9.1) and (9.2) and if* $(G/H;[\ ]')=\underline{G}/H$ *is defined as above, then:*

(i) $\underline{G}/H$ *is a group with the identity* $\overline{e}$,

(ii) nat_H *is a homomorphism, such that* $Ker\,nat_H=H$. □

A subgroup of a (2m,m)-group G is said to be normal in G iff (9.1) and (9.2) hold. Then it is natural to say that G/H is a quotient group of G with respect to H.

As a summary of the above results, we obtain the following:

Theorem 9.6. *Let H be a subgroup of a (2m,m)-group G. Then H is normal in G iff there exists a homomorphism* ξ: G $\to$ G′ *such that* $H = Ker\xi$. □

By the definition of G/H directly one obtains also the following result:

Proposition 9.7. *If H is a normal subgroup in a (2m,m)-group G, then* H^m *is a normal subgroup in* $(G^m;\bullet)$ *and the groups* $(G^m/H^m;\bullet)$, $((G/H)^m;\bullet)$ *are isomorphic.*

In other words, $(G^m/H^m;\bullet)$ *is the associated group of G/H.* □

The following proposition for the trivial (2m,m)-groups is true:

Proposition 9.8. *Let G be a trivial (2m,m)-group induced by a group* $(G;\bullet)$. *Then:*

(i) The usual m-th Cartesian power of $(G;\bullet)$ *is the associated group of G.*

(ii) H is a normal subgroup of G iff H is a normal subgroup of $(G;\bullet)$. □

The trivial (2m,m)-groups are not of a special interest. However, it is desirable to have corresponding abstract descriptions of this class of (2m,m)-groups, which are given by the following proposition (proved in [5]):

Proposition 9.9. *If G is a (2m,m)-group with the identity e, then the following conditions are equivalent:*

(i) G is trivial.

(ii) There exist binary operations $*_1, *_2, \dots, *_m$ *on G such that*

$$[x_1^{2m}] = y_1^m \iff (\forall j \in N_m)\ y_j = x_j *_j x_{m+j}$$

for every x_ν, y_λ.

(iii) For every $x,y \in G$, $x_1^m \in G^m$ *and integers* r,s, $i \in N_m$, $r \neq s$ *the following equalities hold:*

$$[\overset{m-i+1}{e}\, x_1^m\, \overset{i-1}{e}] = x_i^m x_1^{i-1},$$

$$[\overset{r-1}{e}\, x\, \overset{m-1}{e}\, y\, \overset{m-r}{e}]_s = e, \text{ for } s \neq r. \ \square$$

(We note also that there is no problem to formulate and prove corresponding theorems for isomorphisms for the class of (2m,m)-groups.)

Now we will consider the class of (m+1,m)-groups, where $m \geq 2$.

By P.5.5, an (m+1,m)-semigroup $\underline{G}=(G;[\ ])$ induces an (m+k,m)-semigroup for every $k \geq 2$, and thus a corresponding (2m,m)-semigroup. Moreover, $\underline{G}$ is an (m+1,m)-group iff it is a (2m,m)-group. This implies that to any (m+1,m)-group $\underline{G}=(G;[\ ])$ it is possible to join a corresponding associated group $(G^m;\bullet)$; in this case, it is the group $G^{\vee}$.

We will show below that the (m+1,m)-groups can be defined (like groups) by one (m+1,m)-operation, one unary and one nullary operation.

<u>Theorem 9.10</u>. *If* $\underline{G}=(G;[\ ])$ *is an (m+1,m)-semigroup, then the following statements are equivalent:*

(i) $\underline{G}$ *is an (m+1,m)-group.*

(ii) There exists an element $e \in G$ *and a transformation* $h: G \to G$ *on G such that:*

$$[x_1^i\, \overset{m}{e}\, x_{i+1}^m] = x_1^m \qquad (9.6)$$

$$[x\, \overset{m}{e}] = [\overset{m}{e}\, x] \qquad (9.7)$$

$$[x h(x) h^2(x) \dots h^m(x)] = \overset{m}{e} \qquad (9.8)$$

$$h^{m+1} = 1_G \qquad (9.9)$$

for every $x \in G$, $x_1^m \in G^m$, $i \in \{0,1,\dots,m\}$.

