INTRODUCTION TO COMBINATORIAL THEORY OF VECTOR VALUED SEMIGROUPS

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§0. INTRODUCTION

The aim of this paper is to develop a combinatorial theory of vector valued (v.v.) semigroups via their presentations, and to give a satisfactory description of free objects in some varieties of v.v. semigroups. V.v. variants of Post and Cohn-Rebane theorems are obtained as applications of more general results.

In order this paper to be self-contained, we begin with a few necessary definitions, notations and results used in the main text, although they could be found in [2] (this volume). The introduction concludes with a short description of the paper.

Let Q be a nonempty set and r a positive integer. The r-th cartesian power of the set Q, denoted by Q^r, consists of the r-tuples $(a_1,\dots,a_r)$, where $a\in Q$. We will use the following notations: a_1^r, $a_1\dots a_r$, $\underline{a}$ for $(a_1,\dots,a_r)$; a^r for a_1^r when $a_1=\dots=a_r=a$; x_i^j for $x_i\dots x_j$ if $i\le j$, and x_i^j for the empty sequence if $i>j$. The set of all nonempty finite sequences of elements from Q will be denoted by Q^+, and Q^+ together with the empty sequence (usually denoted by 1) will be denoted by Q^*. In fact, Q^+ is a free semigroup, and Q^* is a free monoid, with a basis Q, where the operation is the usual concatenation of sequences. (Sometimes, the elements of Q^* will be called words.)

By N we denote the set of all nonnegative integers, i.e. $N=\{0,1,2,\dots\}$, and by N_r we denote the set $\{1,2,\dots,r\}$. The mapping $d:Q^* \to N$ (called a dimension) is defined by: $d(1)=0$, $d(a_1^r)=r$, $a_\nu \in Q$.

From now on, $n,m,n-m=k$ will be positive integers. Let $Q\neq\emptyset$. A mapping $f:Q^n \to Q^m$ is called an (n,m)-operation (shortly a v.v. operation), and $\underline{Q}=(Q;f)$ is called an (n,m)-groupoid. If, in addition, the equation

$$f(f(a_1^n)a_{n+1}^{n+k}) = f(a_1^j f(a_{j+1}^{j+n})a_{j+n+1}^{n+k}) \qquad (0.1)$$

is satisfied for each $a_\nu \in Q$, $j\in N_k$, then $\underline{Q}$ is an (n,m)-semigroup (shortly a v.v. semigroup).

Denote the set $\bigcup_{s\geq 1} Q^{m+sk}$ by $Q^{(n,m)}$[1]. A mapping $g:Q^{(n,m)} \to Q^m$ is said to be a poly-(n,m)-operation, and $\underline{Q}=(Q;g)$ a poly-(n,m)-groupoid. Moreover, if the equation

$$g(a_1^j g(b_1^{m+rk})a_{j+1}^{sk}) = g(a_1^j b_1^{m+rk} a_{j+1}^{sk}) \qquad (0.2)$$

is satisfied for each $a_\nu, b_\lambda \in Q$, $r,s\geq 1$, $j\in N_{sk}\cup\{0\}$, then $\underline{Q}$ is said to be a poly-(n,m)-semigroup.

To each (n,m)-groupoid $\underline{Q}=(Q;f)$ one can associate a poly-(n,m)-groupoid $\underline{Q}^{\#}=(Q;f^{\#})$ where $f^{\#}$ is defined by induction in the following manner:

$$f^{\#}(a_1^n) = f(a_1^n)\ ,\ f^{\#}(a_1^{m+(s+1)k}) = f(f^{\#}(a_1^{m+sk})a_{m+sk+1}^{m+(s+1)k}) \qquad (0.3)$$

Conversely, to each poly-(n,m)-groupoid $\underline{Q}=(Q;g)$ we can associate an (n,m)-groupoid $\underline{Q}_{\#}=(Q;g_{\#})$ by $g_{\#}(a_1^n)=g(a_1^n)$, i.e. $g_{\#}$ is the restriction of g on Q^n. It is obvious that $(\underline{Q}^{\#})_{\#}=\underline{Q}$, but in general $(\underline{P}_{\#})^{\#}\neq\underline{P}$.

In the case of (n,m)- and poly-(n,m)-semigroups we have the following:

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[1] *If X is a nonepmty set, then by $X^{(n,m)}$ we denote the union* $\bigcup_{s\geq 1} X^{sk+m}$

Proposition 0.1. *(a) An (n,m)-groupoid $\underline{Q}$ is an (n,m)-semigroup iff $\underline{Q}^{\#}$ is a poly-(n,m)-semigroup.*

(b) A poly-(n,m)-groupoid $\underline{P}$ is a poly-(n,m)-semigroup iff $\underline{P}_{\#}$ is an (n,m)-semigroup.

(c) If $\underline{P}$ is a poly-(n,m)-semigroup then $(\underline{P}_{\#})^{\#} = \underline{P}$. □

(For the proof, see [2] p.p. 34-36.)

The notions of (n,m)- and poly-(n,m)-structures are easily thought of as algebras with m n-ary and poly-n-ary operations (called component or scalar operations). Namely, if (Q;f) is an (n,m)- or a poly-(n,m)-groupoid, the component operations $f_1,\dots,f_m:Q^n \to Q$ or $Q^{(n,m)} \to Q$ are defined by

$$f(a_1^{m+rk}) = b_1^m \text{ iff } f_i(a_1^{m+rk}) = b_i, \quad i\in N_m,$$

where r=1 in the first case, and $r \geq 1$ in the second one.

It is easy to interpret the condition (0.1) and (0.2) via the corresponding component operations.

All of the notions such as: subalgebra, congruence, homomorphism, free object in the class of component (n,m)- or poly-(n,m)-algebras (i.e. algebras obtained from (n,m)- or poly-(n,m)-groupoids as above) are considered well known. Using the above notions the following ones can be obtained (without giving their explicit definitions): an (n,m)- and a poly-(n,m)-subgroupoid, a congruence on an (n,m)- and a poly-(n,m)-groupoid, and a homomorphism for (n,m)- and poly-(n,m)-groupoids.

Recall the construction of a free poly-(n,m)-groupoid given in ([2], P.6.3). Let $B\neq\emptyset$ and:

$$B_o=B,\ B_{\alpha+1}=B_\alpha \cup N_m\times B_\alpha^{(n,m)},\ F(B) = \bigcup_\alpha B_\alpha .$$

Define a poly-(n,m)-operation g on F(B) by

$$g(u_1^{m+rk}) = v_1^m \iff (\forall i\in N_m),\ v_i=(i,u_1^{m+rk}) \tag{0.4}$$

Proposition 0.2. *$\underline{F}(B)=(F(B);g)$ is a free poly-(n,m)-groupoid with a basis B.*

Proof. Let $\underline{Q}=(Q;f)$ be an (n,m)-groupoid and $\xi:B \to Q$ a mapping. Then there exists a unique extension $\bar{\xi}$ of ξ such that $\bar{\xi}$ is a homomorphism from $\underline{F}(B)$ into $\underline{Q}$. □

We have already defined an integer valued function, namely the dimension $d:Q^* \to N$. More such functions for $F(B)$ and $F(B)^+$ are defined and used in the main text. Naturally, in the definition of $F(B)$ there is a function denoted by χ and called hierarchy, where $\chi(u)=\min\{\alpha \mid u\in B_\alpha\}$ for $u\in F(B)$.

