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FULLY COMMUTATIVE VECTOR VALUED GROUPOIDS

Ǵ.Čupona, A.Samardžiski, N.Celakoski

Abstract. *The notion of "commutative vector valued operation" is modified in this paper such that the range of the operation is factorized under commutativity. Namely, if Q is a nonempty set and r is a positive integer, then $Q^{(r)}=Q^r/\approx$, where*

$a,b\in Q^r \implies (a \approx b \iff b$ *is a permutation of* a).

Every mapping $f\colon Q^{(n)}\to Q^{(m)}$ *is called a fully commutative (n,m)-operation and $\underline{Q}=(Q;f)$ is called a fully commutative (n,m)-groupoid (shortly: f.c.g.).*

A description of the free generated f.c.g. is given and a result different from the usual algebras is obtained here. Namely, if $\underline{Q}$ is a free f.c. (n,m)-groupoid $(m\geq 2)$ with a basis B, then the identity mapping on B can be extended to infinitely many automorphisms on $\underline{Q}$. We discuss the notion of fully commutative (n,m)-quasigroups (shortly: f.c.q.) and we give a description of the free f.c.q. by using the notion of partial f.c.q. Finally, finite f.c.q. are considered and some examples of finite f.c.q. are given.

1. FULLY COMMUTATIVE (n,m)-OPERATIONS

If Q is a nonempty set and n,m are positive integers, then any mapping $f\colon Q^n\to Q^m$ is called an (n,m)-operation or a vector valued operation. (Here, Q^r is the r-th Cartesian power, i.e.

$$Q^r = \{(a_1,a_2,\ldots,a_r) \mid a_\nu\in Q\};$$

the elements of Q^r will be denoted also by $a_{\alpha+1}^{\alpha+r}$, where $a_\nu\in Q$, $\alpha\geq 0$ and sometimes by a single letter a.)

An (n,m)-operation f is said to be commutative ([4], §2) iff for every permutation σ of the set $\mathbb{N}_n=\{1,2,\ldots,n\}$ the following identity holds:

This paper is in final form and no version of it will be submited for publication elsewhere.

$$f(a_1^n) = f(\sigma(a_1^n)), \text{ where } \sigma(a_1^n) = a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)}.$$

More generally, f is said to be <u>weakly commutative</u> iff for every $a_1^n \in Q^n$ and a permutation b_1^n of a_1^n, the following implication is true:

$$f(a_1^n) = c_1^m,\ f(b_1^n) = d_1^m \Longrightarrow d_1^m \text{ is a permutation of } c_1^m.$$

Here we will consider another kind of vector valued operations which we call "fully commutative (n,m)-operations".

Namely, let $r \geq 1$ and let $\equiv$ be a relation in Q^r defined by:

$$a_1^r \equiv b_1^r \text{ iff } b_1^r \text{ is a permutation of } a_1^r.$$

It is clear that $\equiv$ is an equivalence in Q^r. The factor set $Q^r/\equiv$, denoted by $Q^{(r)}$, will be called "the commutative r-th power of Q". The elements of $Q^{(r)}$ will be denoted again by $a_{\alpha+1}^{\alpha+r}$, where $a_\nu \in Q$ and $\alpha \geq 0$, but now:

$$a_{\alpha+1}^{\alpha+r} = b_{\beta+1}^{\beta+r} \Longleftrightarrow b_{\beta+1}^{\beta+r} \text{ is a permutation of } a_{\alpha+1}^{\alpha+r}.$$

If $n,m \geq 1$, then every mapping $f\colon Q^{(n)} \to Q^{(m)}$ will be called a <u>fully commutative</u> (n,m)-<u>operation</u>.

Let $f\colon Q^n \to Q^m$ be a given (n,m)-operation. It is natural to ask the following question:

Under what conditions there exists a fully commutative (n,m)-operation $f'\colon Q^{(n)} \to Q^{(m)}$ ("induced by f") such that the following diagram is commutative:

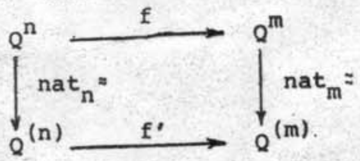

Diagram 1

where $nat_r \equiv$ is the canonical mapping from Q^r into $Q^{(r)}$?

