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POST AND HOSSZÚ-GLUSKIN THEOREM FOR VECTOR VALUED GROUPS

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The notion of an (m+k,m)-group was first introduced in [1], as a generalization of the notion of an n-group. Here we generalize the Post theorem for embedding of an n-group into a group ([6]) and the Hosszú-Gluskin theorem for representation of an n-group by a group ([4], [5]). Namely, in Theorem P we show that every (m+k,m)-group (Q;[ ]) is embeddible into a group (G;·) such that $Q\subseteq G$ and $[a_1^{m+k}]=b_1^m \iff a_1\cdot a_2\cdot\ldots\cdot a_{m+k}=b_1\cdot b_2\cdot\ldots\cdot b_m$ for all $a_\lambda, b_\nu\in Q$. Using this result in Theorem HG we show that every (m+k,m)-group (Q;[ ]) can be represented by a group $(Q^m;*)$. As a corrolary of these results (for m=1) we have that Hosszú-Gluskin theorem is a consequence of Post coset theorem. It is notified (in [3]) that Post had proven the Hosszú-Gluskin theorem in [6], but his proof is, in a way, given in [6] ambiguously.

First we will give some preliminary notations and definitions. If A is a nonempty set, the elements of the n-th Cartesian power A^n of A will be denoted by $(a_1,\ldots,a_n)$, or shortly by a_1^n; for n=0 we define $A^0=\{0\}$. Also, a_r^s is a notation for $(a_r,a_{r+1},\ldots,a_s)$ if $s\geq r$, and the empty symbol if $r>s$. In the case when A is a subset of a semigroup $\underline{S}=(S;\cdot)$, then for $n\geq 1$, we put $A_n=\{a_1\cdot\ldots\cdot a_n \mid a_\nu\in A\}$. This product will be, as usual, written without the operation symbol.

Thus, if $\emptyset\neq A\subseteq S$,

$$A^n=\{a_1^n=(a_1,\ldots,a_n)\mid a_i\in A\},\ n\geq 0$$

$$A_n=\{a_1\ldots a_n\mid a_i\in A\},\ n\geq 1.$$

If $a_1=a_2=\ldots=a_n=a\in A$, then $\overset{n}{a}=a_1^n$, and $a^n=a_1\ldots a_n$.

The free semigroup with a basis A, where A is a nonempty set, is denoted by A^+, and in this case

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This paper is in final form and no version of it will be submited for publication elsewhere.

$$A^{+} = \bigcup_{n\geq 1} A^{n},\ a_1^{i}\cdot a_{i+1}^{i+j} = a_1^{i+j}$$

for each $a_\nu \in A$, $i,j \geq 1$. If $a \in A^{+}$, then we define a <u>length</u> of a, $|a|$, by

$$|a| = n \text{ iff } a \in A^{n}.$$

We denote by $\mathbb{N}$ the set $\{1,2,\ldots\}$ of positive integers, and by $\mathbb{N}_r$ the set $\{1,2,\ldots,r\}$, for each $r \in \mathbb{N}$.

Here we will also give the formulations of the Post and Hosszú-Gluskin theorems and a definition of an $(m+k,m)$-group. Namely, Post theorem states that each n-group $(Q;[\ ])$ can be embedded into a group $(G;\cdot)$ such that $Q \subseteq G$ and for each $a_i \in Q$

$$[a_1^n] = a_1 \cdot a_2 \cdot \ldots \cdot a_n.$$

The Hosszú-Gluskin theorem gives a representation of an n-group $(Q;[\ ])$ by a group $(Q;\cdot)$ with an automorphism θ on $(Q;\cdot)$ and a fixed element $c \in Q$ such that for each $a, a_i \in Q$

$$\theta(c)=c,\ \theta^{n-1}(a)=cac^{-1},\ [a_1^n]=a_1\theta(a_2)\ldots\theta^{n-1}(a_n)c.$$

Let m,k be positive integers, $[\ ]:Q^{m+k} \to Q^{m}$ a mapping. We say that the pair $\underline{Q}=(Q;[\ ])$ is an $(m+k,m)$-<u>group</u> (or a <u>vector valued group</u>) if $[\ ]$ is associative, i.e. for all $a_j \in Q$

$$[[a_1^{m+k}]a_{m+k+1}^{m+2k}] = [a_1^{i}[a_{i+1}^{i+m+k}]a_{i+m+k+1}^{m+2k}],\ i \in \mathbb{N}_k,$$

and the equations

$$[xa_1^k] = b_1^m = [a_1^k y]$$

have solutions $x,y \in Q^m$, for each $a_1^k \in Q^k$, $b_1^m \in Q^m$.