<u>Proof</u>. Let $\underline{G}$ be an (m+1,m)-group. Then $\underline{G}$ induces a corresponding (2m,m)-group. So, if e is the identity of that (2m,m)-

group, then by P.9.1 one obtains that (9.6) holds, and clearly (9.7) is a consequence of (9.6), and the cancellativity in G.

If x is a given element of G, then there exists a unique $y_1^m \in G^m$ such that $[xy_1^m] = \overset{m}{e}$. Let us put $y_1 = h(x)$. Using the fact that $\overset{m}{e}$ is the identity of $(G^m;\bullet)$, and using also (9.6) and (9.7), we obtain

$$\overset{m}{e} = [xy_1^m] = [xy_1^m xy_1^m],$$

by which:

$$y_1^m = [y_1^m xy_1^m] = [y_1^m x]\bullet y_1^m,$$

i.e. $[y_1^m x] = \overset{m}{e}$. By the last equality we have $y_2 = h(y_1) = h^2(x)$. Similarly one obtains that $y_i = h^i(x)$ for every i, and also that $h^{m+1}(x) = x$.

Therefore, (i) $\Longrightarrow$ (ii).

Now suppose that the condition (ii) holds.

By (9.6) it follows that $\overset{m}{e}$ is an identity of the semigroup $(G^m;\bullet)$. Also, by (9.8) one obtains that for every $x_1^m \in G^m$:

$$[x_1^m h(x_m)h^2(x_m)\dots h^m(x_m)h(x_{m-1})\dots h^m(x_{m-1})\dots h(x_1)\dots h^m(x_1)] = e^m,$$

and this implies that $(G^m;\bullet)$ is a group. □

(Note, that (9.9) is a consequence of (9.6) to (9.8)).

If G is an (m+1,m)-group, and e is the identity of the corresponding (2m,m)-group, then we say also that e is the identity of G.

Proposition 9.11. *Let $G=(G;[\ ])$ be an $(m+1,m)$-group and e be the identity of G. If for every $x_1^m \in G^m$ and some $i \in \{0,\dots,m\}$*

$$[x_1^i e x_{i+1}^m] = x_1^m,$$

then $|G|=1$.

Proof. The cancellativity of G and the fact that e is the identity of G, implies that $[ey_1^m] = y_1^m$ for each $y_1^m \in G$. Now, $[\overset{m+1}{e}] = \overset{m}{e}$ implies that

$$\overset{m-1}{e}x = [\overset{m}{e}\,e\,\overset{m-1}{x}] = [\overset{m+1}{e}\,\overset{m-1}{x}] = [\overset{m}{e}\,\overset{m-1}{x}] = [\overset{m-1}{x}\,\overset{m}{e}] =$$
$$= [\overset{m-1}{x}\,\overset{m+1}{e}] = \overset{m-1}{x}e,$$

for each $x \in G$, i.e. $|G|=1$. □

Proposition 9.12. *If G is a nonempty set, then the following statements are equivalent:*

(i) There exists an (m+1,m)-group $\underline{G}=(G;[\ ])$.

(ii) There exists a group $(G^m;\bullet)$ *and a mapping* $\xi: x \mapsto \overline{x}$ *from G into* G^m, *such that*

$$x_1^m = \overline{x}_1 \bullet \overline{x}_2 \bullet \ldots \bullet \overline{x}_m, \qquad (9.10)$$

for every $x_1^m \in G^m$.

(Clearly, every mapping $\xi: G \to G^m$ *with the property (9.10) is injective.)*

Proof. Let $\underline{G}=(G;[\ ])$ be an (m+1,m)-group and let $(G^m;\bullet)$ be the associated group of $\underline{G}$. If we set $\overline{x}=[x\overset{m}{e}]$, where e is the identity of $\underline{G}$, then we obtain that (9.10) holds. Thus: (i) $\Longrightarrow$ (ii).