We define a norm on $F(B)$, i.e. a mapping $|\ |: F(B)^+ \to N$ by induction on χ:

$$|b| = 0, \text{ for } b\in B, \tag{0.5}$$

$$|x| = \sum_{\nu=1}^{\alpha} |u_\nu| \text{ for } x=u_1^\alpha\in(F(B))^\alpha, \tag{0.6}$$

$$|(i,x)| = 1+|x| \text{ for } (i,x)\in F(B). \tag{0.7}$$

Sometimes in the text we will need an alternative definion of a norm. Namely, instead of (0.5) or (0.7) we can take

$$|b| = 1 \text{ for } b\in B, \tag{0.5'}$$

$$|(i,x)| = i+|x| \text{ for } (i,x)\in F(B), \tag{0.7'}$$

$$|(i,x)| = |x| \text{ for } (i,x)\in F(B), \tag{0.7"}$$

and instead of (0.6) we can take

$$|x| = \sum_{\lambda=1}^{\alpha} \prod_{\nu=1}^{\lambda} |u_\nu| \text{ for } x=u_1^\alpha\in(F(B))^\alpha. \tag{0.6'}$$

For the empty sequence 1, we always define $|1|=0$.

The norm used most often in the text is the one defined by (0.5), (0.6) and (0.7). (So, from now on when we say norm, we think of this one.)

A consize review of the results in this paper follows.

In §1 we define the notion of an (n,m)-semigroup determined by a presentation $\langle B;\Delta\rangle$ where $\Delta\subseteq F(B)^2$; it is the quotient structure $\underline{F}(B)/\bar{\Delta}$ where $\bar{\Delta}$ is the least congruence on $\underline{F}(B)$ such that $\Delta\subseteq\bar{\Delta}$ and $\underline{F}(B)/\bar{\Delta}$ is an (n,m)-semigroup. Two presentations $\langle B;\Delta\rangle$ and $\langle B';\Delta'\rangle$ are called equivalent if $F(B)/\bar{\Delta} \cong F(B')/\bar{\Delta}'$, and strongly equivalent if $B=B'$ and $\bar{\Delta}=\bar{\Delta}'$. A presentation $\langle B;\Delta\rangle$

is called a proper presentation if $(a,b)\in B^2\cap\overline{\Delta}$ implies a=b. It is shown that each presentation is equivalent to a proper one. A procedure (in general not sufficiently effective) for determining $\overline{\Delta}$, for given $\langle B;\Delta\rangle$, is described. At the end of §1 several simple examples are presented.

The question about a presentation $\langle B;\Delta\rangle$ which determines an (n,m)-semigroup $\underline{Q}=(Q;f)$ such that $Q\subseteq F(B)$, is investigated in §2. The answer to this question is via a retraction $\psi:F(B)\to F(B)$ (called a reduction for $\langle B;\Delta\rangle$) which satisfies several conditions. Next, we define reductions for the examples from §1 and consider two more examples of presentations together with reductions. One of these examples, the case of $\Delta=\emptyset$ gives a description of free v.v. semigroups.

In §3 we examine a special kind of presentations called vector (n,m)-presentations. It is shown that each (n,m)-presentation is equivalent to a vector one, but there are (n,m)-presentations which are not strongly equivalent with vector ones.

The notions of (n,m)-identities and vector - (n,m)-identidies in the class of poly-(n,m)-groupoids (and so in the class of (n,m)-semigroups as well) are introduced in §4. If θ is a set of (n,m)-identities, then by Varθ we denote the variety of (n,m)-semigroups which satisfy each identity from θ. Next we give the notion of a presentation $\langle B;\Delta;\theta\rangle$ in Varθ and prove some general results. In addition, it is shown that there are varieties of (n,m)-semigroups which could not be determined by vector (n,m)-identities.

In §5, §6 and §7 we consider only vector (n,m)-presentations. We use the fact that each vector (n,m)-presentation $\langle B;\Delta;\theta\rangle$ induces a corresponding presentation $\langle\overline{B;\Delta;\theta}\rangle$ of a semigroup, and investigate the question for a description of $\langle B;\Delta;\theta\rangle$ via a sequence of semigroups. This point of view is highly successful under the condition $(\overline{m})$ (see §5), since in this very case there is a general method for producing a reduction (in most cases - an effective reduction). As a consequence, the proofs of several results are made possible, such as the combinatorial descriptions of the free objects in several varieties of (n,m)-semigroups, and some v.v. variants of Post and Cohn-Rebane Theorems.

Section 8 ends with a commentary on the combinatorial theory of v.v. groups. It is worth mentioning that until now a good description of free v.v. groups is not known, and that in [6] (this volume) a satisfactory description of free (m+1,m)-groups is given.

§1. PRESENTATIONS OF VECTOR VALUED SEMIGROUPS

Let B be a nonempty set, and $\underline{F}(B)=F(B)^{(n,m)}$ be the free poly-(n,m)-groupoid with a basis B (see §0). If Δ is a subset of $F(B)\times F(B)$, i.e. a binary relation of F(B), let $\overline{\Delta}$ be the least congruence on $\underline{F}(B)$ such that $\Delta\subseteq\overline{\Delta}$ and $\underline{F}(B)/\overline{\Delta}$ is an (n,m)-semigroup. Then, we say that Δ is a set of (n,m)-defining (or: defining) relations on B, and $\langle B;\Delta\rangle$ is an (n,m)-presentation (or a presentation) of $\underline{F}(B)/\overline{\Delta}$. The notation $\langle B;\Delta\rangle$ will have the following three conotations: (i) an ordered pair of a set B and a set Δ of (n,m)-defining relations on B; (ii) an (n,m)-semigroup $\underline{F}(B)/\overline{\Delta}$; (iii) the carrier $F(B)/\overline{\Delta}$ of the (n,m)-semigroup $\underline{F}(B)/\overline{\Delta}$.

Let us give a more explicit description of the congruence $\overline{\Delta}$.

First we define a relation $\overset{o}{\vdash}$:

$u \overset{o}{\vdash} v$ iff $(u,v)\in\Delta$ or $u=(i,x'(1,y)\dots(m,y)x'')$, $v=(i,x'yx'')$ where $i\in N_m$, $y\in F(B)^{(n,m)}$, $x'x''\in F(B)^{sk}$ for some $s\geq 1$.