Conversely, if $f'\colon Q^{(n)} \to Q^{(m)}$ is a given fully commutative (n,m)-operation, one can ask the question of existence of (n,m)-operation $f\colon Q^n \to Q^m$, such that the above diagram is commutative.

Consider a more general situation, i.e. the diagram

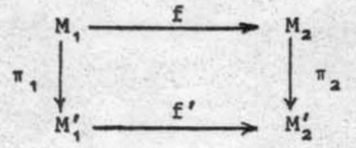

Diagram 2

where M_i, M'_i (i=1,2) are sets. It is easy to show the following:

PROPOSITION 1.1. Let $\pi_i: M_i \to M'_i$ be a surjection for $i=1,2$.

(i) If $f: M_1 \to M_2$ is a mapping, then there exists at most one mapping $f': M'_1 \to M'_2$ such that Diagram 2 is commutative, i.e. $\pi_2 f = f'\pi_1$.

Such a mapping f' do exist iff the following condition is satisfied:

$$(\forall x,y \in M_1)(\pi_1(x)=\pi_1(y) \Rightarrow \pi_2(f(x)) = \pi_2(f(y))). \qquad (1.1)$$

(ii) If $f': M'_1 \to M'_2$ is a mapping, then there exists a mapping $f: M_1 \to M_2$ such that $\pi_2 f = f'\pi_1$.

In general, there are more than one such mappings f, defined in the following way:

$$f = \bigcup_{x'\in M'_1} f_{x'}, \text{ where } f_{x'}: \pi_1^{-1}(x') \to \pi_2^{-1}(f'(x'))$$

is arbitrary. ▯

In the special case when $f: Q^n \to Q^m$ is an (n,m)-operation, the condition (1.1) has the following meaning: if y is a permutation of x, then f(y) is a permutation of f(x). Thus, we have the following:

PROPOSITION 1.2. Let $f: Q^n \to Q^m$ be an (n,m)-operation on a nonempty set Q. There exists at most one fully commutative (n,m)-operation $f': Q^{(n)} \to Q^{(m)}$ on Q such that the diagram 1 is commutative. Such an operation f' exists iff f is weakly commutative.

Conversely, any fully commutative (n,m)-operation f' is induced by a set of weakly commutative (n,m)-operations, between which there are commutative ones.

If $f\colon Q^{(n)} \to Q^{(m)}$ is a fully commutative (n,m)-operation, then $\underline{Q}=(Q;f)$ will be called a <u>fully commutative</u> (n,m)-<u>groupoid</u>. Further on, we will consider fully commutative (n,m)-groupoids (or - operations) only. Therefore we will usually omit the words "fully commutative"; also, the integers n,m will be usually considered as fixed and so we will often say simply "groupoid" (or "operation") instead of "fully commutative (n,m)-groupoid" (or "fully commutative (n,m)-operation").

We will introduce here several concepts which will be used later.

Let $\underline{Q}=(Q;f)$ be a groupoid and H a nonempty subset of Q. H is called a <u>subgroupoid</u> of $\underline{Q}$, in notation $H \le \underline{Q}$, iff

$$a_1^n \in H^{(n)} \implies f(a_1^n) \in H^{(m)}.$$

Clearly, the following proposition is true:

PROPOSITION 1.3. If $\{H_\alpha \mid \alpha \in A\}$ *is a nonempty family of subgroupoids of a groupoid* $\underline{Q}$ *and if* $H = \bigcap_\alpha H_\alpha$ *is a nonempy set, then* H *is a subgroupoid of* $\underline{Q}$. ▯

A subgroupoid H of a groupoid $\underline{Q}$ is said to be <u>generated</u> by a nonempty subset B of Q iff

(i) $B \subseteq Q$, (ii) $K \le \underline{Q}$ & $B \subseteq K \implies H \le K$.

