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POLYADIC OPERATIONS INDUCED BY SEMIGROUPS

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Abstract. Any operation (binary, n-ary or (n,m)-ary) on a set Q can be considered as a partial transformation of the free semigroup Q^+ with a basis Q. Replacing Q^+ by an arbitrary semigroup $\underline{S}=(S;\cdot)$ such that $Q\subseteq S$, one can consider ($\underline{S}$;n,m)-groupoids on Q. The notions of subgroupoids and homomorphisms (and some related topics) are considered.

1. ($\underline{S}$;n,m)-groupoids. Let $\underline{S}=(S;\cdot)$ be a semigroup, Q a nonempty subset of S, and p a positive integer. Denote by Q_p the set $\{a_1\cdot a_2\cdot\ldots\cdot a_p \mid a_\nu\in Q\}$. If n and m are two positive integers, and $f:Q_n\to Q_m$ a mapping, then we say that f is an ($\underline{S}$;n,m)-operation on Q (or an $\underline{S}$-polyadic operation), and the pair (Q;f) will be called an (S;n,m)-groupoid.

Example 1.1. Let Q^+ be the free semigroup with a basis Q, i.e. $Q^+=\bigcup\{Q^p \mid p\geq 1\}$, where Q^p is the p-th cartezian power of Q, and the operation on Q^+ is the usual concatenation of words. Then a (Q^+;n,m)-groupoid is the same as an (n,m)-groupoid ([2]).

Example 1.2. Let $Q^{\oplus}$ be the free commutative semigroup with a basis Q, i.e. $Q^{\oplus}=Q^+/\approx$ where $\approx$ is the least congruence on Q^+ such that $Q^+/\approx$ is commutative. (Namely, if $a_\nu, b_\lambda\in Q$ then:

$$a_1a_2\ldots a_r \approx b_1b_2\ldots b_s$$

iff r=s and $b_1b_2\ldots b_r$ is a permutation of $a_1a_2\ldots a_r$.) A ($Q^{\oplus}$;n,m)-groupoid is the same as a fully commutative (n,m)-groupoid ([3], [5]).

Example 1.3. If SL(Q) is a free semilattice with a basis Q, then an (SL(Q);n,m)-groupoid (Q;f) will be called a semilattice (n,m)-groupoid.

Let us give a more convenient description of semilattice (n,m)-groupoids. Denote first by F(Q) the family of the finite nonempty subsets of Q, and if $p\geq 1$, then put $F_p(Q)=\{X\in F(Q) \mid |X|\leq p\}$. Then, if $f:F_n(Q)\to F_m(Q)$ we obtain a semilattice (n,m)-groupoid (Q;f).

Let $(Q;f)$ be an $(S;n,m)$-groupoid. Then, there exists a unique homomorphism $\pi:Q^+ \to S$ such that $\pi(b)=b$, for every $b\in Q$, and moreover we have $\pi(Q^p)=Q_p$, for every $p\geq 1$, and thus π induces a surjective mapping $\pi_p:Q^p \to Q_p$.

<u>Proposition 1.4</u>. If $g:Q^n \to Q^m$ is an (n,m)-operation on Q, then there exists at most one $(\underline{S};n,m)$-operation $\overline{g}:Q_n \to Q_m$ on Q such that the following diagram commutes:

$$\begin{array}{ccc} Q^n & \xrightarrow{\ g\ } & Q^m \\ {\scriptstyle \pi_n}\downarrow & & \downarrow{\scriptstyle \pi_m} \\ Q_n & \xrightarrow{\ \overline{g}\ } & Q_m \end{array}$$

Such an operation $\overline{g}$ do exists iff the following implication holds:

$$\pi_n(x) = \pi_n(y) \implies \pi_m(g(x)) = \pi_m(g(y)),$$

for every $x,y\in Q^n$. ⌘

(We say that $\overline{g}$ is <u>induced</u> by g.)

