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# FREE COMMUTATIVE (2m,m)-GROUPS

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Abstract. A commutative (2m,m)-group is a pair (G,[ ]) where G is a nonempty set and $[\ ]:G^{2m}\to G^m$ is a (2m,m)-operation on G, such that $G^m$ with the operation $x\circ y=[xy]$ is a commutative group, and for each $x\in G^i$, $y\in G^{2m-i}$, $z\in G^i$ and $v\in G^{m-i}$, $[[xy]zv]=[x[yz]v]$. In this paper we give a combinatorial description of free commutative (2m,m)-groups (for $m\geq 2$), inspired by the characterization of the commutative (2m,m)-groups as algebras with one (2m,m)-operation, one unary and one nulary operation.

## 1. Introduction

Let $m\geq 1$ and let $G\neq 0$. Let $[\ ]:G^{2m}\to G^m$ be a map satisfying the following two conditions:

$$[x_1^i[x_{i+1}^{2m+i}]x_{2m+i+1}^{3m}] = [[x_1^{2m}]x_{2m+1}^{3m}],\ \text{for each } i\in\mathbb{N}_m; \tag{1.1}$$

for each $a,b\in G^m$, there exist $x,y\in G^m$, such that

$$[ax] = b = [ya]. \tag{1.2}$$

Then, the pair (G;[ ]) is called a (2m,m)-group ([Č]). Above, $G^m$ denotes the m-th Cartesian power of G, $(x_1^t)$ denotes the vector $(x_1,x_2,\ldots,x_t)\in G^m$, $[x_1^{2m}]$ denotes the image of $(x_1^{2m})$ under the map [ ], $a\in G^t$ denotes a vector from $G^t$, and $\mathbb{N}_t$ denotes the set $\{1,2,\ldots,t\}$.

Let (G;[ ]) be a (2m,m)-group. Then $(G^m,\circ)$, where $a\circ b=[ab]$, is a group with a neutral element $e=(e^m)=(e,e,\ldots,e)\in G^m$ ([Č.D]). We say that a (2m,m)-group is commutative, if the associated group $(G,\circ)$ is commutative. Note, that for m=1, the notions of a (2,1)-group and a commutative (2,1)-group coincide with the usual notions of a group and a commutative group. So, from now on we assume that $m\geq 2$.

In this paper we give a combinatorial description of free commutative (2m,m)-groups, inspired by the following characterization of commutative (2m,m)-groups.

Proposition 1.1. Let $G\neq 0$ and $[\ ]:G^{2m}\to G^m$. Then, $(G;[\ ])$ is a commutative (2m,m)-group if and only if there exist $g:G\to G$ and $e\in G$ such that:

(a) $[[xy]z] = [x[yz]]$ for every $x,y,z\in G^m$;

(b) $[xey] = xy$ for each $xy\in G^m$, where $e=(e^m)\in G^m$;

(c) $[x^m(g(x))^m] = e$, for each $x\in G$, where $(x^m)=(x,x,\dots,x)\in G^m$;

(d) $[xayubv] = [xbyuav]$ for each $xy,uv,xv\in G^{m-1}$, $a,b\in G$;

(e) $[xyuv] = wz \iff [yxvu] = zw$, for each $xy,uv,wz,xv,xz\in G^m$.

Proof. Conditions (a), (b) and (e) imply that $[\ ]$ satisfies the associativity condition (1.1), and then, the rest of the proof follows from Propositions 1.1., 1.3, and 1.4 from [D,I]. ◇

We will use the following notions, conventions and notations.

Let A be a nonempty set.