<u>1</u>. Let $\underline{G}=(G;\cdot)$ be a group with the unity e, and m,k be positive integers. We say that the subset $Q \subseteq G$ is an $(m+k,m)$-<u>subgroup</u> of $\underline{G}$ iff the following conditions hold:

(I) The mapping $a_1^m \mapsto a_1 \ldots a_m$ is a bijection from Q^m into Q_m.

Note that if (I) holds then the mappings $a_1^r \mapsto a_1 \ldots a_r$ are bijections from Q^r into Q_r, for each $r \in \mathbb{N}_m$. In this case we will identify a_1^r and $a_1 \ldots a_r$, i.e. we will assume $Q_r = Q^r$, for each $r \in \mathbb{N}_m$.

(II) $(\forall a \in Q_k)\, aQ_m = Q_m$,

where $aQ_m = \{aa_1 \ldots a_m \mid a_i \in Q\}$.

To each (m+k,m)-subgroup Q of a group G we associate a mapping $[\![\ ]\!] : Q^{m+k} \to Q^m$ defined by:

$$[\![a_1^{m+k}]\!] = b_1^m \iff a_1 \ldots a_{m+k} = b_1 \ldots b_m \qquad (1.1)$$

for each $a_\nu, b_\lambda \in Q$. $[\![\ ]\!]$ is a well defined (m+k,m)-operation on Q, as (I) and (II) hold for Q. The pair $(Q;[\![\ ]\!])$ is said to be an (m+k,m)-groupoid. We will show that $(Q;[\![\ ]\!])$ is an (m+k,m)-group, but first we will give some properties of (m+k,m)-subgroups of a group G.

1.1°. (a) $Q_{m+k} = Q_m$.

(b) $Q_i Q_j = Q_{i+j}$, for each $i,j \geq 1$.

(c) $i+j = m+k+r,\ r \geq 0 \implies Q_i Q_j = Q_{m+r}$, for each $i,j \geq 1$. □

1.2°. (a) $sk \geq m \implies (e \in Q_{sk}\ \&\ (\forall a \in Q)\, a \in Q_{sk+1})$.

(b) $sk > m,\ a \in Q \implies a^{-1} \in Q_{sk-1}$.

(c) $H = \bigcup_{i \geq 1} Q_i$ is a subgroup of the group G.

(d) $H = Q^m \cup Q_{m+1} \cup \ldots \cup Q_{m+k-1}$.

Proof. (a) By 1.1° (c) and (II), for each $a \in Q$ there exist $x_1, \ldots, x_m \in Q$ such that

$$a^{sk} x_1 \ldots x_m = a^m,$$

where $s \geq 1$. If $sk \geq m$ we have

$$a^{sk-m} x_1 \ldots x_m = e. \qquad (1.2)$$

Thus $e \in Q_{sk}$ and

$$a = a^{sk-m+1} x_1 \ldots x_m \in Q_{sk+1}.$$

(b) If $sk > m$ we obtain from (1.2)

$$a^{-1} = a^{sk-m-1}x_1 \dots x_m \in Q_{sk-1}.$$

(c) is a consequence of (a) and (b).

(d) is a consequence of 1.1° (c), (a) and the fact that there exist an s such that $m \le sk < m+k$. □

<u>1.3°</u>. (a) $i \ge 1,\ j \ge m,\ a \in Q_i \Longrightarrow aQ_j = Q_j a = Q_{i+j}$.

(b) $(\forall a \in Q_k)\ aQ_m = Q_m = Q_m a$.