<u>Proposition 1.5</u>. If $(Q;f)$ is an $(\underline{S};n,m)$-groupoid then there exists an (n,m)-groupoid $(Q;g)$ such that $\overline{g}=f$, and the above diagram is commutative.

<u>Proof</u>. Let $x\in Q^n$. Then $f(\pi_n(x))\in Q_m$, and $\pi_m^{-1}f(\pi_n(x))\subseteq Q^m$. Choose a $y\in\pi_m^{-1}f(\pi_n(x))$ and put $y=g(x)$. Then, $f\pi_n=\pi_n g$. ⌘

<u>Remark 1.6</u>. In the case when $\underline{S}$ is a commutative semigroup (a semilattice), we can also state analogically propositions as in <u>1.4</u> and <u>1.5</u>.

<u>2</u>. <u>Subgroupoids</u>. Let $(Q;f)$ be an $(\underline{S};n,m)$-groupoid, and P a nonempty subset of Q. We say that P is a <u>subgroupoid</u> of $(Q;f)$ iff $f(P_n)\subseteq P_m$. And, P is called a <u>strong subgroupoid</u> of $(Q;f)$ iff for all $a_i\in P$, $b_j\in Q$

$$f(a_1\cdot a_2\cdot\ldots\cdot a_n)=b_1\cdot b_2\cdot\ldots\cdot b_m \implies b_1,b_2,\ldots,b_n\in P.$$

<u>Proposition 2.1</u>. Every strong subgroupoid of $(Q;f)$ is a subgroupoid of $(Q;f)$. ⌘

Proposition 2.2. If $\{P_i \mid i\in I\}$ is a family of strong subgroupoids of $(Q;f)$ and if $P=\bigcap\{P_i \mid i\in I\}\neq\emptyset$, then P is a strong subgroupoid of $(Q;f)$. ⌘

Example 2.3. Let $Q=\{a,b,c\}$, and let $\underline{S}$ be the semigroup given by the following presentation $\langle a,b,c;b^2=c^2\rangle$. Define an $(\underline{S};1,2)$-groupoid $(Q;f)$ by:

$$f(a) = f(b) = f(c) = b^2 \quad (=c^2).$$

Then, $B=\{a,b\}$ and $C=\{a,c\}$ are subgroupoids of $(Q;f)$, but $A=B\cap C=\{a\}$ is not a subgroupoid. (Thus, a nonempty intersection of subgroupoids is not necessarily a subgroupoid.)

Proposition 2.2 implies that every nonempty subset D of an $(\underline{S};n,m)$-groupoid generates a uniquelly determined strong subgroupoid $\langle D\rangle$, and from Ex. 2.3 it follows that this is not true in the case of subgroupoids. (Namely, we have

$$\langle a\rangle = \langle a,b\rangle=\langle a,c\rangle=\langle a,b,c\rangle = \{a,b,c\},$$

$$\langle b\rangle = \langle c\rangle=\langle b,c\rangle = \{b,c\},$$

if we consider strong subgroupoids, but there does not exist a (unique) subgroupoid of $(Q;f)$ generated by $\{a\}$.)

Remark 2.4. If $(Q;f)$ is an (n,m)-groupoid (a fully commutative (n,m)-groupoid, a semilattice (n,m)-groupoid) then $P\subseteq Q$ is a subgroupoid iff it is a strong subgroupoid.

Remark 2.5. When we say that a subset B of Q is a generating subset of $(Q;f)$, then we mean that if P is a subgroupoid of $(Q;f)$ such that $B\subseteq P$, then $P=Q$. And, B is called weakly generating subset of $(Q;f)$ iff $\langle B\rangle=Q$, i.e. if B generates $(Q;f)$ as a strong subgroupoid.