By $A^+$ we will denote the free, and by $A^{(+)}$ the free commutative semigroup generated by A. For a positive integer p, we can identify the p-th Cartesian power $A^p$ with the subset $\{a_1a_2\dots a_p \mid a_i\in A\}$, of $A^+$, and instead of writing $(a_1,a_2,\dots,a_p)$ for a vector from $A^p$, we will use the notations $a_1^p$ and $a_1a_2\dots a_m$. (Here $a_1a_2\dots a_p$ denotes the product of $a_1,a_2,\dots,a_p$ in $A^+$.) For a positive integer p, let $A^{(p)}$ be the subset $\{a_1a_2\dots a_p \mid a_i\in A\}$ of $A^{(+)}$, where $a_1a_2\dots a_p$ is the product of $a_1,a_2,\dots,a_p$ in $A^{(+)}$. As above, we will use the notation $a_1^p$ instead of $a_1a_2\dots a_p$, keeping in mind that $a_1^p=b_1^p$ in $A^{(p)}$, for $a_i,b_i\in A$ if and only if $b_1,b_2,\dots,b_p$ is a permutation of $a_1,a_2,\dots,a_p$.

## 2. A combinatorial description of free commutative (2m,m)-groups

Let A be an arbitrary set. We will construct a free commutative (2m,m)-group $(Q;[\ ])$ with a basis A.

For every $a\in A$ let $a'$ be a new element, such that, for $A'=\{a' \mid a\in A\}$ the map $f:A \to A'$, defined by $f(a)=a'$, is a bijection and $A\cap A'=\emptyset$. Let e be an element not in $A\cup A'$, i.e. $e\notin A\cup A$, and let $B=\{e\}\cup A\cup A'$.

We define a sequence of sets $B_0, B_1, \ldots, B_\alpha, \ldots$ by induction on $\alpha$, as follows. Let $B_0=B$. Suppose that $B_\alpha$ is defined, and then define $B_{\alpha+1}$ by

$$B_{\alpha+1} = B_\alpha \cup \{(x_1,x_2,\ldots,x_m) \mid x_i\in B_\alpha^{(n)},\ n\geq 2\}. \qquad (2.1)$$

Now, let $D = \bigcup_{\alpha\geq 1} B_\alpha$.

Remark. By definition, $u\in D$ if and only if $u\in B_0$, or $u=(x_1,x_2,\ldots,x_m)$, where $x_1,x_2,\ldots,x_m\in B_\alpha^{(n)}$ for some $n\geq 2$ and $\alpha\geq 0$. Moreover, if $u=(x_1,x_2,\ldots,x_m)$ and $v=(y_1,y_2,\ldots,y_m)$, where $x_i\in B_\alpha^{(n)}$ and $y_i\in B_\beta^{(k)}$, then $u=v$ if and only if $n=k$ and there is a $\gamma\geq\alpha,\beta$ such that $x_i=y_i$ in $B_\gamma^{(n)}$.

Let $|\ |:D \to \mathbb{N}$ be the map defined by induction on $\alpha$ as follows:

(i) $|b| = 1$, for $b\in B_0$;

(ii) $|x| = |u_1|+|u_2|+\ldots+|u_n|$ for $x=u_1^n\in D^{(n)}$; and

(iii) $|u| = |x_1|+|x_2|+\ldots+|x_m|$ for $u=(x_1,x_2,\ldots,x_m)\in D$.

We say that $|u|$ is a lenght of u.

Next, by induction on the lenght, we define a map $\phi:D \to D$, called reduction, as follows:

(a) $\phi(b)=b$ for $b\in B_0$;

(b) Suppose that for each $u\in D$ with $|u|<t$, $\phi$ is well defined,

$$\phi(u) \neq u \iff |\phi(u)| < |u|, \text{ and} \qquad (2.2)$$

$$\phi^2(u) = \phi(u). \qquad (2.3)$$

(c) Next, let $u=(x_1,x_2,\ldots,x_m)\in D$ with $|u|=t$, where for each $i\in\mathbb{N}_m$, $x_i\in D^{(n)}$ for some $n\geq 2$. We define $\phi(u)$ by the first possible application of one of the following steps:

(I) If there is $i\in\mathbb{N}_m$, such that $x_i=zx$ and $\phi(x)\neq x$, where $z\in D^{(n-1)}$ and $x\in D$, then $\phi(u)=\phi(\phi_1(x_1),\ldots,\phi_1(x_m))$, where $\phi_1(y_1^t)$ denotes the element $\phi(y_1)\ldots\phi(y_t)$ in $D^{(t)}$, for $y_1^t\in D^{(t)}$.