Conversely, suppose that $(G^m;\bullet)$ is a group and $\xi: x \mapsto \overline{x}$ a mapping from G into G^m such that (9.10) holds. If we set

$$[x_0^m] = \overline{x}_0 \bullet \overline{x}_1 \bullet \ldots \bullet \overline{x}_m,$$

then we obtain an (m+1,m)-group for which the given group $(G^m;\bullet)$ is the associated group. □

An (m+1,m)-group $\underline{G}=(G;[\ ])$ is said to be trivial iff $|G|=1$. The question for the existence of nontrivial (m+1,m)-groups comes naturally.

Theorem 9.13. *If G is an infinite set, then there exists an (m+1,m)-group* $\underline{G}=(G;[\ ])$.

Proof. This proposition is a consequence of the main result of the paper [14], by which if B is a nonempty set, then a free (m+1,m)-group $\underline{F_m}(B)=(F_m(B);[\ ])$ with a basis B has the cardinality

$$|F_m(B)| = \max\{|B|, \aleph_0\}. \ \square$$

(We note that in the mentioned paper [14] a satisfactory combinatorial description of a free (m+1,m)-group $\underline{F_m}(B)$ is given.)

It remains the case when G is a finite set.

Theorem 9.14. *If* $m \geq 2$, *then there is no nontrivial finite (m+1,m)-group.*

This proposition is a direct consequence of the following result, which was proved by Prof. John Thompson (and he kindly provided us with that proof):

Proposition 9.15. *If $(H;\cdot)$ is a finite group such that there exists a subset S of H with the properties:*

$$S\cdot S = \{x\cdot y \mid x,y\in S\} = H \text{ and } |S|^2=|H|, \tag{9.11}$$

then $|H|=1$. □

Professor Thomson's proof of P.9.15 is via the gorup algebra $\mathbb{C}[H]$ over the field of complex numbers, Wedderburn's theorem for a decomposition of $\mathbb{C}[H]$, and characters of finite groups.[1)]

It is possible to generalize P.9.15. to:

Proposition 9.16. *If $(H;\cdot)$ is a finite group and $S\subseteq H$ such that*

$$\underbrace{S\cdot\ldots\cdot S}_{m} = \{x_1\cdot\ldots\cdot x_m \mid x_\nu\in S\} = H \text{ and } |S|^m = |H|,$$

where $m\geq 2$, then $|H|=1$. □

The conclusion in T.9.14. comes easily as a consequence of P.9.16.

Indeed, let $\underline{G}=(G;[\ ])$ be a finite $(m+1,m)$-group. By P.9.12., if $(G^m;\bullet)$ is the associated group of G and if $S=\{[\overset{m}{xe}] \mid x\in G\}$, where e is the identity of $\underline{G}$, then

$$|S| = |G|,\ |S|^m = |G^m| = |G|^m \text{ and } \underbrace{S\cdot\ldots\cdot S}_{m} = G^m.$$

By P.9.16., if follows that $|G^m|=1$, i.e. $|G|=1$.

This completes the proof of T.9.14.

We note also that, as a consequence of T.9.13, one obtains the following generalization:

Corollary 9.17. *If G is an infinite set and n,m,k are positive integers such that $n-m=k\geq 1$, then there exists an (n,m)-group $\underline{G}=(G;[\ ])$.*

Proof. By T.9.13., there exists an $(m+1,m)$-group $\underline{G}=(G;[\ ])$ which by T.5.8. induces a corresponding (n,m)-group. □

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1) *See p. 72.*

Next we are going to describe a method for production of examples of $(2m+s,m)$-groups, $s \geq 1$.

Let $(G;\cdot)$ be a group with a neutral element e. Suppose that there exists a homomorphism $*$ from the product group $(G^{m+s};\cdot)$ into the product group $(G^m;\cdot)$ such that:

(i) $(\overset{s}{e}x_1^m)^* = x_1^m$, (ii) $x_1^{m+s} \in \ker * \Longrightarrow x_2^{m+s}x_1 \in \ker *$.