Proposition 1.3 implies that:

PROPOSITION 1.4. If $\underline{Q}$ *is a groupoid and B is a nonempty subset of Q, then there exists a uniquely determined subgroupoid of* $\underline{Q}$ *which is generated by B.* ▯

A description of the subgroupoid of $\underline{Q}$ generated by a set $B \subseteq Q$ can be given in the following way. Let $B_0, B_1, B_2, \ldots$ be a sequence of subsets of Q defined as follows:

$$B_0 = B, \quad B_{p+1} = B_p \cup C_{p+1},$$

where $C_{p+1} = \{b \in Q \setminus B_p \mid (\exists a_1^n \in B_p^{(n)}) f(a_1^n) = b\, b_2^m\}$. Then the set $\langle B\rangle = \bigcup_{p \ge 0} B_p$ is the subgroupoid of $\underline{Q}$ generated by B.

To every element $c \in \langle B\rangle$ we assign a number $\chi_B(c)$, called the <u>hierarchy</u> of c, <u>relative</u> to B, defined by:

$$\chi_B(c) = \min\{p \mid c \in B_p\}.$$

The notion of homomorphism can be introduced in a usual way. Namely, let $\underline{Q}=(Q;f)$ and $\underline{Q}'=(Q';f')$ be groupoids and ϕ a mapping from Q into Q'. We say that ϕ is a homomorphism from $\underline{Q}$ into $\underline{Q}'$ iff

$$f(a_1^n) = b_1^m \Longrightarrow f'(\bar{a}_1^n) = \bar{b}_1^m,$$

where $\bar{c}=\phi(c)$, $c\in Q$. If, in addition, ϕ is bijective, then ϕ is called an isomorphism. It is easy to show that:

PROPOSITION 1.5. ϕ: $\underline{Q} \to \underline{Q}'$ *is an isomorphism iff* ϕ^{-1}: $\underline{Q}' \to \underline{Q}$ *is an isomorphism.* ▯

The notions of an endomorphism and automorphism have the usual meanings.

2. FREE FULLY COMMUTATIVE (n,m)-GROUPOIDS

We will give here a description of free fully commutative (n,m)-groupoids which we will call, shortly again, free groupoids.

A groupoid $\underline{Q}=(Q;f)$ is said to be free with a basis B iff the following conditions are satisfied:

(i) B is a generating set for $\underline{Q}$;

(ii) if $\underline{Q}'=(Q';f')$ is a groupoid and ψ: $B \to Q'$, then there exists a homomorphism ϕ: $\underline{Q} \to \underline{Q}'$ which is an extension of ψ.

In order to give a description of free groupoids, let B be a nonempty set and let $(B_p \mid p \geq 0)$ be a sequence of sets defined as follows:

$$B_o = B, \quad B_{p+1} = B_p \cup \mathbb{N}_m \times B_p^{(n)},$$

where $\mathbb{N}_m$ denotes the set $\{1,2,\ldots,m\}$. If $u\in B_{p+1}\setminus B_p$, then we say that u has the hierarchy $p+1$ and we write $\chi(u)=p+1$; if $b\in B$, then we set $\chi(b) = 0$.

Let $[B]=\bigcup_{p\geq o} B_p$ and define an (n,m)-operation $f:[B]^{(n)} \to [B]^{(m)}$ by:

$$f(u_1^n) = (1,u_1^n)(2,u_1^n)\ldots(m,u_1^n). \qquad (2.1)$$

So we obtain a groupoid $\underline{[B]}=([B];f)$ with a generating set B. Here, the notion of hierarchy of $u\in[B]$ coincides with the notion of the hierarchy relative to B introduced in 1.

Suppose now that $\underline{Q}'=(Q';f)$ is a groupoid and $\psi: B \to Q'$ an arbitrary mapping from B into Q'. We will show that there exists a homomorphism $\phi: [\underline{B}] \to \underline{Q}'$ which is an extension of ψ.