<u>Proof</u>. (a) It is obvious that $aQ_j \subseteq Q_{i+j}$, $Q_j a \subseteq Q_{i+j}$ for $a \in Q_i$. Let $a = a_1 \dots a_i$, $a_\lambda \in Q$, and $b_1 \dots b_{i+j} \in Q_{i+j}$, $b_\nu \in Q$. The equation $xa_1 \dots a_i = b_1 \dots b_{i+j}$ has a solution $x \in G$, and $x \in H$, since H is a subgroup of <u>G</u>. Let $sk > m$. Then $x = b_1 \dots b_{i+j} a_i^{-1} \dots a_1^{-1} \in$ $\in Q_{i+j+i(sk-1)} = Q_{j+isk} = Q_j$. Thus $Q_{i+j} \subseteq Q_j a$, and by symmetry $Q_{i+j} \subseteq aQ_j$. □

<u>1.4°</u>. <u>If</u> Q <u>is</u> <u>an</u> (m+k,m)-<u>subgroup</u> <u>of</u> <u>a</u> <u>group</u> <u>G</u> <u>then</u> <u>the</u> <u>induced</u> (m+k,m)-<u>groupoid</u> $(Q;[\![\ ]\!])$ (<u>defined</u> <u>by</u> (1.1)) <u>is</u> <u>an</u> (m+k,m)-<u>group</u>.

<u>Proof</u>. Let $a_\lambda \in Q$ and let $[\![a_1^{m+k}]\!] = b_1^m$, $[\![a_{i+1}^{i+m+k}]\!] = c_1^m$, i.e. $a_1 \dots a_{m+k} = b_1 \dots b_m$, $a_{i+1} \dots a_{i+m+k} = c_1 \dots c_m$ in <u>G</u>. Then we have

$$[\![[\![a_1^{m+k}]\!]a_{m+k+1}^{m+2k}]\!] = [\![b_1^m a_{m+k+1}^{m+2k}]\!] = d_1^m \Longleftrightarrow d_1 \dots d_m =$$
$$= b_1 \dots b_m a_{m+k+1} \dots a_{m+2k} = a_1 \dots a_{m+k} a_{m+k+1} \dots a_{m+2k} =$$
$$= a_1 \dots a_i c_1 \dots c_m a_{m+k+i+1} \dots a_{m+2k} \Longleftrightarrow d_1^m =$$
$$= [\![a_1^i\ [\![a_{i+1}^{m+k+i}]\!] a_{m+k+i+1}^{m+2k}]\!]$$

for each $i \in \mathbb{N}_k$.

The solubility of the equations

$$[\![xa_1^k]\!] = b_1^m = [\![a_1^k y]\!]$$

for $a_\nu, b_\lambda \in Q$ is a consequence of 1.3° (b). □

<u>1.5°</u>. <u>If</u> $Q=\{a\}$ <u>is</u> <u>a</u> <u>one</u> <u>element</u> <u>subset</u> <u>of</u> <u>a</u> <u>group</u> <u>G</u>, <u>then</u> Q <u>is</u> <u>an</u> (m+k,m)-<u>subgroup</u> <u>of</u> G <u>iff</u> <u>the</u> <u>order</u> <u>of</u> a <u>devides</u> k.

Proof. For each $m \geq 1$, $a^{m+k}=a^m$ iff $a^k=1$ iff the order of a devides k. □

1.6°. (a) If $|Q| \geq 2$ and Q is an (m+k,m)-subgroup of the group G, then $aQ_{m-1} \subset Q_m$, for each $a \in Q$.

(b) If $|Q| \geq 2$ and Q is an $(m'+k',m')$-, and $(m''+k'',m'')$-subgroup of a group G, then $m'=m''$.

Proof. (a) Let $a,b \in Q$, $a \neq b$. It is clear that $aQ_{m-1} \subseteq Q_m$. Suppose $aQ_{m-1}=Q_m$. Then, there exist $a_\lambda \in Q$, such that $aa_1 \ldots a_{m-1}=b^m$, which, by (I), implies $a=b$.