(II) If for each $i\in\mathbb{N}_m$, $x_i=z_i\pi_i(v)$ for some $v=(y_1,y_2,\ldots,y_m)$ in D and some $z_i\in E^{(n-1)}$, then $\phi(u)=\phi(z_1y_1,z_2y_2,\ldots,z_my_m)$, where $\pi_i(y_1,y_2,\ldots,y_m)=(y_i,y_{i+1},\ldots,y_m,y_1,y_2,\ldots,y_{i-1})\in D$.

Remark. Although $(x_1,x_2,\ldots,x_m)$ does not belong to D for $x_1,x_2,\ldots,x_m\in D$, sometimes we will denote $\phi(x_1)$ by $\phi(x_1,x_2,\ldots,x_m)$.

(III) If for each $i\in\mathbb{N}_m$, $x_i=z_ie$, then $\phi(u)=\phi(z_1,z_2,\ldots,z_m)$.

(IV) If for each $i\in\mathbb{N}_m$, $x_i=z_iaa'$ for some $a\in A$, then $\phi(u)=\phi(z_1e,z_2e,\ldots,z_me)$. (V) $\phi(u)=u$.

The following three propositions, whose proofs will be given later, show that the map $\phi$ is well defined and give several of its properties which will be used in the proof of the main theorem.

Proposition 2.1. (a) The map $\phi$ is well defined and satisfies the conditions (2.2) and (2.3).

(b) For every $b\in B$, $\phi(b)=b$. (2.4)

(c) For every $u\in D$, $\phi(u)\leq|u|$. ◊ (2.5)

Proposition 2.2. Let $u=(x_1,x_2,\ldots,x_m)\in D$, and $x_i=z_it_i$. Then:

(a) $\phi(u)=\phi(\phi_1(x_1),\phi_1(x_2),\ldots,\phi_1(x_m))$. (2.6)

(b) $\phi(u)=\phi(z_1\phi_1(t_1),z_2\phi_1(t_2),\ldots,z_m\phi_1(t_m))$. ◊ (2.7)

Proposition 2.3. For every $z_i\in D^{(n)}$, $n\geq 1$, $v=(y_1,y_2,\ldots,y_m)$, $y_i\in D^{(k)}$, $k\geq 2$, $i\in\mathbb{N}_m$, $a\in A$:

(a) $\phi(z_1\pi_1(v),\ldots,z_m\pi_m(v))=\phi(z_1y_1,\ldots,z_my_m)$. (2.8)

(b) $\phi(z_1e,\ldots,z_me)=\phi(z_1,\ldots,z_m)$. (2.9)

(c) $\phi(z_1aa',\ldots,z_maa')=\phi(z_1e,\ldots,z_me)$. ◊ (2.10)

Let $Q=\phi(D)$. The condition (2.3) implies that $Q=\{u\in D\mid\phi(u)=u\}$. We define a $(2m,m)$-operation $[\ ]:Q^{2m}\to Q^m$ by:

$$[u_1^{2m}]_i = \phi(\pi_i(u_1u_{m+1},\dots,u_mu_{2m})), \text{ for } i\in N_m, \qquad (2.11)$$

where $[\ ]_i$ is the i-th component operation of $[\ ]$, i.e.

$[x_1^{2m}] = y_1^m \iff [x_1^{2m}]_i = y_i$ for each $i\in N_m$.

The following theorem is the main result of the paper.

<u>Theorem 2.4</u>. $(Q,[\ ])$ is a free commutative $(2m,m)$-group with a basis A.

<u>Proof</u>. (A) The definition of $[\ ]$ and the fact that $uv=vu$ in $D^{(+)}$ for $u,v\in D$, imply that: $[xy]=[yx]$ for $x,y\in D^m$, i.e. $(Q,\circ)$ is a commutative groupoid; and condition (d) of Proposition 1.1. holds.