First, for $b\in B$, we set $\phi(b)=\psi(b)$. Suppose that $\phi(u)=\bar{u}\in Q'$ is a well defined element of Q' if $u\in Q$ has a hierarchy $\le p$. If $v\in[B]$ has a hierarchy p+1, then v has the form $v=(i,u_1^n)$, where $i\in N_m$, $u_1^n\in[B]^{(n)}$, $\chi(u_\nu)\le p$ and $\chi(u_\alpha)=p$ for some α. Then, setting $v_j=(j,u_1^n)$, we obtain that $\chi(v_j)=p+1$ for every $j\in N_m$. Since $\bar{u}_1^n\in Q'^{(n)}$, there exists $c_1^m\in Q'^{(m)}$ such that $f'(\bar{u}_1^n)=c_1^m$. Then, if we put $\phi(v_j)=c_j$, we obtain that $\phi(v_j)\in Q'$ is a well-defined element for every $j\in N_m$.

(Note that, in general, there are many ways of defining $\phi(v_1),\phi(v_2),\ldots,\phi(v_m)$, but not more than m!)

Thus, by induction on hierarchy, we defined a mapping $\phi: [B] \to Q'$ which is an extension of ψ.

By the definitions of $f: [B]^{(n)} \to [B]^{(m)}$ and $\phi: [B] \to Q'$, it is clear that $\phi: [\underline{B}] \to \underline{Q}'$ is a homomorphism. Thus, we proved the following:

PROPOSITION 2.1. *$[\underline{B}]$ is a free groupoid with a basis B.* ▯

Now we will prove that:

PROPOSITION 2.2. If ξ is an endomorphism on $[\underline{B}]$ such that

$$(\forall b\in B)\ \xi(b) = b, \tag{2.2}$$

then ξ is an automorphism on $[\underline{B}]$.

<u>Proof</u>. If $p\ge 0$, then we denote by S_p the set $\{u\in[B] \mid \chi(u)=p\}$. By the above assumption, ξ induces the identity bijection from S_o into S_o. Suppose that ξ induces a bijection from the set S_p into S_p. Let $v\in S_{p+1}$. Then v has the form $v=(i,u_1^n)$ for some $i\in N_m$ and $u_1^n\in S_p^{(n)}$, where there exists $\nu\in N_m$, such that $\chi(u_\nu)=p$. Put $v_\alpha=(\alpha,u_1^n)$. Then $f(u_1^n)=v_1^m$, and thus $f(\bar{u}_1^n)=\bar{v}_1^m$, where $\xi(u_\nu)=\bar{u}_\nu\in S_p$, $\bar{v}_\alpha=\xi(v_\alpha)=\xi(v)$. Now, $\xi(v)=(j,\bar{u}_1^n)$, where $\xi(u_\nu)=\bar{u}_\nu$ and $j\in N_m$. Therefore, using the hypothesis that $\xi(S_p)=S_p$, we have $\xi(v)\in S_{p+1}$. This implies that if $\xi(v)=\xi(w)$, then $w=(s,u_1^n)$ for some $s\in N_m$. Setting $v_\alpha=(\alpha,u_1^n)$, we obtain that $\xi(v_\alpha)=(\ell_\alpha,\bar{u}_1^n)$, where $\alpha\mapsto\ell_\alpha$ is a permutation of N_m, and this implies that

$$\xi(v) = \xi(w) \Longrightarrow v = w.$$

Thus the restriction of ξ on S_{p+1} is an injection. It remains to show that this restriction is a surjection. Let $u=(i,u_1^n)\in S_{p+1}$. Then $u_\mu \in S_p$, and thus there exist $v_\mu \in S_p$ such that $\xi(v_\mu)=(u_\mu)$. If we put $w_\beta=(\beta,v_1^n)$, then we obtain that there exists $\gamma\in\mathbb{N}_m$ such that $\xi(w_\gamma)=u$. This completes the proof that ξ is a bijection and thus an automorphism.

(Note that the set of automorphisms ξ on $[\underline{B}]$, with (2.2) is infinite.) ▯

If $\underline{Q}=(Q;f)$ is an another free (n,m)-groupoid with a basis B, then there exist homomorphisms ζ: $[\underline{B}] \to \underline{Q}$, η: $\underline{Q} \to [\underline{B}]$, such that

$$(\forall b\in B)\ \zeta(b) = \eta(b) = b. \tag{2.3}$$

Clearly, $\xi=\eta\zeta$ is an endomorphism on $[\underline{B}]$ with the property (2.2). Thus ξ is an automorphism on $[\underline{B}]$, which implies that ζ is an injective homomorphism.