(b) Let $m' < m''$, i.e. $m' \leq m''-1$. Then $m'+t=m''-1$, for some $t \geq 0$, and for each $a \in Q$

$$aQ_{m''-1} = aQ_{m'+t} = Q_{m'+t+1} = Q_{m''}$$

which contradicts the result in (a). □

1.7°. If Q is an (m+k,m)-subgroup of a group G and $m \leq sk < m+k$, $sk=m+p$, then Q_{m+p} is an invariant subgroup of the subgroup H of G.

Proof. By 1.1° (c) $Q_{m+p}Q_{m+p} = Q_{m+p+sk} = Q_{m+p}$, and by 1.2° (a), $e \in Q_{m+p}$. Let $a_\lambda \in Q$ and $tk > m$. Then by 1.2° (b) we have $a_\lambda^{-1} \in Q_{tk-1}$ and thus

$$(a_1 \ldots a_{sk})^{-1} = a_{sk}^{-1} \ldots a_1^{-1} \in Q_{sk(tk-1)} = Q_{sk}.$$

$Q_{sk} = Q_{m+p}$ is an invariant subgroup of H by 1.3° (a). □

1.8°. Let k be the least positive integer such that Q is an (m+k,m)-subgroup of G. Then

(a) Q is an $(m+k',m)$-subgroup of G iff $k|k'$.

(b) $m \leq i < j < m+k \Longrightarrow Q_i \cap Q_j = \emptyset$.

(c) $H/Q_{m+p} \cong \mathbb{Z}_k$, where $\mathbb{Z}_k$ is the cyclic group of order k.

Proof. (a) If $k|k'$ then Q is obviously an $(m+k',m)$-subgroup of G. Let Q be an $(m+k',m)$-subgroup of G, where $k'=rk+t$, $0 < t < k$. Then for each $a \in Q_t$

$$aQ_m = Q_{m+t} = Q_{m+t+rk} = Q_{m+k'} = Q_m$$

i.e. Q is an (m+t,m)-subgroup of G, contradicting the choice of k.

(b) Since Q_{m+p} is an invariant subgroup of H, by 1.3° we have

$$H/Q_{m+p} = \{xQ_{m+p} \mid x\in H\} = \{Q_{m+i} \mid 0 \le i < k\},$$

which implies that the sets $Q_m, Q_{m+1}, \ldots, Q_{m+k-1}$ are either equal or pairwise disjoint. Let $Q_{m+t} = Q_{m+r}$, $k > t > r \ge 0$. Then for some $a\in Q_r$

$$aQ_{m+t-r} = Q_{m+t} = Q_{m+r} = aQ_m,$$

which implies $Q_{m+t-r} = Q_m$. Thus Q is an (m+t-r,m)-subgroup of G, contradicting the choice of k.

(c) By 1.2° (d) and (b) we obtain that the mapping $\phi: a_1 \ldots a_{m+i} \mapsto i-p$ is an epimorphism from H into $\mathbb{Z}_k$, with $\ker\phi = Q_{m+p}$. □

We say that a group G is a <u>covering group</u> of its (m+k,m)-subgroup Q if G is generated by Q, i.e.

(III) $G = Q_m \cup Q_{m+1} \cup \ldots \cup Q_{m+k-1}$.

If, moreover, G satisfies the following condition

(IV) $m \le i < j < m+k \implies Q_i \cap Q_j = \emptyset$

then we say that G is a <u>universal covering group</u> of its (m+k,m)-subgroup Q. The universal covering group of Q will be denoted by Q^V.

<u>1.9°</u> <u>Let Q be an (m+k,m)-subgroup of $G=Q^V$, Q' an (m+k,m)-subgroup of a group (G';*) and $\lambda: Q \to Q'$ a map, such that for all $a_i, b_j \in Q$</u>

$$a_1 \ldots a_{m+k} = b_1 \ldots b_m \iff \lambda(a_1) * \ldots * \lambda(a_{m+k}) = \lambda(b_1) * \ldots * \lambda(b_m).$$

<u>Then there exists a unique homomorphism $\xi: G \to G'$ which is an extension of λ.</u> □