3. Homomorphisms. Consider an $(\underline{S};n,m)$-groupoid $(Q;f)$ and an $(\underline{S}';n,m)$-groupoid $(Q';f')$. A mapping $\phi:c\mapsto c'$ from Q into Q' is called a homomorphism from $(Q;f)$ into $(Q';f')$ iff for any $a_i,b_j\in Q$ the following implication is true:

$$f(a_1\cdot\ldots\cdot a_n) = b_1\cdot\ldots\cdot b_m \Longrightarrow f'(a_1'\cdot\ldots\cdot a_n') = b_1'\cdot\ldots\cdot b_m'.$$

If, furthermore, ϕ is bijective and ϕ^{-1} is a homomorphism, then ϕ is called an isomorphism.

Example 3.1. Let Q be a set with at least two distinct elements, and let n,m be two positive integers such that $m \geq 2$. Denote by π the canonical homomomorphism from Q^+ into $Q^{\oplus}$. Assume that $g:Q^n \to Q^m$ is a $(Q^+;n,m)$-operation such that

$$x,y \in Q^n,\ \pi(x)=\pi(y) \implies g(x)=g(y)$$

and that there exist an $x \in Q^n$ and $a,b \in Q$, $a \neq b$, such that $g(x)=aby$, where $y \in Q^{m-2}$. (In the case m=2, y is the "empty" symbol.) Then g induces a fully commutative (n,m)-operation $\overline{g}$, and the identity transformation $c \mapsto c$ of G is a bijective homomorphism from $(Q;g)$ into $(Q;\overline{g})$, but it is not an isomorphism.

Proposition 3.2. Let $(Q;f)$ be an $(S;n,m)$-groupoid, $(Q';f')$ be an $(S';n,m)$-groupoid, and $\phi:Q \to Q'$ be a homomorphism.

(i) If P is a subgroupoid of $(Q;f)$, then $\phi(P)$ is a subgroupoid of $(Q';f')$.

(ii) If P' is a strong subgroupoid of $(Q';f')$ and if $\phi^{-1}(P') \neq \emptyset$, then $\phi^{-1}(P')$ is a strong subgroupoid of $(Q;f)$.

Proof. (i) Let $a_1',a_2',\ldots,a_n' \in P'$. Then there exist $a_1,a_2,\ldots,a_n \in P$ and $b_1,b_2,\ldots,b_m \in P$ such that $a_i'=\phi(a_i)$, $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$. From the second equality it follows $f(a_1'\cdot\ldots\cdot a_n')=b_1'\cdot\ldots\cdot b_m'$, where $b_j'=\phi(b_j)$.

(ii) Let $a_1,a_2,\ldots,a_n \in \phi^{-1}(P')$, $b_1,b_2,\ldots,b_m \in Q$ are such that $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$. Then we have $f'(a_1'\cdot\ldots\cdot a_n')=$ $=b_1'\cdot\ldots\cdot b_m'$, where $\phi(a_i)=a_i' \in P'$, and this implies that $b_1',\ldots,b_m' \in P'$, where $\phi(b_i)=b_i'$, i.e. $b_1,\ldots,b_m \in \phi^{-1}(P')$. ∎

Example 3.3. Let $Q=\{a,b,c\}$, $Q'=\{\alpha,\beta\}$ and let V be the variety of commutative semigroups, satisfying the identity $x^2=y^2$. Let $\underline{S}$ ($\underline{S}'$) be a free semigroup in V with a basis $Q(Q')$. Then we have:

$$Q_2 = \{ab,ac,bc,a^2=b^2=c^2\}$$

$$Q_4=\{a^4=b^4=c^4=a^2b^2=a^2c^2=b^2c^2,a^2bc=b^3c=bc^3,ab^2c=a^3c=ac^3,abc^2=ab^3=a^3b\}$$

$$Q_2' = \{\alpha^2=\beta^2,\alpha\beta\},\quad Q_4' = \{\alpha^4=\beta^4=\alpha^2\beta^2,\ \alpha^3\beta=\alpha\beta^3\}.$$