We extend $*$ to a homomorphism (denoted again by $*$) $*$: $G^{t(m+s)} \longrightarrow G^m$ by:

$$(x_1^{t(m+s)})^* = (x_1^{m+s})^* \cdot (x_{m+s+1}^{2(m+s)})^* \dots (x_{(t-1)(m+s)+1}^{t(m+s)})^*.$$

Next, we extend $*: \bigcup_{t\geq 1} G^{t(m+s)} \longrightarrow G^m$ to $*: \bigcup_{\lambda\geq 0} G^{m+\lambda} = \overline{G} \longrightarrow G^m$ by:

$$(x_1^{t(m+s)+p})^* = (\overset{m+s-p}{e}x_1^p)^* \cdot (x_{p+1}^{t(m+s)+p})^*,$$

where $t \geq 0$, $0 \leq p < m+s$.

The following theorem is proved in [10]:

Theorem 9.18. *Let* $[\ ]: G^{2m+s} \longrightarrow G^m$ *be defined by:*

$$[x_1^{2m+s}] = (\overset{s}{e}x_1^m)^*(x_{m+1}^{m+s})^* = (x_1^{2m+s})^*.$$

Then $(G;[\ ])$ *is a* $(2m+s,m)$*-group.* □

The v.v. group in E.2.7. 2) is obtained by the above procedure. In this example, the homomorphism $*: G^{m+1} \longrightarrow G^m$ is given by $(xx_1^m)^* = (x_1-x,\dots,x_m-x)$.

Let us examine the universal covering semigroup for $(G;[\ ])$ as above.

From the assumptions about $(G;\cdot)$ and $*$, it follows that the neutral element in the group $(G^m;\cdot)$, where $x_1^m \cdot y_1^m = [x_1^m \overset{s}{e} y_1^m]$ (see C.8.4.), is of the form $\overset{m}{e}$. So, by E.8.13.,

$$G^\wedge = G \cup \dots \cup G^m \cup eG^m \cup \dots \cup \overset{m+s-1}{e} G^m,$$

and

$$x_1^r \cdot y_1^t = \begin{cases} x_1^r y_1^t & \text{if } r+t \leq m \\ \overset{p}{e}(\overset{m+s-p}{e} x_1^r y_1^t)^* & \text{if } r+t > m, \end{cases}$$

where $0 \leq p = r+t-m-q(m+s) < m+s$.

We will prove the following:

Proposition 9.19. *If* $\underline{G}=(G;[\ ])$ *is a* $(5,3)$*-group and if* $1<|G|<+\infty$*, then* $|G|$ *is an even number.*

Proof. Let $(G;[\ ])$ be a $(5,3)$-group and for $a\in G$, let $f(a)g(a)h(a)$ be the neutral element in the group $(G^3;*)$ (see C.8.4.). It is easy to check out that $f(f(a))=g(a)$, $f(g(a))=h(a)$ and $f(h(a))=a$, i.e. $g=f^2$, $h=f^3$ and $f^4=1_G$. Moreover, there are only three possibilities:

$$a = f(a) = g(a) = h(a);\ \text{or}\ a = g(a) \neq f(a) = h(a)$$

$$\text{or}\ |\{a,f(a),g(a),h(a)\}| = 4.$$

If for some $a\in G$, $a=f(a)=g(a)=h(a)$, then $[\overset{5}{a}]*[\overset{5}{a}]=[\overset{11}{a}]=[\overset{7}{a}]=\overset{3}{a}$. If $[\overset{5}{a}]=\overset{3}{a}$, then

$$[x_1^3\overset{4}{a}] = x_1^3*\overset{3}{a} = x_1^3*[\overset{5}{a}] = [x_1^3\overset{6}{a}]$$

implies that $x_1^3=[x_1^3\overset{2}{a}]$. Symmetrically, $[\overset{2}{a}x_1^3]=x_1^3$. Now, for $x_1^3=axa$, $[ax\overset{3}{a}]=[\overset{3}{a}xa]$ implies that $x=a$ for each $x\in G$, i.e. $|G|=1$. Hence, if $1<|G|<\infty$, then $[\overset{5}{a}]$ is an element of order 2 in $(G^3;*)$, which implies that 2 is a divisor of $|G|$.