Assume that $\overset{\nu}{\vdash}$ is well defined, and define $\overset{\nu+1}{\vdash}$ as follows

$u,v\in F(B) \Longrightarrow (u \overset{\nu+1}{\vdash} v \Longleftrightarrow u=(i,xu'y),\ v=(i,xv'y),\ u' \overset{\nu}{\vdash} v')$.

Define a relation $\vdash$ on F(B) by:

$$u\vdash v \text{ iff } (\exists\lambda\geq 0)\ u \overset{\lambda}{\vdash} v.$$

Finally, let $\sim$ be the symmetric extension of $\vdash$ and $\approx$ be the reflexive and transitive extension of $\sim$. That is, $u\sim v$ iff $u\vdash v$ or $v\vdash u$, and $u\approx v$ iff there exist $t\geq 0$, $u_0,u_1,\dots,u_t\in F(B)$ such that $u=u_0$, $v=u_t$ and $u_{\lambda-1}\sim u_\lambda$ for any $\lambda\in N_t$.

Proposition 1.1. *If $u, v \in F(B)$, then: $(u,v) \in \overline{\Delta}$ iff $u \approx v$.* □

The (n,m)-semigroup $\underline{F}(B)/\overline{\Delta} = \langle B;\Delta\rangle$ can be more abstractly characterized by the notion of realizations of the pair (B,Δ) in (n,m)-semigroups.

Assume that $\underline{Q}=(Q;f)$ is an (n,m)-semigroup, and $\xi: b \mapsto \overline{b}$ a mapping from B into Q. By P.0.2, there is a unique homomorphism $\overline{\xi}:\underline{F}(B) \to Q$ which is an extension of ξ. We say that ξ is a realization of (B,Δ) in $\underline{Q}$ iff $\overline{\xi}(u)=\overline{\xi}(v)$ for every pair $(u,v) \in \Delta$. Moreover, if ξ is such that for every realization $\xi':B \to Q'$ of (B,Δ) in an (n,m)-semigroup $\underline{Q}'=(Q';f')$ there exists a unique homomorphism $\zeta:\underline{Q} \to Q'$ satisfying the equality $\xi'=\zeta\xi$, then we say that ξ is a universal realization of (B,Δ).

It is clear that the following two statements hold.

Proposition 1.2. *If ξ, η are universal realizations of (B,Δ) in $\underline{Q}, \underline{P}$ - respectively, then there exists a unique isomorphism $\zeta:\underline{Q} \to \underline{P}$ such that $\eta=\zeta\xi$.* □

Proposition 1.3. *The natural mapping $nat\Delta: b \mapsto b^{\Delta}$ is a universal realization of (B,Δ) in $\underline{F}(B)/\approx$. (Here, if $u \in F(B)$, we denote by u^{Δ} the $\approx$ - equivalence class containing u, i.e. $u^{\Delta}=\{v \in F(B) \mid u \approx v\}$. Also, instead of $\overline{\Delta}$ we write $\approx$.)* □

Because of the last two properties, from now on, we will denote by $\langle B;\Delta\rangle$ any (n,m)-semigroup $\underline{Q}$ such that there exists a universal realization of (B,Δ) in $\underline{Q}$.

We say that a presentation $\langle B;\Delta\rangle$ is proper iff

$$(\forall a,b \in B)(a \approx b \Longrightarrow a=b).$$

In this case we may assume that B is a subset of $\langle B;\Delta\rangle$.

Proposition 1.4. *A presentation $\langle B;\Delta\rangle$ is proper iff there exists an injective realization of (B,Δ) in an (n,m)-semigroup.* □

Proposition 1.5. *If $\Delta \subseteq F(B)\times F(B)$ is such that $|u|\cdot|v| \geq 1$, for every pair $(u,v) \in \Delta$, then the presentation $\langle B;\Delta\rangle$ is proper.* □

Consider some trivial examples.

Example 1.6. If Δ is such that $\overline{\Delta}=F(B)\times F(B)$, then $<B;\Delta>$ is a one element (n,m)-semigroup.

Example 1.7. If $\Delta=\{(u,v)\in F(B)\times F(B) \mid |u|\cdot|v|\geq 1\}$, then $<B;\Delta>=B\cup\{o\}$, where $o\notin B$, and $f(c_1^n)=o^m$, for any $c_\nu\in B\cup\{o\}$.

If Δ_1 is a subset of Δ such that:

$$(u,v)\in\Delta_1 \iff (\exists i\in N_m)u=(i,x),\ v=(i,y),\ x,y\in F(B)^{(n,m)},$$

then:

$$<B;\Delta_1> = B\cup\{o_1,o_2,\dots,o_m\}$$

is a constant (n,m)-semigroup, such that $f(c_1^n)=o_1^m$ for any $c_\nu\in B\cup\{o_1,\dots,o_m\}$, where $B\cap\{o_1,\dots,o_m\}=\emptyset$.

Example 1.8. If $\Delta=\{((i,b_1^n),b_i) \mid b_\nu\in B,\ i\in N_m\}$, then $<B;\Delta>=B$ is the left zero (n,m)-semigroup on B, i.e. $f(b_1^n)=b_1^m$ for any $b_\nu\in B$.

Example 1.9. Let $\underline{Q}=(Q;f)$ be an (n,m)-semigroup and $\Gamma(\underline{Q})\subseteq F(Q)\times F(Q)$ be defined as follows:

$$\Gamma(\underline{Q}) = \{((i,a_1^n),b_i) \mid f(a_1^n)=b_1^m \text{ in } \underline{Q},\ i\in N_m\}.$$

Then, the identity transformation $1:a\longmapsto a$ is a universal realization of $(Q,\Gamma(\underline{Q}))$ in $\underline{Q}$. Moreover, if $\underline{P}=(P;g)$ is an (n,m)-semigroup and $\xi:a\longmapsto\xi(a)$ is a mapping from Q in P, then ξ is a realization of $(Q,\Gamma(\underline{Q}))$ in $\underline{P}$ iff $\xi:\underline{Q}\to\underline{P}$ is a homomorphism. Thus $\underline{Q}=<Q;\Gamma(\underline{Q})>$. We say that $\Gamma(\underline{Q})$ is the graph of $\underline{Q}$ and that $<Q;\Gamma(\underline{Q})>$ is the graphical presentation of $\underline{Q}$.

Note that E.1.8 is a special case of E.1.9.

Proposition 1.10. *The presentations in E.1.7, E.1.8 and E.1.9 are proper, and if $|B|\geq 2$ then the presentation in E.1.6 is not proper.* □

We say that two presentations $<B;\Delta>$ and $<B';\Delta'>$ are equivalent iff the corresponding (n,m)-semigroups are isomorphic, i.e. if $\underline{F}(B)/\overline{\Delta}$ is isomorphic to $\underline{F}(B')/\overline{\Delta}'$. Then we write $<B;\Delta>\cong<B';\Delta'>$.