Define an $(\underline{S};4,2)$-groupoid $(Q;f)$ by $f(u)=bc$, for all $u\in Q_4$, and an $(\underline{S}';4,2)$-groupoid $(Q';g)$ by $f'(u')=\alpha^2$, for all $u'\in Q'_4$. Then, the mapping $\phi=\binom{a\ b\ c}{\alpha\ \beta\ \beta}$ is a homomorphism from $(Q;f)$ into $(Q';f')$, $A'=\{\alpha\}$ is a subgroupoid of $(Q';f')$, but $A=\{a\}=\phi^{-1}(A')$ is not a subgroupoid of $(Q;f)$. On the other hand, $\{b,c\}=D$ is a strong subgroupoid of $(Q;f)$, but $\phi(D)=D'=\{\beta\}$ is not a strong subgroupoid of $(Q';f')$.

4. Equivalences. It is natural to ask the question when we should say that an $(\underline{S};n,m)$-groupoid $(Q;f)$ is equal to an $(\underline{S}';n,m)$-groupoid $(Q';f')$. Certainly, we have the follwing two trivial answers:

a) $S = S'$, $Q = Q'$, $f = f'$;

b) $(Q;f)$ and $(Q';f')$ are isomorphic.

But, the first condition is too strong, and the second one is too "abstract". We chose the following "meadle way".

We say that an $(\underline{S};n,m)$-groupoid $(Q;f)$ is equivalent to an $(\underline{S}';n,m)$-groupoid $(Q';f')$ iff $Q=Q'$ and the identity transformation of Q is an isomorphism from $(Q;f)$ into $(Q';f')$. Then, we write $(Q;f) \equiv (Q;f')$. In other words:

$$(Q;f) \equiv (Q';f') \text{ iff } Q = Q'$$

and, for any $a_i,b_j\in Q$,

$$f(a_1\cdot\ldots\cdot a_n) = b_1\cdot\ldots\cdot b_m \iff$$
$$f(a_1*\ldots*a_n) = b_1*\ldots*b_m,$$

where $\underline{S} = (S;\cdot)$, $\underline{S}' = (S';*)$.

Proposition 4.1. Let $(Q;f)$ be an $(\underline{S};n,m)$-groupoid, and let $\underline{T}$ be the subsemigroup of $\underline{S}$ generated by Q. Define a $(\underline{T};n,m)$-groupoid $(Q;f')$ by:

$$f'(a_1\cdot a_2\cdot\ldots\cdot a_n) = f(a_1\cdot a_2\cdot\ldots\cdot a_n),$$

for any $a_i\in Q$. Then: $(Q;f) \equiv (Q';f')$. ■

(So, we can always assume that Q is a generating subset of $\underline{S}$, whenever an $(\underline{S};n,m)$-groupoid $(Q;f)$ is considered.)

Example 4.2. Let $(Q;f)$ be an $(\underline{S};n,m)$-groupoid, and let $p=\max\{m,n\}$, $0\notin S$. Assume also that $R=Q\cup Q_2\cup\ldots\cup Q_p\neq S$. Define on operation $*$ on $T=R\cup\{0\}$ as follows:

$$x*y=\begin{cases} z, \text{ if } xy=z \text{ in } \underline{S} \text{ and } z\in R,\\ 0, \text{ otherwise.}\end{cases}$$

Then $\underline{T}=(T;*)$ is a semigroup generated by Q, and if a $(\underline{T};n,m)$-operation f' is defined on Q by:

$$f'(a_1*\ldots*a_n)=f(a_1\cdot\ldots\cdot a_n),$$

then we obtain that $(Q;f)\equiv(Q;f')$. ∎

Assume now that $(Q;f)$ is an arbitrary $(\underline{S};n,m)$-groupoid, and let $\pi:Q^+\to S$, $\pi_n:Q^n\to Q_n$, $\pi_m:Q^m\to Q_m$, be defined as in part $\underline{1}$. If

$$\sigma_n:Q^n/\ker\pi_n\to Q_n,\quad \sigma_m:Q^m/\ker\pi_m\to Q_m$$

are corresponding canonical bijections, then we can define a mapping:

$$\overline{f}:Q^n/\ker\pi_n\to Q^m/\ker\pi_m,$$

as follows:

$$\overline{f}=\sigma_m^{-1}f\sigma_n.$$

Proposition 4.3. If $(Q;f)$ is an $(\underline{S};n,m)$-groupoid and $(Q;f')$ is an $(\underline{S}';n,m)$-groupoid such that $\overline{f}=\overline{f'}$, then $(Q;f)\equiv(Q;f')$.