If $|G|<\infty$ and for some $a\in G$, $a\neq f(a)$, then there is a partition of G into (disjoint) subsets with 2 or 4 elements, which implies that 2 is a divisor of $|G|$, i.e. $|G|$ is an even number. □

• The above proof shows that finite sets with an odd number of elements (bigger than 1) do not admit a (5,3)-group structure.

We note that it is enough to consider the existence questions only for (n,m)-groups where n and m are relatively prime, as the following propositions states.

Proposition 9.20. *If there exists an* (n,m)*-group* $\underline{G}=(G;[\ ])$ *and if* $t\geq 1$*, then there exists an* (nt,mt)*-group* $\underline{G}'=(G;[\ ]')$ *too.*

Proof. If $\underline{G}=(G;[\ ])$ is an (n,m)-group, then $\underline{G}'=(G;[\ ]')$, where $[\ ]': G^{nt}\longrightarrow G^{mt}$ is defined by

$$[x_1^t y_1^t \dots z_1^t]'_{rt+i} = [x_i y_i \dots z_i]_{r+1},$$

$0\leq r\leq m-1$, $1\leq i\leq t$, is an (nt,mt)-group. □

Proposition 9.21. *If there exists a (tn,tm)-group $(G;[\ ])$, then there exists an (n,m)-group $(G^t;[\ ]')$ too.*

Proof. If $(G;[\ ])$ is a (tn,tm)-group, then $(G^t;[\ ]')$, where $[\ ]': (G^t)^n \longrightarrow (G^t)^m$ is defined by

$$[x_1^t y_1^t \ldots z_1^t]_i' = [x_1^t y_1^t \ldots z_1^t]_i$$

is an (n,m)-group. □

§10. NOTES AND COMMENTS

The notion of vector valued operation is treated for the first time, in our knowledge, in [22] (aside from the use of vector valued functions in the analysis and its applications). Namely, the paper [22] is concerned with the problem of characterization for some algebras of partial v.v. operations. Similar questions are considered in [35], where a class of algebras with countably many partial binary operations is examined and it is shown that every such algebra is a subalgebra of an algebra of v.v. operations. Different questions connected to the composition algebras, especially their completeness, are treated in the extentive paper [17], which appeared several years ago. (Here, if Op(A) is the set of v.v. operations on a set A, $\bullet$ is the usual composition and $\times$ the direct product of mappings, then $\mathcal{A} = (\mathrm{Op}(A);\bullet,\times,1_A)$ is called a composition algebra on A. If $F \subseteq \mathrm{Op}(A)$, then $(A;F)$ is a v.v. algebra. A special attention is given to the case when F is a generating set of the composition algebra $\mathcal{A}$.)

The definitions of v.v. groupoids and v.v. semigroups for the first time are given in [34]. The notion of weak v.v. quasigroups is given in [33], under the name "(n,m)-quasigroups", while the notion of v.v. quasigroups, defined in §2, for the first time is introduced in [2]. (Also, the notion of a partial v.v. quasigroup is given there, but in a more general content.) Several interpretations of v.v. quasigroups are given in [2], where the most interesting is the geometric one. A review of the known results about v.v. quasigroups is given in [27] (in this book). Therefore we will mention briefly here only that v.v. groupoids, v.v. quasigroups and their geometric interpretations are treated in the papers: [2], [20], [21], [23], [24], [25], [26], [29], [30], [31], [32], [38].