Proposition 1.11. *Every (n,m)-presentation is equivalent to a proper (n,m)-presentation.*

Proof. If $<B;\Delta>=\underline{Q}$, then $<B;\Delta>\cong<Q;\Gamma(\underline{Q})>$. □

Two presentations $\langle B;\Delta\rangle$ and $\langle B';\Delta'\rangle$ are called *strongly equivalent* iff $\overline{\Delta}=\overline{\Delta}'$. Thus:

Proposition 1.12. *The presentations* $\langle B;\Delta\rangle$ *and* $\langle B;\overline{\Delta}\rangle$ *are strongly equivalent.* □

Clearly, if two presentations are strongly equivalent then they are equivalent as well.

At the end of this section we note the following. If $\langle B;\Delta\rangle$ is not a proper (n,m)-presentation, then there exists a proper (n,m)-presentation $\langle B';\Delta'\rangle$ obtained as follows. Choose a unique element b′ from $b^{\Delta}\cap B$, for each $b\in B$, and put $B'=\{b' \mid b\in B\}$. Construct $\Delta'\subseteq F(B')\times F(B')$ by replacing each appearence of b in $(u,v)\in\Delta$ with the unique corresponding element $b'\in B'$. Then $\langle B;\Delta\rangle$ is equivalent to $\langle B';\Delta'\rangle$, and $\langle B',\Delta'\rangle$ is a proper (n,m)-presentation.

§2. REDUCTIONS

The (n,m)-semigroup $\underline{F}(B)/\approx\; = \langle B;\Delta\rangle$ is a "quotient structure" and, it is usually desirable to find an (n,m)-semigroup isomorphic to $\underline{F}(B)/\approx$ whose carrier is a subset of F(B). This can be achieved by a choice of one and only one element $\psi(u)$ from each $\approx$-equivalence class $u^{\Delta}=\{v \mid u \approx v\}$. Or, equivalently, by a mapping $\psi:F(B) \to F(B)$ with the following properties:

(i) $(u,v)\in\Delta \Longrightarrow \psi(u)=\psi(v)$;

(ii) $\psi(i,x'(1,y)(2,y)\ldots(m,y)x") = \psi(i,x'yx")$;

(iii) $\psi(i,x'wx") = \psi(i,x'\psi(w)x")$;

(iv) $u \approx \psi(u)$;

(v) $\psi^2 = \psi$;

for every $u,v,w,\ (i,x'wx"),(i,x'(1,y)\ldots(m,y)x")\in F(B)$.

A mapping $\psi:F(B) \to F(B)$ is said to be a *reduction* for $\langle B;\Delta\rangle$ if (i) to (v) are satisfied. We say that $u\in F(B)$ is *reduced* if $\psi(u)=u$, and *reducible* otherwise.

Proposition 2.1. *Let ψ be a reduction for $\langle B;\Delta\rangle$ and let $Q=\psi(F(B))$. If an (n,m)-operation g is defined on Q by*

$$g(u_1^{m+sk}) = v_1^m \iff (\forall i\in N_m) v_i = \psi(i,u_1^{m+sk}),$$

then $\underline{Q}=(Q;g)$ is an (n,m)-semigroup and the restriction of ψ on B is a universal realization of (B,Δ) in $\underline{Q}$. Therefore, $\underline{Q}=\langle B;\Delta\rangle$.

Proof. First, $Q=\psi(F(B))$ implies that g is a well defined (n,m)-operation on Q, and we may consider ψ as a surjective homomorphism from $\underline{F}(B)$ onto $\underline{Q}$ such that $\overline{\Delta}\subseteq\ker\psi$. If $(u,v)\in\ker\psi$, i.e. $\psi(u)=\psi(v)$, then $u \approx \psi(u)=\psi(v) \approx v$, whence $(u,v)\in\overline{\Delta}$. Thus, $\ker\psi=\overline{\Delta}$, and therefore $\underline{Q}$ and $\underline{F}(B)/\approx$ are isomorphic. □

Note that the condition (v) is a consequence of (i) - (iv).

A reduction ψ for $\langle B;\Delta\rangle$ is called a proper one iff $\psi(b)=b$ for every $b\in B$.

Proposition 2.2. *A presentation $\langle B;\Delta\rangle$ is proper iff it admits a proper reduction.* □

In general, there exist many reductions for a presentation $\langle B;\Delta\rangle$. Usually, we look for a reduction which satisfies some conditions of "effectivness".

Let us consider E.1.6 to E.1.9.

First, if u_0 is an arbitrary element of $F(B)$, and if we put $\psi(u)=u_0$ for all $u\in F(B)$, then we obtain a reduction for $\langle B;\Delta\rangle$, where $\overline{\Delta}=F(B)\times F(B)$.

Let $\langle B;\Delta\rangle$ and $\langle B;\Delta_1\rangle$ be as in E.1.7, and let $u_0\in F(B)\setminus B$, $x_i\in F(B)^{(n,m)}$ for every $i\in N_m$ be fixed. Define two mappings $\psi,\psi_1:F(B) \to F(B)$ as follows:

$\psi(b) = \psi_1(b) = b$ for every $b\in B$,

$\psi(i,x) = u_0$, $\psi_1(i,x)=(i,x_i)$, for every $(i,x)\in F(B)\setminus B$.

Then, ψ is a reduction for $\langle B;\Delta\rangle$, and ψ_1 is a reduction for $\langle B;\Delta_1\rangle$.

If $\langle B;\Delta\rangle$ is as in E.1.8, then we can define a reduction ψ by induction on the norm in the following manner: $\psi(b)=b$ for every $b\in B$, and $\psi(u)=\psi(u_i)$, for every $u=(i,u_1^{m+sk})\in F(B)\setminus B$. More generally, if $\underline{Q}=(Q;f)$ is an (n,m)-semigroup then a reduction for the graphical presentation $\langle Q;\Gamma(\underline{Q})\rangle$ of $\underline{Q}$ (E.1.9) can be defined as follows. First, $\psi(b)=b$, for every $b\in Q$. Assume that $u=(i,u_1^{m+sk})\in F(Q)\setminus Q$, and that $\psi(v)\in Q$ is well defined for every $v\in F(B)$ such that $|v|<|u|$. Then $\psi(u)$ is defined by:

$$\psi(u) = f_i(a_1^{m+sk}),$$

where $a_\nu=\psi(u_\nu)$.

Note that we do not have any particular use of the corresponding reductions in the above examples, as we know wery well their structure. However, in the next two examples the corresponding reductions are of substantial use.

Example 2.3. Let $B\neq\emptyset$, and $\Delta=\emptyset$. In this case $\langle B;\emptyset\rangle$ is the free (n,m)-semigroup with a basis B.

A reduction $\psi:F(B)\to F(B)$ will be defined by induction on the norm as follows.

(0) $(\forall b\in B)\psi(b) = b$.