Proof. The equality $\overline{f}=\overline{f'}$ implies

$$\sigma_m^{-1}f\sigma_n=\sigma_m'^{-1}f'\sigma_n'$$

where $\pi':Q^+\to S'$. This means that $Q^n/\ker\pi_n=Q^n/\ker\pi_n'$, i.e. $a_1\cdot\ldots\cdot a_n=a_1*\ldots*a_n$ for every $a_i\in Q$. Now, let $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$, where $a_i,b_j\in Q$. Then

$$f'(a_1*\ldots*a_n)=\sigma_m'\sigma_m^{-1}f(a_1\cdot\ldots\cdot a_n)=$$
$$=\sigma_m'\sigma_m^{-1}(b_1\cdot\ldots\cdot b_m)=\sigma_m'((b_1,\ldots,b_m)^{\ker\pi}n),$$

where $(b_1,\ldots,b_m)\in Q^m$. It follows that

$$\pi_n(b_1,\ldots,b_m)=\pi_m'(b_1,\ldots,b_m)=b_1*\ldots*b_m, \text{ i.e.}$$
$$f'(a_1*\ldots*a_n)=b_1*\ldots*b_m.\ ∎$$

Example 4.4. Let $Q=\{a,b\}$, $0\notin Q$, and let $S=\{a,b,(a,a),(a,b),(b,a),(b,b),0\}$, $S'=\{a,b,0\}$, and let the semigroups $\underline{S}=(S;\cdot)$, $\underline{S}'=(S';*)$ be defined as follows:

$$x*y = 0 \text{ for any } x,y\in S';$$

$$x\cdot y = \begin{cases}(x,y), \text{ if } x,y\in Q\\ 0, \text{ otherwise.}\end{cases}$$

Define an $(\underline{S};2,3)$-operation f, and an $(\underline{S}';2,3)$-operation f' by

$$f(u) = 0, \quad f'(u') = 0$$

for every $u\in Q_2=\{(a,a),(a,b),(b,a),(b,b)\}$, $u'\in Q_2'=\{0\}$. Then we have: $(Q;f) \equiv (Q;f')$, but $\overline{f}\neq\overline{f'}$. (Namely, $Q_2=Q^2$, and π_2 is the identity mapping, which implies that:

$$\overline{f} = \begin{pmatrix}\{(a,a)\}, & \{(a,b)\}, & \{(b,a)\}, & \{(b,b)\}\\ Q^3 & Q^3 & Q^3 & Q^3\end{pmatrix}.$$

We also have $Q^2/\ker\pi_2' = Q^2$, and thus $\overline{f'} = \begin{pmatrix}Q^2\\Q^3\end{pmatrix}$.)

5. Regularity. An $(\underline{S};n,m)$-groupoid $(Q;f)$ is said to be trivial iff $Q_n\subseteq Q_m$, and $f(u)=u$, for every $u\in Q_n$, i.e. f is the imbedding from Q_n into Q_m. And, $(Q;f)$ is said to be regular iff there is a trivial $(\underline{S}';n,m)$-groupoid $(Q;f')$ such that $(Q;f) \equiv (Q;f')$.