(B) Let $u_1^{3m}\in Q^{3m}$ and $i\in N_m$. Then, the definition of $[\ ]$ and conditions (2.7) and (2.8) imply that: $[[u_1^{2m}]u_{2m+1}^{3m}]_i=$

$=[\pi_1(u_1u_{m+1},\dots,u_mu_{2m})\dots\pi_m(u_1u_{m+1},\dots,u_mu_{2m})u_{2m+1}^{3m}]_i=$

$=\phi(\pi_i(\pi_1(u_1u_{m+1},\dots,u_mu_{2m})u_{2m+1},\dots,\pi_m(u_1u_{m+1},\dots,u_mu_{2m})u_{3m}))=$

$=\phi(\pi_i(u_1u_{m+1}u_{2m+1},u_2u_{m+2}u_{2m+2},\dots,u_mu_{2m}u_{3m}))=$

$=\phi(\pi_i(u_1\pi_1(u_{m+1}u_{2m+1},\dots,u_{2m}u_{3m}),\dots,u_m\pi_m(u_{m+1}u_{2m+1},\dots,u_{2m}u_{3m})))=$

$=[u_1^m[u_{m+1}^{3m}]]_i$.

(C) Let $u_1^m\in Q^m$, $i\in N_m$, $j\in N_{m+1}$. Then (A) and (2.9) imply that

$[u_1^{j-1}e^mu_j^m]_i = [u_1^me^m]_i = \phi(\pi_i(u_1e,u_2e,\dots,u_me)) = u_i$.

(D) Let $u_1^{2m}\in Q^{2m}$, $j\in N_m$ and $[u_1^{2m}]=z_1^m$. Then, $z_{j+1}^mz_1^j[e^{m-j}z_1^me^j]$ implies that $z_{j+1}=[e^{m-j}z_1^me^j]_1$. Moreover,

$z_{j+1}=[e^{m-j}[u_1^{2m}]e^j]_1=[e^{m-j}\phi(\pi_1(u_1u_{m+1},\dots,u_mu_{2m}))..\phi(\pi_m(u_1u_{m+1},\dots,$

$,\dots,u_mu_{2m}))e^j]_1=\phi(\pi_{j+1}(u_1u_{m+1},\dots,u_mu_{2m})) =$

$= [u_{j+1}^mu_1^ju_{m+j+1}^{2m}u_{m+1}^{m+j}]_1$, i.e.

$$z_{j+1}^mz_1^j = [u_{j+1}^mu_1^ju_{m+j+1}^{2m}u_{m+1}^{m+j}].$$

(E) We will define by induction on the lenght, a mapping $g$: $Q\to Q$ such that $[u^m(g(u))^m]=e^m$ for every $u\in Q$.

Let $g(e)=e$, $g(a)=a'$, and $g(a')=a$. Suppose that for every $v\in Q$ such that $|v|<|u|$, $g(v)$ is defined and $[v^m(g(v))^m]=e^m$. Let $u=(x_1,x_2,\dots,x_m)$. Then for each $i\in\mathbb{N}_m$, $x_i=x_{i1}x_{i2}\dots x_{in}=z_iy_i\in Q^{(n)}$ for some $n\geq 2$. We will put $G(x_1^m)=g(x_{11})\dots g(x_{1n})g(x_{21})\dots g(x_{mn})$ and

$$g(u) = \phi(\pi_2(u)\dots\pi_m(u)G(x_1^m),\dots,\pi_2(u)\dots\pi_m(u)G(x_1^m)). \quad (2.12)$$