As a corollary of 1.8° we have

1.10°. *If* k *is the least positive integer such that* Q *is an* $(m+k,m)$*-subgroup of the group* $\underline{G}$, *and* $\underline{G}$ *is a covering group for* Q, *then* $\underline{G}$ *is a universal covering group for* Q. □

Let us note that by 1.3° the following is also true:

1.11°. *If* $\underline{G}$ *is a universal covering group of its* $(m+k,m)$*-subgroup* Q, *then for each* $a \in Q$

$$G = Q^m \cup aQ^m \cup \ldots \cup a^{k-1}Q^m = $$
$$= Q^m \cup Q^m a \cup \ldots \cup Q^m a^{k-1}. \quad \square$$

2. Let $\underline{Q}=(Q;[\ ])$ be a given $(m+k,m)$-group. We will construct a group $\underline{G}=(G;\cdot)$ such that $Q\subseteq G$ is an $(m+k,m)$-subgroup of $\underline{G}$, and $\underline{G}$ is its universal covering group. The $(m+k,m)$-operation $[\![\ ]\!]$ induced by the $(m+k,m)$-subgroup Q, defined by (1.1), will coincides with the operation $[\ ]$ of the given $(m+k,m)$-group $\underline{Q}$.

Further on by $\underline{Q}=(Q;[\ ])$ a given $(m+k,m)$-group will be denoted. By ([2], pg. 27) $\underline{Q}$ satisfies the general associative law, and the "product" $[a_1^{m+sk}]$ is defined for all $s \geq 1$. Also, $\underline{Q}$ is cancellative, i.e.

$$[a_1^{i-1} x_1^m a_i^k] = [a_1^{i-1} y_1^m a_i^k] \Longrightarrow x_1^m = y_1^m \qquad (2.1)$$

for each $i \in \mathbb{N}_{k+1}$, and $a_\lambda, x_\nu, y_\mu \in Q$ (see [2], pg. 54). By (2.1), for each $x,y \in Q^i$, $ab, cd \in Q^{m+sk-i}$, $i \geq 1$,

$$[axb] = [ayb] \Longrightarrow [cxd] = [cyd], \qquad (2.2)$$

(see [2], pg. 37).

Let $\underline{Q}=(Q;[\ ])$ be a given $(m+k,m)$-group. Define a relation $\sim$ on Q^+ by:

$$(\forall u,v \in Q^+)(u \sim v \iff (\exists w \in Q^+)[uw]=[vw]), \qquad (2.3)$$

where $[uw]$ and $[vw]$ denote that $uw \in Q^{m+sk}$, $vw \in Q^{m+tk}$ for some $s,t \geq 0$, and we put $[a_1^m] = a_1^m$ for $a_i \in Q$.

2.1°. (a) $u \sim v \Longrightarrow |u| \equiv |v| \pmod{k}$.

(b) The relation $\sim$ is a congruence on Q^+.

(c) $Q^+/\sim$ is a group.

(d) $a,b \in Q$, $a \sim b \Longrightarrow a=b$ (i.e. we can concider Q as a subset of $Q^+/\sim$).

Proof. (a) $u \sim v \Longrightarrow (\exists w \in Q^+)[uw]=[vw] \Longrightarrow |uw| \equiv |vw| \pmod{k} \Longrightarrow |u| \equiv |v| \pmod{k}$.

(b) Note that by (2.2) and (2.3) it follows that

$$u \sim v \Longrightarrow [tuw] = [tvw] \qquad (2.4)$$

for all $t,w \in Q^+$ such that $|tuw| \equiv |tvw| \equiv m \pmod{k}$. Now by (2.4) we obtain that $\sim$ is a congruence on Q^+.

(c) We will show that the equations $ux \sim v$ and $zu \sim v$ have solutions on x and z for every $u,v \in Q^+$. If $|v| < m$ then for some $w,t \in Q^+$ we have $|wv| = m$ and $|wut| = sk$, $s \geq 1$. Now, since in the $(m+k,m)$-group $\underline{Q}$ the equation $[wuty] = wv$ has a solution $y \in Q^m$, we obtain that $x=ty$ is a solution of $ux \sim v$. In the other case, when $|v| \geq m$, we have $v=v'v''$ where $|v'| = m$, and the equation $[uty] = v'$ has a solution $y \in Q^m$ for some $t \in Q^r$. Now $x=tyv''$ is a solution of $ux \sim v$. Similarly we solve the equation $zu \sim v$.