Assume that $u=(i,x)\in F(B)\setminus B$ and that $\psi(v)\in F(B)$ is well defined for every $v\in F(B)$ such that $|v|<|u|$. Moreover, assume that the following condition is satisfied:

$$\psi(v) \neq v \Longrightarrow |\psi(v)| < |v|. \tag{2.1}$$

If $x=u_1^{m+sk}$, $u_\nu\in F(B)$, then $v_\lambda=\psi(u_\lambda)$ is well-defined, and thus $v=(i,v_1^{m+sk})\in F(B)$. If there exists a λ such that $v_\lambda\neq u_\lambda$, then $|v|<|u|$. Consequently, we can define $\psi(u)$ by:

(1) $\psi(u) = \psi(v)$.

If $\psi(u_\lambda)=u_\lambda$ for every λ and if $x=x'(1,y)(2,y)\ldots(m,y)x''$, where $x',x''\in F(B)$, $(\nu,y)\in F(B)$, and x' has the least possible dimension then we define $\psi(u)$ by:

(2) $\psi(u) = \psi(i,x'yx'')$.

And if $\psi(u)$ could not be defined by (1) or (2) then we put

(3) $\psi(u) = u$.

Clearly, if $\psi(u)$ is defined by (1) or (2) then we have $|\psi(u)| < |(u)|$ and this implies that $\psi: F(B) \to F(B)$ is a well-defined mapping. Moreover, (2.1) holds for every $v \in F(B)$.

By induction on the norm it can be checked that (i)-(v) are satisfied, i.e., that ψ is a reduction for $\langle B;\emptyset\rangle$. (see [5], [2]).

We also note that we have a good description of $S(B)=\psi(F(B))$. Namely, $u \in S(B)$, i.e. u is reduced, iff $u \in B$ or $u=(i,u_1^{m+sk})$ where $u_\nu \in S(B)$ for every ν and there is no $j \in N_{sk}$ such that $u_{j+\lambda}=(\lambda,y)$, for every $\lambda \in N_m$. Moreover, if u is a given element of F(B) then $\psi(u)$ is determined in a finite number of steps.

Further on we will always denote the above reduction ψ by ψ_o. Thus, $\underline{S}(B)=(S(B);f)$ is a free (n,m)-semigroup with a basis B, where f is defined by:

$$f(u_1^{m+sk}) = v_1^m \iff (\forall i \in N_m) v_i = \psi_o(i,u_1^{m+sk}).$$

In the case m=1 we have the following well known result.

Proposition 2.4. *The (n,1)-semigroup $\underline{S}(B)=(S(B);f)$ where*

$$S(B) = \{u \in B^+ \mid d(u) = 1 + sk,\ s \geq 1\}, \text{ and}$$

$$f(u_1^n) = u_1u_2\ldots u_n,$$

is a free (n,1)-semigroup, i.e. a free n-semigroup with a basis B. □

We say that a presentation $\langle B;\Delta\rangle$ is reduced iff $\Delta \subseteq S(B)\times S(B)$.

Proposition 2.5. *Let $\Delta \subseteq F(B)\times F(B)$ and*

$$\Delta_o = \{(\psi_o(u),\psi_o(v)) \mid (u,v)\in\Delta\}.$$

Then $\langle B;\Delta\rangle$ and $\langle B;\Delta_o\rangle$ are strongly equivalent and $\langle B;\Delta_o\rangle$ is a reduced presentation. □

From now on we will usually deal with reduced (n,m)-presentations.

Example 2.6. Let B be a nonempty set, $m \geq 3$ and let Δ be defined by:

$$\Delta = \{(u,v) \mid u=(1,x),\ v=(2,x)\in F(B)\}.$$

We will define a reduction ψ for $<B;\Delta>$ in the same way as in E.2.3. Namely, assume that (0), (1), (2) and (3) are as in E.2.3, and:

(1-) $\psi(2,x) = \psi(1,x)$.

In (2) it is assumed that $x=x'(1,y)(2,y)(3,y)\ldots(m,y)x''$. The proof that (1-), (0), (1), (2) and (3) define a reduction for $<B;\Delta>$ is by an induction on a norm defined by (0.5), (0.6) and (0.7') (see §0).

Note that in E.2.6 it is possible to take m=2, but then for the definition of a reduction one more step is needed, that is

$$\psi(i,x'(1,y)yx'') = \psi(i,x'y(1,y)x''),$$

and in the proof that ψ is indeed a reduction we need a norm defined by (0.5'), (0.6') and (0.7') (see §0).

Another remark about E.2.6 is that it is possible to take

$$\Delta = \{(u,v) \mid u = (i,x),\ v = (j,x),\ u,v\in F(B)\},$$

for $1 \le i < j \le m$. The cases $1 \le i < j < m$ and $1 < i < j \le m$ are the same as the case i=1, j=2, $m \ge 3$, and the case i=1, j=m, $m \ge 2$ is the same as the case i=1, j=m=2.

§3. VECTOR (n,m)-PRESENTATIONS

In the next part of the paper we will usually deal with a special kind of (n,m)-relations which will be called "vector (n,m)-relations".

Assume that B is a nonempty set and Λ a subset of $B^+\times B^+$ such that for every $(a_1^p,b_1^q)\in\Lambda$ we have $m \le q \le p$, and $q \equiv p \equiv m \pmod k$. Then we say that Λ is a set of vector (n,m)-relations on B. (Note that the assumption $q \le p$ is not essential.)

We can associate a set $\Lambda_{\#}$ of (n,m)-relations to a set Λ of vector (n,m)-relations in the following way.

Firstly, the preceeding notation is modified. Namely, below (i,u_1^m) will be another sign for u_i. Thus, $u\in F(B)$ iff $u=(i,u_1^p)$, where $i\in N_m$, $u_\nu\in F(B)$, $m \le p$, $p \equiv m(\mathrm{mod}\ k)$.

Now, if $\Lambda \subseteq B^+ \times B^+$ is a set of vector (n,m)-relations, then $\Lambda_\#$ is defined by:

$\Lambda_\# = \{(u,v) \mid u=(i,a_1^p),\ v=(i,b_1^q),\ (a_1^p,b_1^q)\in\Lambda,\ i\in N_m\}$.

If $\underline{Q}=(Q;f)$ is an (n,m)-semigroup and $\xi: B \to Q$ is a mapping, then we say that ξ is a (universal) realization of (B,Λ) in $\underline{Q}$ iff ξ is a (universal) realization of $(B,\Lambda_\#)$ in $\underline{Q}$. Then $\langle B;\Lambda_\#\rangle$ is also denoted by $\langle B;\Lambda\rangle$. We say that $\langle B;\Lambda\rangle$ is a vector (n,m)-presentation or simply a vector presentation.