The paper [4] is entirely concerned with the v.v. semigroups. The notion of v.v. group for the first time is introduced in that paper. It is proved there Post Theorem for v.v. semigroups, and also several other results connected to this theorem. The question about the existence of nontrivial v.v. groups (i.e. v.v. groups with more than one element) is also considered in [4]. Examples of nontrivial (sm,m)-groups are given too, by using ordinary groups and a theorem (that, for each m, the free (m+1,m)-group is nontrivial) is stated, which implies that for each $n > m$, nontrivial (n,m)-groups do exist. Although the above theorem is true, which follows from the main result in [14], its proof given in [4] is not complete, i.e. we do not know a direct proof that the identity $x = y$ is not a consequence of the axioms for (m+1,m)-groups. At the end of [4], a list of problems is given, some of which are solved. For example, in [11], a satisfactory combinatorial description of free v.v. semigroups is given, and this made possible to prove more general v.v. variants of Post and Cohn-Rebane Theorems, which is done in [6] and [7].

The result about the non-existence of nontrivial finite (m+1,m)-groups came successively. Namely, at the beginning, in [10], some non-existence conditions for (m+1,m)-groups were obtained, which implied, for example, that the number of elements of a finite (3,2)-group had to be divisible by 6, and later in [12] an elementary proof that (3,2)-groups with 6 and 12 elements do not exist was given. At this moment, we do not have a general answer to the question: when do nontrivial finite (n,m)-groups exist, if m is not a divisor of n?

Besides this, the question about the existence of nontrivial v.v. groups in some classes of v.v. groups is of special interest. we note that [10] contains some answers to this question. It is known that there are no nontrivial commutative (n,m)-groups for $m \geq 2$ [4], but for some classes of v.v. groups it is usefull to weaken the commutativity condition uch that a large class of v.v. groups is obtained. Thus, in [15], $(2m,m)$-groups whose associated group is commutative are examined and examples of such "nontrivial" groups are obtained. Moreover, it is shown in [15] that finite $(4,2)$- and $(6,3)$-groups whose associated group is cyclic must be "trivial". The interesting question about the existence of finite "nontrivial" $(2m,m)$-groups with a prime number of elements, which is analogous to the fact that a finite group with prime number of elements must be cyclic, is still open.

Similarly as for the semigroups and groups, it is of interest to consider and examine continuous v.v. semigroups and v.v. groups. It is shown that continuous $(3,2)$-groups over the real numbers do not exist [28]. The question about some continuous v.v. groups is treated in [16].

We noted in the introduction that the presentation of semigroups is of use for the examination of v.v. groups, since to each v.v. groupoid $\underline{Q}$ we associate its universal semigroup $\underline{Q}^{\wedge}$ via a corresponding presentation. Here we have the similar algorithmical problems as for the usual semigroups and groups [40], [42], [43]. These problems are not examined in details in this work but we note that a sufficiently effective reduction in the semigroup presentation (see page 17) gives an algorithm for solving the word problem in this presentation. Thus, if $\underline{A} = (A;F)$ is a "sufficiently effective" partial v.v. algebra, then the semigroup that contains $\underline{A}$, in the proof of Cohn-Rebane Theorem (page 19) is also "sufficiently effective". We note that the construction of the free (n,m)-semigroups (considered as poly-(n,m)-semigroups) in §6 is also effective. It was mentioned in §6 (by E.6.4) that the given reduction in the free (n,m)-groupoids $\overline{B}$ was not good, but it is possible to alter the definition of reduced elements and to obtain an effective construction of free (n,m)-semigroups via free (n,m)-groupoids.

In [9], presentations of v.v. semigroups are examined; descriptions of free v.v. semigroups in some varieties (for example, the variety of commutative v.v. semigroups) are obtained; and corresponding Post Theorems for these varieties are proven.

The general associative law is characterized by the fact that $|\mathcal{P}_\alpha(f)| = 1$, for each α (see page 29). This suggests a generalization of v.v. semigroups to a more general class of v.v. algebras, namely v.v. associatives. If F is a set of v.v. operations on $\underline{Q}$, then it is possible, in a similar manner, to define sets $\mathcal{P}_{n,m}(F)$ of polynomial operations on Q. An algebra (Q;F) is called a <u>v.v. associative</u> if $|\mathcal{P}_{n,m}(F)| \le 1$ for each $n,m \ge 1$. A special kind of v.v. associatives is considered in [1].