(d) Let $a,b \in Q$, $a \sim b$. Then by (2.4) $[\overset{m}{a}]=[b\overset{m-1}{a}]$, i.e. $\overset{m}{a}=b\overset{m-1}{a}$ in Q^m, which implies $a=b$. □

2.2°. $Q^+/\sim = Q^V$.

Proof. We will show that the conditions (I)-(IV) are fulfiled for $Q^+/\sim$. It is clear that (I) holds for $Q^+/\sim$, as if $a_1^m \sim b_1^m$ in $Q^+/\sim$ it follows that $[a_1^m w] = [b_1^m w]$, which (by cancellativity of the $(m+k,m)$-operation [ ]) implies $a_1^m = b_1^m$.

Let $a=a_1\ldots a_k \in Q_k$ and $b=b_1\ldots b_m \in Q_m$ $(a_\nu, b_\lambda \in Q)$. Now, as $[a_1^k b_1^m] = c_1^m \in Q^m$, $ab \sim c_1 \ldots c_m$, and thus $ab \in Q_m$, i.e. $aQ_m \subseteq Q_m$. If $c=c_1\ldots c_m \in Q_m$ is given, then for each $a \in Q_k$ the equation

$[ax]=c$ has a solution $x \in Q_m$, i.e. $c \sim ax$. Thus $c \in aQ_m$, i.e. $Q_m \subseteq aQ_m$, i.e. (II) is satisfied. As Q generates $Q^+/\sim$ we have (III), and (IV) follows by 2.1° (a). □

Theorem P.[1] Let $(Q;[\ ])$ be an $(m+k,m)$-group. Then there exists a group $(G;\cdot)$ such that $Q \subseteq G$ and for each $a_i, b_j \in Q$

$$[a_1^{k+m}] = b_1^m \iff a_1 \dots a_{k+m} = b_1 \dots b_m.$$

Proof. Take $G = Q^+/\sim$. □

We note that for each $(m+k,m)$-group $(Q;[\ ])$, as a consequence of the results above, the $(m+k,m)$-operation $[\![\ ]\!]$ (defined by (1.1)) and $[\ ]$ coincides, i.e. for each $a_i \in Q$,

$$[\![a_1^{m+k}]\!] = [a_1^{m+k}]. \tag{2.5}$$

3. We have seen in 2 that the $(m+k,m)$-group $(Q;[\ ])$ coincides with the $(m+k,m)$-group $(Q;[\![\ ]\!])$ induced by the $(m+k,m)$-subgroup Q of $Q^V = Q^+/\sim$. From now on we will denote by a a fixed element from Q and Q^V will be given in the form

$$Q^V = Q^m \cup Q^m a \cup Q^m a^2 \cup \dots \cup Q^m a^{k-1}. \tag{3.1}$$

The product in Q^V is defined by

$$x_1^m a^i \cdot y_1^m a^j = z_1^m a^{i \oplus j}$$

where $\oplus$ is the operation in the cyclic group Z_k, and z_1^m is a solution of the equation

$$[x_1^m a^i y_1^m a^{k+j-(i \oplus j)}] = [z_1^m a^k].$$

The inverse of the element $x \in Q^V$ will be denoted by x^{-1}; and $Q_{m+p} = Q^m a^p$ is the invariant subgroup of Q^V, where $m+p \equiv 0 \pmod{k}$, $0 \le p < k$.

3.1° (a) Defining an operation $*$ on Q^m by

$$x_1^m * y_1^m = [x_1^m a^p y_1^m], \tag{3.2}$$

for each $x_1^m, y_1^m \in Q^m$, a group $(Q^m;*)$ is obtained, with a unity a^{-p} and the inverse b^{-*} of $b \in Q^m$ defined by $b^{-*} = a^{-p} b^{-1} a^{-p}$.

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[1] This property is given in [2].