Proposition 3.1. *Let Λ be a set of vector (n,m)-relations on B, and $\underline{Q}=(Q;f)$ be an (n,m)-semigroup. Then:*

(i) A mapping $\xi: B \to Q$ is a realization of (B,Λ) in $\underline{Q}$ iff $f(\overline{a}_1^p)=f(\overline{b}_1^q)$ for every $(a_1^p,b_1^q)\in\Lambda$ where $\overline{c}=\xi(c)$.

(ii) $\langle B;\Lambda\rangle=\underline{Q}$ iff there is a universal realization ζ of (B,Λ) in $\underline{Q}$. □

Proposition 3.2. *Let $\Delta\subseteq F(B)\times F(B)$ be a set of (n,m)-relations on B with the following properties:*

If $(u,v)\in\Delta$ and $u\notin B$ (or $v\notin B$) then $u=(i,a_1^p)$ $(v=(i,b_1^q))$ where $i\in N_m$, $a_\nu,b_\lambda\in B$, $m\le q<p$, $q\equiv p\equiv m (\mathrm{mod}\ k)$, and for every $j\in N_m$, $(u',v')\in\Delta$, where $u'=(j,a_1^p)$, $v'=(j,b_1^q)$.

Define a subset $\Delta^\#$ of $B^+\times B^+$ by:

$$\Delta^\# = \{(\overset{m}{a},\overset{m}{b}) \mid (a,b)\in\Delta,\ a,b\in B\}\ \cup$$
$$\cup\ \{(a_1^p,b_1^q) \mid (\forall i\in N_m)((i,a_1^p),(i,b_1^q))\in\Delta,\ m\le q\le p,\ p>m\}.$$

Then $\Delta^\#$ is a set of vector (n,m)-relations such that $(\Delta^\#)_\#=\Delta$. □

If $\Delta\subseteq F(B)\times F(B)$ is such that $\Delta=\Lambda_\#$ for a set Λ of vector (n,m)-relations on B, then we also say that $\langle B;\Delta\rangle$ is a vector (n,m)-presentation. Moreover, no distinction is being made between the two (n,m)-presentations, $\langle B;\Delta\rangle$, $\langle B;\Lambda\rangle$.

Proposition 3.3. *Every (n,m)-presentation is equivalent to a vector (n,m)-presentation.*

Proof. If $\langle B;\Delta\rangle=\underline{Q}=(Q;f)$, then $\langle B;\Delta\rangle$ and $\langle Q;\Gamma(Q)\rangle$ are equivalent, and, moreover, $\Gamma(\underline{Q})=\Lambda_\#$, where:

$$\Lambda = \{(a_1^n, b_1^m) \mid f(a_1^n) = b_1^m \text{ in } \underline{Q}\}. \quad \square$$

It is natural to ask for an (n,m)-presentation which is not strongly equivalent to a vector (n,m)-presentation.

Proposition 3.4. *The presentation <B;Δ> given in E.2.6 is not strongly equivalent to a vector (n,m)-presentation.*

Proof. It is sufficient to show that there does not exist a vector (n,m)-presentation Λ on B such that $\overline{\Delta}=\overline{\Lambda}_{\#}$. Namely, if $(a_1^p, b_1^q) \in B^+ \times B^+$ is a vector (n,m)-relation on B such that $a_1^p \neq b_1^q$, then it can be easily seen that

$$(\exists i \in N_m)\ ((i, a_1^p), (i, b_1^q)) \notin \overline{\Delta}. \quad \square$$

As we have noticed in §2, if <B;Δ> is an (n,m)-presentation and if $(u,v) \in \Delta$, we can assume that u and v are reduced. And, if m=1 then $u \in F(B)$ is reduced iff $u \in B$ or $u=(1, a_1^{m+sk})$ where $s \geq 1$, $a_\nu \in B$. The last assertion implies the following:

Propisition 3.5. *Every (n,1)-presentation is strongly equivalent to a vector (n,1)-presentation.* □

Further on we will always assume that n,m and k are given positive integers such that n-m=k, $m \geq 2$. We will also assume that B is a nonempty set and Λ a set of vector (n,m)-relations on B. Then we can also consider Λ as a set of vector (2,1)-relations on B. This is the reason behind the use of different notations. Namely, we denote by <B;Λ> the corresponding (n,m)-presentation, and by $\overline{<B;\Lambda>}$ the same presentation but now considered as a (2,1)-presentation. Thus, <B;Λ> is an (n,m)-semigroup, and $\overline{<B;\Lambda>}$ is a semigroup.

Proposition 3.6. *Let <B;Λ> be a vector (n,m)-presentation, where $m \geq 2$, and let $\vdash$, ~, $\overset{\Lambda}{=}$ be relations in B^+ defined as follows.*

$u \vdash v$ iff $u=u_1u'u_2$, $v=u_1v'u_2$, $(u',v') \in \Lambda$, $u_\nu \in B^*$;

$u \sim v$ iff $u \vdash v$ or $v \vdash u$;

$u \overset{\Delta}{=} v$ iff there exist $t \geq 0$, $u_0, \ldots, u_t \in B^+$ such that $u=u_0$, $v=u_t$, $u_{\lambda-1} \sim u_\lambda$ for any $\lambda \in N_t$.

Then $\overset{\Lambda}{=}$ *is a congruence on* B^+ *such that* $B^+/\underset{=}{\Lambda} = \langle\overline{B;\Lambda}\rangle$.

Moreover:

(i) $u \in B^+$, $d(u) < m \Longrightarrow (u \overset{\Lambda}{=} v \Longleftrightarrow u = v)$

(ii) $u \overset{\Lambda}{=} v \Longrightarrow d(u) \equiv d(v) \pmod{k}$. □

It follows from (i) that we can assume that $B \cup B^2 \cup \ldots \cup B^{m-1}$ is a subset of $B^+/\underset{=}{\Lambda} = \langle\overline{B;\Lambda}\rangle$. In general, two different elements u,v of B^m can define the same element in $\langle\overline{B;\Lambda}\rangle$, i.e.

$$u \overset{\Lambda}{=} v,\ u \neq v.$$

Let $\langle B;\Lambda\rangle$, n,m,k and $\overset{\Lambda}{=}$ be as above. If $u \in B^+$, then the set $\{d(v) \mid u \overset{\Lambda}{=} v\}$ is denoted by $\overline{d}(u)$. Clearly,

$$u \overset{\Lambda}{=} v \Longrightarrow \overline{d}(u) = \overline{d}(v)$$

and therefore if we put $\overline{d}(u^\Lambda) = \overline{d}(u)$ we get that for every $x \in \langle\overline{B;\Lambda}\rangle$ $\overline{d}(x)$ is a well defined set of positive integers. (Here $u^\Lambda = \{v \mid u \overset{\Lambda}{=} v\}$.)