A part of the results, stated and proved in this work, are published earlier, mainly in the following papers: [34], [4], [5], [10], [11], [12], [13], [7], [8], [1]. In many cases, new simpler proofs are given here. (For example, GAL is supposed in all of these papers, but an explicit proof is given for the first time in this work.) In the main text, we usually do not quote the source where a corresponding result is given for the first time. On the other hand, many results are stated in this work for the first time. They are: 1.1, 1.2; 2.1, 2.2, 2.4, 2.12, 2.13; 3.3, 3.7; 4.12, 4.13; 5.1-5.4, 5.7-5.12; 6.4, 6.8-6.12; 7.5, 7.6, 7.9-7.14; 8.2, 8.4-8.16; 9.10, 9.11, 9.12, 9.14, 9.16, 9.19, 9.20, 9.21.

Finally, we make a note of a terminological (and historical) nature with respect to the terms "Post Theorem" and "Cohn-Rebane Theorem". First, in his extentive work [41], Post proved the following result (stated here in our terminology): "Every (n,1)-group is an (n,1)-subsemigroup of a group". Later on, in several papers (see, for ex., [38]) there is a proof of the Post Theorem for (n,1)-semigroups and also for some classes of (n,1)-semigroups. Any result of this kind we call "Post Theorem".

In the beginning of the sixties, the Soviet matematician Rebane proved a result which (again in terms of this work) could be stated in the following way: "If F is a set of finitary operations on a set Q and if Λ is the set of semigroup defining relations, defined by:

$$\Lambda = \{(b, fa_1^n) \mid b = f(a_1^n),\ a_\nu, b \in Q\},$$

then the presentation $\langle Q \cup F; \Lambda \rangle$ is pure". In the meantime, a similar result appeared in the monograph [36]. Results of this kind are known usually as "Cohn-Rebane Theorems" ([46]). Other vector valued variants of Post Theorems and Cohn-Rebane Theorems are given in [9].

We give here *Prof.* John Thomson's proof of P.9.15, in the same form as it was provided to us.

"Let A be the group algebra H over the field of complex numbers. Then by Wedderburn's theorem, $A=A_1\oplus\ldots\oplus A_d$, where each A_i is a full matrix algebra of f_i by f_i matrices over C. For each subset T of H, set $[T] = \sum_{t\in T} t$ (this is an element of A). The hypothesis imply that $[S]\cdot[S]=\sum_{g\in H} g=N$, say. Each element a in A is uniquely $a=a_1+\ldots+a_d$, $a_i\in A_i$, and the maps $\chi_i: H \to C$, $g \mapsto \mathrm{trace}_{A_i} g_i$, $i=1,\ldots,d$, are the irreducible characters of H. Choose notation so that $\chi_1=1_H$, the trivial character which assumes the value 1 at each group element. Then it is a basic and easy result that $N_i=0$ if $i>1$. Then $[S]=[S]_1+\ldots+[S]_d$, and if $i>1$, then $[S]_i^2=0$, so that $[S]_i$ is a nilpotent f_i by f_i matrix for each $i>1$, whence $\chi_i([S])=0$, $i>1$. On the other hand, $\chi_i([S])=\sum_{s\in S}\chi_i(s)$. Hence

$$\sum_{i=1}^{d}\chi_i(1)\cdot\chi_i([S])=\chi_1(1)\cdot\chi_1([S])=\mathrm{card}S=\mathrm{card}H\cdot\delta_{1,S},$$

where $\delta_{1,S}=1$ if $1\in S$, 0 otherwise. This is so since $\sum_i \chi_i(1)\chi_i$ is the character of the regular representation of H, so vanishes at each element of H - {1} and has value cardH at the identity element of H. So $n=n^2\cdot\delta_{1,S}$, whence n=1."

B I B L I O G R A P H Y

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I N D E X

SYMBOLS AND NOTATIONS