(b) <u>The mapping</u> $\theta: x \mapsto a^{-p}xa^{p}$ <u>is an automorphism of</u> $(Q^m;*)$ <u>such that</u>

$$\theta^n(x) = a^{-np}xa^{np} \tag{3.3}$$

<u>for each</u> $x \in Q^m$, $n \geq 1$.

<u>Proof</u>. (a) By (2.5) we have

$$x_1^m * y_1^m = x_1 \ldots x_m a^p y_1 \ldots y_m$$

which implies the associativity of $*$. It is easy to check that a^{-p} is the unity, and $a^{-p}b_m^{-1}\ldots b_1^{-1}a^{-p}$ is the inverse of $b_1^m \in Q^m$. □

Note that if $b \in Q$, then $b \in Q^m a^{p+1}$. Now, if $k=tm+r$, $0 \leq r < m$, then for each $x \in Q^r$ there exists $y \in Q^m$ such that $x = ya^{p+r}$. Define a mapping $\phi: Q^r \to Q^m$ by

$$\phi(x) = ya^{r-tp}. \tag{3.4}$$

(If $r=0$, then $Q^0 = \{0\}$, and $\phi(0) = a^{-tp-p}$.)

<u>3.2°</u>. ϕ <u>is an injection from</u> Q^r <u>into</u> Q^m. □

Let $z \in Q^r$, $x_\lambda \in Q^m$ and $x = x_0 x_1 \ldots x_t z \in Q^{m+k}$. Using θ and ϕ defined as above we obtain

$$\begin{aligned}
& x_0 * \theta(x_1) * \theta^2(x_2) * \ldots * \theta^t(x_t) * \theta^{t+1}(\phi(z)) = \\
&= x_0 a^p a^{-p} x_1 a^p a^p a^{-2p} x_2 a^{2p} a^p \ldots \\
&\quad \ldots a^p a^{-tp} x_t a^{tp} a^p a^{-(t+1)p} \phi(z) a^{(t+1)p} = \\
&= x_0 x_1 \ldots x_t z a^{-(p+r)} a^{r-tp} a^{(t+1)p} = \\
&= x_1 \ldots x_t z \\
&= [x].
\end{aligned}$$

Thus we have proven the following

<u>Theorem HG</u>. <u>Let</u> $(Q;[\ ])$ <u>be an</u> $(m+k,m)$-<u>group, where</u> $k=tm+r$, $0 \leq r < m$. <u>Then there exists a group</u> $(Q^m;*)$, <u>an automorphism</u> $\theta \in Aut(Q^m;*)$ <u>and an injection</u> $\phi: Q^r \to Q^m$ <u>such that for each</u> $x_i \in Q^m$, $z \in Q^r$ <u>the equality</u>

$$[x_0 \ldots x_t z] = x_0 * \theta(x_1) * \theta^2(x_2) * \ldots * \theta^t(x_t) * \theta^{t+1}(\phi(z)) \quad (3.5)$$

holds. Furthermore, if $r=0$, then

$$\theta(\phi(0)) = \phi(0), \quad (3.6)$$

$$\theta^t(x) = \phi(0) * x * \phi(0)^{-*} \quad (3.7)$$

for each $x \in Q^m$. □

In the case $m=1$, $k=n-1$, the notion of $(n,1)$-group coincides with the notion of n-group. Thus the theorem of Hosszú-Gluskin for representation of an n-group by a group is a special case of Theorem HG. In the case of n-groups the converse is also valid, i.e. if $(G;*)$ is a group, θ an automorphism of $(G;\cdot)$ and $\phi(0) \in G$, such that (3.6) and (3.7) are valid then by (3.5) an n-ary operation [ ] on G is defined such that (G;[ ]) is an n-group.

In the vector valued version of Hosszú-Gluskin theorem the converse is not generally valid, because even when $r=0$, the $(m+k,m)$-operation [ ] defined by (3.5) need not be associative (although it satisfies the condition for solubility of equations when $t \geq 1$).

Note that Theorem HG is a consequence of Theorem P. Thus in the n-ary case (when $m=1$) we obtain that the Post coset theorem implies the Hosszú-Gluskin Theorem.

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