In what follows we suppose that $(Q;f)$ is an $(\underline{S};n,m)$-groupoid, and $Q^*=Q^+\cup\{1\}$, $(1\notin S)$ is the free monoid with a basis Q (and 1 as its unit). If $a_i,a_{i+1},\dots,a_{i+p}\in Q$ then a_i^{i+p} is the corresponding product of $a_i,\dots,a_{i+p}$ in Q^+, and $a_{i+1}^i=1$.

Define a relation $\approx$ on Q^+ as follows.

If $u,v\in Q^+$, then:

$$u \vdash v \iff (\exists u',u''\in Q^*,a_i,b_j\in Q)(u=u^1a_1^nu'',v=u'b_1^mu'',$$
$$f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m),$$

$$u \sim v \iff u\vdash v \text{ or } v\vdash u,$$

$$u \approx v \iff (\exists p\geq 0,u_0,\dots,u_p\in Q^+)\ u=u_0\sim u_1\sim\ldots\sim u_p = v.$$

(Thus, $\sim$ is the symmetric closure of $\vdash$, and $\approx$ is the reflexive and transitive closure of $\sim$.)

Proposition 5.1. The relation $\approx$ is a congruence on Q^+. ✗

Denote by $\underline{Q}^\wedge$ the corresponding quotient semigroup $Q^+/\approx$. Note that $\underline{Q}^\wedge$ is the semigroup given by the presentation

$$\langle Q;\{a_1\ldots a_n=b_1\ldots b_m \mid f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m\}\rangle \qquad (5.1)$$

in the class of semigroups.

If $u\in Q^+$, then we will denote by $\overline{u}$ the $\approx$-equivalence class containing u, i.e. $\overline{u}=\{v\in Q^+ \mid u \approx v\}$. In this sence the set $\{\overline{a} \mid a\in Q\}$, will be denoted by $\overline{Q}$. This implies that, for each $p\geq 1$, we have $\overline{Q}_p=\{a_1^p \mid a_1,\ldots,a_p\in Q\}$, and that $\overline{Q}$ is a generating subset of $\underline{Q}^\wedge$.

Proposition 5.2. $\overline{Q}_n \subseteq \overline{Q}_m$.

Proof. If $a_1,\ldots,a_n\in Q$, and $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$ then $a_1^n \approx b_1^m$, and thus $a_1^n=b_1^m\in\overline{Q}_m$. ✗

Thus, we can define a $(\underline{Q}^\wedge;n,m)$-operation $\hat{f}$ on $\overline{Q}$, by: $\hat{f}(u)=u$, for every $u\in\overline{Q}_n$.

Proposition 5.3. $(\overline{Q};\hat{f})$ is a trivial $(\underline{Q}^\wedge;n,m)$-groupoid, and the mapping $-:a\mapsto\overline{a}$ is a homomorphism from $(Q;f)$ into $(\overline{Q};\hat{f})$. ✗

We will show now that $(\overline{Q};\hat{f})$ admits a corresponding universal property.

Proposition 5.4. Let ϕ be a homomorphism from $(Q;f)$ into a trivial $(\underline{S}';n,m)$-groupoid $(Q';f')$. Then, by $\overline{\phi}(\overline{a})=\phi(a)$ is defined a homomorphism $\overline{\phi}$ from $(\overline{Q};\hat{f})$ into $(Q';f')$.

Proof. Denote the operation of $\underline{S}'$ by $*$. If $a_1,\ldots,a_n$, $b_1,\ldots,b_m\in Q$ are such that $f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m$ in $(Q;f)$, then we have:

$$\phi(a_1)*\ldots*\phi(a_n)=f'(\phi(a_1)*\ldots*\phi(a_n))=\phi(b_1)*\ldots*\phi(b_m),$$

because ϕ is a homomorphism, and $(Q';f')$ is trivial. This, and the fact that $\underline{Q}^\wedge$ has a presentation (5.1), implies that there exists a unique homomorphism $\hat{\phi}:\underline{Q}^\wedge\to\underline{S}'$ such that $(\forall a\in Q)\,\hat{\phi}(a)=\phi(a)$.