Then, the definition of [ ], conditions (2.7), (2.8), (2.9), the inductive hypothesis and the fact that $u=\pi_1(u)$ imply that

$$[u^m(g(u))^m]_i=[u^m(\phi(\pi_2(u)\dots\pi_m(u)G(x_1^m),\dots,\pi_2(u)\dots\pi_m(u)G(x_1^m)))^m]_i=$$

$$=\phi(\pi_i(u\phi(\pi_2(u)\dots\pi_m(u)G(x_1^m),\dots,\pi_2(u)\dots\pi_m(u)G(x_1^m),\dots,u\phi(\pi_2(u)\dots$$

$$\dots\pi_m(u)G(x_1^m),\dots,\pi_2(u)\dots\pi_m(u)G(x_1^m))))=\phi(u\pi_2(u)\dots\pi_m(u)G(x_1^m),\dots,$$

$$,\dots,u\pi_2(u)\dots\pi_m(u)G(x_1^m))=\phi(x_1^mG(x_1^m),\dots,x_1^mG(x_1^m))=$$

$$=\phi(x_1^{m-1}z_mG(x_1^{m-1}z_m)y_mg(y_m),\dots,x_1^{m-1}z_mG(x_1^{m-1}z_m)y_mg(y_m))=$$

$$=\phi(x_1^{m-1}z_mG(x_1^{m-1}z_m),\dots,x_1^{m-1}z_mG(x_1^{m-1}z_m))=\dots=e.$$

Now, the steps (A)-(E) imply that $(Q;[\ ])$ is a commutative $(2m,m)$-group.

(F) Since, $\phi(b)=b$ for every $b\in B$ and $A\subset B$ it follows that $A\subset Q$. Let $u=(x_1,x_2,\dots,x_m)$ be an element of $Q$, with $x_i=x_{i1}x_{i2}\dots x_{in}$, $x_{ij}\in\langle A\rangle$, for each $i\in\mathbb{N}_m$, $j\in\mathbb{N}_n$, where $\langle A\rangle$ denotes the $(2m,m)$-subgroup of $(Q;[\ ])$ generated by $A$. Then,

$[x_{11}x_{21}\dots x_{m1}x_{12}x_{22}\dots x_{m2}\dots x_{1n}x_{2n}\dots x_{mn}]=v_1^m\in\langle A\rangle$ and

$a_i=\phi(\pi_i(x_1,x_2,\dots,x_m))$ imply that $\phi(u)\in\langle A\rangle$.

Since $u=\phi(u)$, it follows that $u\in\langle A\rangle$. Hence, $Q=\langle A\rangle$.

(G) Let $(P;[\ ])$ be a commutative $(2m,m)$-group, and $h:A\to P$ be a given map. We denote by $\lambda:B\to P$ the map defined by:

$\lambda(a)=h(a)$, for every $a\in A$,

$\lambda(e)=f$, where $f$ is the neutral element of $(P;[\ ])$,

$\lambda(a')=\psi(\lambda(a))$, where $\psi:P \to P$ is the bijection with the property $[x^m(\psi(x))^m]=f^m$, for every $x\in P$.

Let $\xi:Q \to P$ be defined, inductively, by:

$\xi(b)=\lambda(b)$, for every $b\in B$,

$$\xi(x_1,x_2,\ldots,x_m)=[\xi(x_{11})\xi(x_{21})\ldots\xi(x_{m1})\ldots\xi(x_{1n})\xi(x_{2n})\ldots \ldots\xi(x_{mn})]_1,$$

where $x_i=x_{i1}\ldots x_{in}$ for each $i\in N_m$.

Let $\eta:D \to P$ be the map $\xi o\phi$. It is obvious that, $\eta$ is extension of $\lambda$. By induction on the lenght, using the definition and the properties of the reduction $\phi$, and the properties of commutative (2m,m)-groups, it is easy to show that

$$\eta(x_1,x_2,\ldots,x_m)=[\xi(x_{11})\xi(x_{21})\ldots\xi(x_{m1})\ldots\xi(x_{1n})\xi(x_{2n})\ldots \ldots\xi(x_{mn})]_1 \qquad (2.13)$$

for every $(x_1,x_2,\ldots,x_m)\in D$ with $x_i=x_{i1}x_{i2}\ldots x_{in}$, $x_{ij}\in Q$, for each $i\in N_m$, $j\in N_n$. This implies that $\xi$ is a (2m,m)-homomorphism from $(Q;[\ ])$ into $(P;[\ ])$. ◊