<u>Proposition 3.7</u>. *If* $x \in \langle\overline{B;\Lambda}\rangle$ *and if* $\alpha \geq m$ *for some* $\alpha \in \overline{d}(x)$ *then* $\beta \geq m$ *for every* $\beta \in \overline{d}(x)$ *and moreover:* $\beta, \gamma \in \overline{d}(x) \Longrightarrow \beta \equiv \gamma \pmod{k}$. □

<u>Proposition 3.8</u>. *Let* $\langle B;\Lambda\rangle$ *be a vector (n,m)-presentation and* $\xi: c \longmapsto \overline{c}$ *be a realization of* (B,Λ) *in an (n,m)-semigroup* $\underline{P} = (P;g)$. *If* $a_\nu, b_\lambda \in B$, $r,s \geq 0$ *are such that* $a_1^{m+rk} \overset{\Lambda}{=} b_1^{m+sk}$, *then*

$$g(\overline{a}_1^{m+rk}) = g(\overline{b}_1^{m+sk}).$$

<u>Proof</u>. Let $u = a_1^{m+rk}$, $v = b_1^{m+sk}$, $\overline{u} = \overline{a}_1^{m+rk}$, $\overline{v} = \overline{b}_1^{m+sk}$.

If $u \sim v$ then $u = v_1 u' v_2$, $v = v_1 v' v_2$, where $v_1, v_2 \in B^*$ and $(u',v') \in \Lambda$ or $(v',u') \in \Lambda$; thus $g(\overline{u}') = g(\overline{v}')$. This implies that

$$\begin{aligned} g(\overline{u}) &= g(\overline{v}_1 \overline{u}' \overline{v}_2) = g(\overline{v}_1 g(\overline{u}') \overline{v}_2) \\ &= g(\overline{v}_1 g(\overline{v}') \overline{v}_2) = g(\overline{v}_1 \overline{v}' \overline{v}_2) \\ &= g(\overline{v}). \end{aligned}$$

If $u_0, u_1, \ldots, u_t \in B^+$ are such that $t \geq 2$ and

$$u = u_0 \sim u_1 \sim \ldots \sim u_t = v$$

then

$$g(\overline{u}) = g(\overline{u}_0) = g(\overline{u}_1) = \ldots = g(\overline{u}_t) = g(\overline{v}). \quad \square$$

§4. PRESENTATIONS IN VARIETIES OF VECTOR VALUED SEMIGROUPS

A class of (n,m)-semigroups is said to be a variety of (n,m)-semigroups if it has an axiom system which is a set of identities.

First we give a more precise definition of an identity in the class of poly-(n,m)-groupoids.

Denote by N the set of nonnegative integers and consider the free poly-(n,m)-groupoid F(N) with a basis N. The elements of F(N) will be denoted by $\rho, \omega, \tau, \ldots$.

Let $\underline{Q}=(Q;f)$ be a poly-(n,m)-groupoid. Every element

$$\rho \in F(N_t) \subset F(N)$$

define a t-ary operation $\rho^{\underline{Q}}$ on Q as follows.

(i) If $\rho=j\in N_t$ then $\rho^{\underline{Q}}(a_1^t)=a_j$;

(ii) If $\rho=(i,\rho_1^{m+sk})$ and $\rho_\nu^{\underline{Q}}(a_1^t)=b_\nu$

then

$$\rho^{\underline{Q}}(a_1^t) = f_i(b_1^{m+sk}).$$

We note that if $t<q$ and $\rho\in F(N_t)$ then $\rho\in F(N_q)$ and thus ρ defines a q-ary operation $\rho'^{\underline{Q}}$ on Q as well. Clearly, we have

$$\rho'^{\underline{Q}}(a_1^q) = \rho^{\underline{Q}}(a_1^t)$$

for any $a_\nu \in Q$. Further on, we ommit the upperscript, i.e. we write $\rho(a_1^t)$ instead of $\rho^{\underline{Q}}(a_1^t)$.

Let $\rho, \omega \in F(N_t)$. We say that a poly-(n,m)-groupoid $\underline{Q}$ satisfies the (n,m)-identity (ρ,ω), and write

$$\underline{Q} \models (\rho,\omega),$$

iff

$$\rho(a_1^t) = \omega(a_1^t) \text{ for any } a_1^t \in Q^t.$$

The reduction $\psi_0 : F(N) \to F(N)$ is defined as in the general case. By induction on the norm it can be easily shown that:

Proposition 4.1. *If $\underline{Q}$ is an (n,m)-semigroup and $\rho \in F(N)$ then $\underline{Q} \models (\rho, \psi_0(\rho))$.* □

We say that an (n,m)-identity (ρ,ω) is reduced if $\psi_o(\omega)=\omega$ and $\psi_o(\rho)=\rho$. Since we are interested only in poly-(n,m)-groupoids which are (n,m)-semigroups, from now on we consider only reduced (n,m)-identities.

Proposition 4.2. *If (ρ,ω) is a reduced (n,m)-identity and $\underline{Q} \models (\rho,\omega)$ for every (n,m)-semigroup $\underline{Q}$, then $\rho=\omega$.*

Proof. If B is a nonempy set and $\rho\neq\omega$ then the free (n,m)-semigroup S(B) with a basis B does not satisfy the (n,m)-identity (ρ,ω). □

(We note that the conclusion of the proof holds in the case m=1 only if we assume that $|B|>2$. Namely the free (n,1)-semigroup with a free generator is commutative.)

If $\theta \subseteq F(N\setminus\{0\})\times F(N\setminus\{0\})$ then we say that θ is a set of (n,m)-identities. By $\mathcal{V}=\mathrm{Var}\theta$ we denote the class of (n,m)-semigroups $\underline{Q}$ such that $\underline{Q}\models\theta$, where: $\underline{Q}\models\theta$ iff $\underline{Q}\models(\rho,\omega)$ for every $(\rho,\omega)\in\theta$. We say that $\mathrm{Var}\theta$ is the variety of (n,m)-semigroups generated by θ.

Let B be a nonempty set and let $\rho\in F(N_t)$, $u_1^t\in F(B)^t$. Then $\rho(u_1^t)\in F(B)$, because ρ induces a t-ary operation on F(B). If θ is a set of (n,m)-identities then we put:

$$\theta(F(B)) = \{(\rho(u_1^t),\ \omega(u_1^t))\ |\ (\rho,\omega)\in\theta,\ \rho,\omega\in F(N_t),\ u_1^t\in F(B)^t,\ t\in N\}.$$

Proposition 4.3. *$\langle B;\theta(F(B))\rangle$ is a free object in $\mathrm{Var}\theta$ with a basis B.*

Proof. If $\Delta=\theta(F(B))$ then $\overline{\Delta}$ is the least congruence on $\underline{F}(B)$ such that $\underline{F}(B)/_{\overline{\Delta}}\in\mathrm{Var}\theta$. □

If Δ is a set of (n,m)-relations on B and θ is a set of (n,m)-identities then the (n,m)-semigroup $\langle B;\Delta\cup\theta(F(B))\rangle$ will be denoted by $\langle B;\Delta;\theta\rangle$. This (n,m)-semigroup can be characterized in the following way:

Proposition 4.4. *$\underline{Q}=(Q;f)=\langle B;\Delta;\theta\rangle$ iff the following conditions are satisfied:*

(i) $\underline{Q}\in\mathrm{Var}\theta$;

(ii) There is a realization ξ of (B,Δ) in $\underline{Q}$ such that for

any realization ξ' of (B,Δ) in an (n,m)-semigroup $\underline{Q}'\in Var\theta$ there is a unique homomorphism $\zeta:\underline{Q} \to \underline{Q}'$ such that $\zeta\xi=\xi'$. □

We say that $<B;\Delta;\theta>$ is an (n,m)-presentation in Varθ.