The restriction $\overline{\phi}$ of $\hat{\phi}$ on $\overline{Q}$ is a homomorphism from $(\overline{Q};\hat{f})$ into $(Q';f')$ defined by $(\forall a\in Q)\ \overline{\phi}(a)=\phi(a)$. ✗

Proposition 5.5. $(Q;f)$ is regular iff $-:a \mapsto \bar{a}$ is an isomorphism from $(Q;f)$ into $(\bar{Q};\hat{f})$.

Proof. Let $(Q;f)$ be regular, i.e. there exists a trivial $(\underline{S}';n,m)$-groupoid $(Q;f')$ such that $1=1_a:a \mapsto a$ is an isomorphism. By Pr. 4.4, $\bar{1}:\bar{a} \mapsto a$ is a homomorphism from $(\bar{Q};\hat{f})$ into $(Q;f')$. This implies that $\bar{1}$ is bijective, and therefore $-:a \mapsto \bar{a}$ is a bijective homomorphism. Moreover, we have that $\bar{a} \overset{1}{\mapsto} a \overset{1}{\mapsto} a$, and this implies that $-:a \mapsto \bar{a}$ is an isomorphism from $(Q;f)$ into $(\bar{Q};\hat{f})$. Conversely, let $-:a \mapsto \bar{a}$ be an isomorphism from $(Q;f)$ onto $(\bar{Q};f)$. Then, we can assume that $Q=\bar{Q} \subseteq Q^{\wedge}$, and thus $(Q;f) \equiv (Q;f)$. ⌘

Proposition 5.6. $(Q;f)$ is regular iff the following conditions are satisfied:

(i) $(\forall a,b \in Q)(a \approx b \Longrightarrow a=b)$;

(ii) $(\forall a_i,b_j \in Q)(a_1^n \approx b_1^m \Longrightarrow f(a_1\cdot\ldots\cdot a_n)=b_1\cdot\ldots\cdot b_m))$. ⌘

Although Pr. 5.5 and Pr. 5.6 give complete descriptions of regular $(\underline{S};n,m)$-groupoids, these descriptions are not satisfactory enough. In the case when $\underline{S}=Q^+$ and $n>m$ we have a convenient answers, and namely the following statement holds:

Proposition 5.7. ([4]) If $n>m$, then a $(Q^+;n,m)$-groupoid $(Q;f)$ is regular iff it is an (n,m)-semigroup. ⌘

This result suggests to define the notion of $(\underline{S};n,m)$-semigroup, for arbitrary semigroup $\underline{S}$ and positive integers n,m, such that $n>m$. In the paper [4] it is also given a satisfactory description of regular $(Q^+;1,m)$-groupoids, but until now we do not have a convenient result in the case $1<n<m$.

The answer in the case $n=m$ is quite clear. Namely, we have the following

Proposition 5.8. An $(S;n,n)$-groupoid $(Q;f)$ is regular iff it is trivial.

Proof. If $(Q;f)$ is regular, then by Pr. 5.6 (ii) we have:

$$a_1^n = a_1^n \Longrightarrow f(a_1\cdot\ldots\cdot a_n)=a_1\cdot\ldots\cdot a_n,$$

i.e. (Q;f) is trivial. ∎

R E F E R E N C E S

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ПОЛИАДИЧНИ ОПЕРАЦИИ ИНДУЦИРАНИ ОД ПОЛУГРУПИ

Ѓ. Чупона и С. Марковски

Секоја операција (бинарна, n-арна или (n,m)-арна) на множество Q може да се разгледува како делумна трансформација на слободна полугрупа Q^+ со база Q. Заменувајќи ја Q^+ со произволна полугрупа $\underline{S}=(S;\cdot)$ таква што $Q\subseteq S$, се доаѓа до поимот $(\underline{S};n,m)$-групоид над Q. Во овој труд се разгледуваат поимите подгрупоиди и хомоморфизми на $(\underline{S};n,m)$-групоиди.