3. Proofs of Proposition 2.1, 2.2, 2.3

Proof of Proposition 2.1. (a) In (1)-(4) the right hand sides have a form $\phi(v)$ where $|v| < t$ and by the inductive hypothesis $\phi(v)$ is well defined, wich imply that $\phi(u)$ is well defined. By the inductive hypothesis $|\phi(v)| \le |v|$, $\phi^2(v)=\phi(v)$, and so $|\phi(u)| < |u|$ and $\phi^2(u)=\phi(u)$.

(b) It follows directly from the definition.

(c) Follows from (a). ◊

Proof of Proposition 2.2. (a) If $\phi_1(x_i)=x_i$ for each $i\in N_m$, the conclusion is obvious. If there is $i\in N_m$, such that $x_i=zx$ and $\phi(x)\neq x$, then (a) follows directly from (I).

(b) Follows directly from (a) and the condition (2.3). ◊

Proof of Proposition 2.3. The proof is given via the following four lemmas, whose proofs are by induction on the length.

Lemma 3.1. If $u=(x_1\pi_1(v),x_2\pi_2(v),\dots,x_m\pi_m(v))$ and $\phi(v)=v$, where $x_i \in D^{(n)}$, $n\geq 1$, $v=(y_1,y_2,\dots,y_m)$, $y_i\in D^{(k)}$, $k\geq 2$, for each $i\in N_m$, then

$$\phi(u) = \phi(x_1y_1,x_2y_2,\dots,x_my_m).$$

Proof. (A) If (I) is applicable on u then $(\phi_1(x_1),\phi_1(x_2),\dots,\dots,\phi_1(x_m))\neq(x_1,x_2,\dots,x_m)$, and the conclusion follows from (I), the fact that $\phi(v)=v$ and the inductive hypothesis.

(B) Let $(\phi_1(x_1),\phi_1(x_2),\dots,\phi_1(x_m))=(x_1,x_2,\dots,x_m)$. Then (II) is applicable on u and the conclusion follows from (II). ◊

Remark. If (II) is applicable on an element $u=(x_1,x_2,\dots,x_m)$, we write $u\neq(\phi_2(x_1),\phi_2(x_2),\dots,\phi_2(x_m))$, where $(\phi_2(x_1),\phi_2(x_2),\dots,\dots,\phi_2(x_m))$ denotes the element obtained from u by one application of (II).

Lemma 3.2. If $u=(x_1e,x_2e,\dots,x_me)$, $x_i\in D^{(n)}$, $n\geq 1$, $i\in N_m$, then

$$\phi(u) = \phi(x_1,x_2,\dots,x_m).$$

Proof. (A) If (I) is applicable on u, then $(\phi_1(x_1),\phi_1(x_2),\dots,\dots,\phi_1(x_m))\neq(x_1,x_2,\dots,x_m)$, and the conclusion follows from (I), the condition (2.4) and the inductive hypothesis.

(B) Let $(\phi_1(x_1),\phi_1(x_2),\dots,\phi_1(x_m)=(x_1,x_2,\dots,x_m)$. If (II) is applicable on u, then $(x_1,x_2,\dots,x_m)\neq(\phi_2(x_1),\phi_2(x_2),\dots,\phi_2(x_m))$, and the conclusion follows from (II) and the inductive hypothesis.

(C) Let $(x_1,x_2,\dots,x_m)=(\phi_2(x_1),\phi_2(x_2),\dots,\phi_2(x_m))$. Then (III) is applicable on u, and the conclusion follows from (III). ◊

Remark. If (III) is applicable on $u=(x_1,x_2,\dots,x_m)$, we write $u\neq(\phi_3(x_1),\phi_3(x_2),\dots,\phi_3(x_m))$, where $(\phi_3(x_1),\phi_3(x_2),\dots,\phi_3(x_m))$ is only a notation for the element obtained from u by one application od (III).