Every set θ of (n,m)-identities induces an (n,m)-presentation $<N;\theta>$, and we say that θ is a set of vector (n,m)-identities iff $<N;\theta>$ is a vector (n,m)-presentation.

Vector (n,m)-identities can be defined directly, as well. Namely, let p=m+sk, q=m+rk, where $r,s \geq 0$, and let $(i_1^p, j_1^q)\in N^+ \times N^+$. We say that an (n,m)-semigroup $\underline{Q}=(Q;f)$ satisfies the vector (n,m)-identity (i_1^p, j_1^q), and write $\underline{Q} \models (i_1^p, j_1^q)$, iff for every $a_1^t \in Q^t$ we have $f(b_1^p)=f(c_1^q)$, where $t=\max_{\nu,\lambda}\{i_\nu, j_\lambda\}$ and $b_\nu = a_{i_\nu}$, $c_\lambda = a_{j_\lambda}$, for any ν,λ. From now on we assume that $p \geq q$.

Proposition 4.5. *If $i_1^m \neq j_1^m$ and if $\underline{Q} \models (i_1^m, j_1^m)$ then $|Q| = 1$.* □

We say that a variety $\mathbb{V}$ of (n,m)-semigroups is a vector variety iff there exists a set of vector (n,m)-idnetities θ such that $\mathbb{V}$=Varθ. Also, $<B;\Lambda;\theta>$ is a vector (n,m)-presentations iff $<B;\Lambda>$ and $<N;\theta>$ are vector (n,m)-presentations.

Let us consider some examples.

Example 4.6. If there exists an $(i_1^m, j_1^m)\in\theta$ such that $i_1^m \neq j_1^m$ then Varθ= $\mathbb{O}(n,m)=\mathbb{O}$ is the least variety of (n,m)-semigroups, and $\underline{Q}=(Q;f)\in\mathbb{O}$ iff $|Q|=1$. Therefore for every B and Δ,$<B;\Delta;\theta>$ is a one element (n,m)-semigroup.

From now on we assume that Var$\theta \neq \mathbb{O}$. Therefore, we can also assume that in θ there are no identities of the form (i_1^m, j_1^m).

In the case $\theta=\{(i_1^p, j_1^q)\}$ we write Var(i_1^p, j_1^q) and $<B;\Delta;(i_1^p, j_1^q)>$ instead of Var$\{(i_1^p, j_1^q)\}$ and $<B;\Delta;\{(i_1^p, j_1^q)\}>$, respectively.

Example 4.7. Let $i_\nu=\nu$ for each $\nu\in N_n$. Then Var(i_1^n, i_1^m)= LZ is the variety of left zero (n,m)-semigroups, i.e., (Q;f)$\in$LZ iff $f(a_1^{m+sk})=a_1^m$ for any $a_\nu \in Q$, $s \geq 0$. (See E.1.8.) Any (n,m)-semigroup $\underline{Q}=(Q;f)\in$LZ is a free object in LZ with a basis Q. If $<B;\Delta;(i_1^n, i_1^m)>$ is a vector (n,m)-presentation in LZ, then it determines the left zero (n,m)-semigroup on $B/\approx$ where $\approx$ is the least equivalence on B such that

$$(a_1^p, b_1^q) \in \Delta \implies (\forall i \in N_m),\ a_i \approx b_i.$$

More generaly, let $\Delta \subseteq F(B) \times F(B)$ and let us define a mapping $\psi : F(B) \to B$ by induction on norm in the following way:

$\psi(b) = b$ for every $b \in B$

$\psi(i, u_1^{m+sk}) = \psi(u_i)$, for every $u_\nu \in F(B)$, $i \in N_m$, $s \geq 1$.

Then, if $\approx$ is the least equivalence in B such that

$$(u,v) \in \Delta \implies \psi(u) \approx \psi(v),$$

we obtain that $\langle B; \Delta; (i_1^n, i_1^m)\rangle$ is the left zero (n,m)-semigroup on $B/\approx$.

Example 4.8. Let $i_\nu = \nu$, $j_\nu = n+\nu$ for any $\nu \in N_n$. Then, $\mathrm{Var}(i_1^n, j_1^n) = Z(n,m)$ is the variety of constant (n,m)-semigroups (E.1.7.). We recall that an (n,m)-semigroup $\underline{Q} = (Q; f)$ is a constant one iff there exists a $o_1^m \in Q^m$ such that $f(a_1^n) = o_1^m$ for any $a_\nu \in Q$. Then we also have $f(a_1^{m+sk}) = o_1^m$ for any $s \geq 1$, $a_\nu \in Q$. As in the first two examples it is easy to give a description of an (n,m)-semigroup $\langle B; \Delta; (i_1^n, j_1^n)\rangle$, where $\Delta \subseteq F(B) \times F(B)$. Let $\{o_1, o_2, \ldots, o_m\}$ be a set disjoint with B and let $B_o = B \cup \{o_1, \ldots, o_m\}$, $o_i \neq o_j$ if $i \neq j$. Define a subset Δ_o of $B_o \times B_o$ by:

$$\Delta_o = \{(a,b) \mid (a,b) \in \Delta \cap B^2\}$$
$$\cup \{(o_i, a) \mid a \in B \text{ and } (a,u) \in \Delta \text{ or } (u,a) \in \Delta \text{ for some } u = (i,x) \in F(B)\}$$
$$\cup \{(o_i, o_j) \mid (u,v) \in \Delta \text{ for some } u = (i,x),\ v = (j,y) \in F(B)\}.$$

Let $\approx$ be the least equivalence in B_o containing Δ_o. If $c \in B_o$ then we denote by $\overline{c}$ the $\approx$-equivalence class containing c. Then $\langle B; \Delta; (i_1^n, j_1^n)\rangle$ is the constant (n,m)-semigroup $(B_o/\approx; f)$ defined by $f(\overline{c}_1^n) = o_1^m$.

We note that in E.4.6 we do not need any construction of F(B), and the same is true in the other two examples iff Δ is a set of vector (n,m)-relations on B.

Each one of the varieties considered above is an example of vector variety of (n,m)-semigroups, and certainly the class of all (n,m)-semigroups is such a variety. Below we give an example of a variety which is not a vector variety.