Lemma 3.3. If $u=(x_1aa',x_2aa',\dots,x_maa')$, then $\phi(u)=$ $=\phi(x_1e,x_2e,\dots,x_me)$.

Proof. (A) If (I) is applicable on u, then $(x_1,x_2,\ldots,x_m)\neq$ $\neq(\phi_1(x_1),\phi_1(x_2),\ldots,\phi_m(x_m))$, and the conclusion follows from (I), the condition (2.4) and the inductive hypothesis.

(B) Let $(x_1,x_2,\ldots,x_m)=(\phi_1(x_1),\phi_1(x_2),\ldots,\phi_1(x_m))$. If (II) or (III) is applicable on u, then, $(x_1,x_2,\ldots,x_m)\neq(\phi_j(x_1),\phi_j(x_2),\ldots,$ $,\ldots,\phi_j(x_m))$, $j\in\{2,3\}$, and the conclusion follows from Lemma 3.(j-1), and the inductive hypothesis.

(C) Let $(x_1,x_2,\ldots,x_m)=(\phi_j(x_1),\phi_j(x_2),\ldots,\phi_j(x_m))$. Then (IV) is applicable on u, and the conclusion follows directly from (IV). ◊

Remark. If (IV) is applicable on $u=(x_1,x_2,\ldots,x_m)$, we write $u\neq(\phi_4(x_1),\phi_4(x_2),\ldots,\phi_4(x_m))$, where $(\phi_4(x_1),\phi_4(x_2),\ldots,\phi_4(x_m))$ is only a notation for the element obtained from u by one applications of (IV).

Lemma 3.4. If $u=(x_1\pi_1(v),x_2\pi_2(v),\ldots,x_m\pi_m(v)$, $x_i\in D^n$, $n\geq 1$, $v=(y_1,y_2,\ldots,y_m)$, $y_i\in D^{(k)}$, $k\geq 2$, $i\in\mathbb{N}_m$, then

$$\phi(u) = \phi(x_1y_1,x_2y_2,\ldots,x_my_m).$$

Proof. (A) If $\phi(v)=v$, then conclusion follows directly from Lemma 3.1.

(B) If $\phi(v)\neq v$, then $v\neq(\phi_j(y_1),\phi_j(y_2),\ldots,\phi_j(y_n))$, for some $j\in\mathbb{N}_4$.

(B.1) For j=1, Proposition 2.2 and the inductive hypothesis imply that

$$\phi(u)=\phi(x_1\phi(\pi_1(v)),x_2\phi(\pi_2(v)),\ldots,x_m\phi(\pi_m(v)))=$$
$$=\phi(x_1\phi_1(y_1),x_2\phi_1(y_2),\ldots,x_m\phi_1(y_m))=\phi(x_1y_1,x_2y_2,\ldots,x_my_m).$$

(B.2) If (I) is not applicable on u, then $j\in\{2,3,4\}$ and, similarly to case (B.1), the conclusion follows from Proposition 2.2, Lemma 2.(j-1), and the inductive hypothesis. ◊

## R E F E R E N C E S


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## СЛОБОДНИ КОМУТАТИВНИ (2m,m)-ГРУПИ

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Р е з и м е

Комутативна (2m,m)-група е пар (G,[ ]) каде што G е непразно множество, а $[\ ]:G^{2m} \to G^m$ е (2m,m)-операција на G, така што $G^m$ со операцијата $x \circ y = [xy]$ е комутативна група, и за секој $x \in G^i$, $y \in G^{2m-i}$, $z \in G^i$ и $v \in G^{m-i}$, важи $[[xy]zv]=[x[yz]v]$.

Во оваа работа даваме комбинаторен опис на слободни комутативни (2m,m)-групи (за $m \geq 2$), инспириран од карактеризацијата на комутативните (2m,m)-групи како алгебри со една (2m,m)-операција, една унарна и една нуларна операција.