By induction on hierarchy of elements of $Q=\langle B\rangle$ we will show that ζ is surjective as well. Let $c\in Q$ has the hierarchy p+1 (relative to B). Then there exist $c_1^m\in Q^{(m)}$ such that $c_i=c$ for some $i\in\mathbb{N}_m$, and $d_1^n\in Q^{(n)}$ such that $g(d_1^n)=c_1^m$ and $d_1^n\in T_p^{(n)}$, where $T_p=\{d\in Q \mid \chi_B(d) \le p\}$. These assumptions imply that there exists $u_1^n\in[B]^{(n)}$ such that $\zeta(u_\nu)=d_\nu$. If $f(u_1^n)=v_1^m$, then $f(d_1^n)=\overline{v}_1^m$, where $\overline{v}_\alpha=\zeta(v_\alpha)$. Thus $c_1^m=\overline{v}_1^m$, i.e. $c=\overline{v}_\alpha=\zeta(v_\alpha)$ for some α, which proves that ζ is surjective.

We will restate the above results (P.2.1, P.2.2 and the last one) as the following:

THEOREM 2.3. (i) Every nonempty set B is a basis of a free fully commutative (n,m)-groupoid.

(ii) If B is a basis of a free fully commutative (n,m)-groupoid, then the set of its automorphisms which fix the all elements of B is infinite.

(iii) Free groupoids with a same basis are isomorphic. ▯

(We note that (ii) is something different from the usual algebras.)

3. FULLY COMMUTATIVE VECTOR VALUED QUASIGROUPS

In this section we will assume that $n-m = k \geq 1$ and $m \geq 2$.

A groupoid $\underline{Q}=(Q;f)$ is said to be cancellative iff for every $a \in Q^{(k)}$, $x,y \in Q^{(m)}$ the following implication is true:

$$f(ax) = f(ay) \Longrightarrow x = y. \qquad (3.1)$$

A groupoid $\underline{Q}$ is called a fully commutative (n,m)-quasigroup or, shortly, a quasigroup iff for every $a \in Q^{(k)}$, $b \in Q^{(m)}$ the equation

$$f(ax) = b$$

is uniquely solvable on x in $Q^{(m)}$.

Clearly, every quasigroup is a cancellative groupoid, and every finite cancellative groupoid is a quasigroup.

We will show below that every cancellative groupoid is a subgroupoid of a quasigroup.

First we will consider a more general concept of fully commutative partial (n,m)-groupoid. Namely, if $Q \neq \emptyset$, $\mathcal{D} \subseteq Q^{(n)}$ and $f: \mathcal{D} \to Q^{(m)}$, then we call $(Q;\mathcal{D};f)=\underline{Q}$ a fully commutative partial (n,m)-groupoid. As in §1 we will omit the words "fully commutative" and "(n,m)-".

A partial groupoid $\underline{Q}=(Q;\mathcal{D};f)$ is said to be cancellative iff for every $a \in Q^{(k)}$, $x,y \in Q^{(m)}$ such that $ax, ay \in \mathcal{D}$, the following implication is true:

$$f(ax) = f(ay) \Longrightarrow x = y.$$

In this case we say also that $\underline{Q}$ is a partial quasigroup, i.e. a partial groupoid $\underline{Q}$ is a partial quasigroup iff $\underline{Q}$ is cancellative. In particular: every cancellative groupoid is a partial quasigroup.

We will prove first the following more general result: every partial quasigroup is a partial subgroupoid of a quasigroup. (Note that $(Q;\mathcal{D};f)$ is a partial subgroupoid of a partial groupoid $(Q';\mathcal{D}';f')$ iff

$$Q \subseteq Q',\ \mathcal{D} \subseteq \mathcal{D}' \text{ and } a_1^n \in \mathcal{D} \Longrightarrow f(a_1^n) = f'(a_1^n).)$$

For this purpose we will consider first two kinds of extensions of partial groupoids.

Let $\underline{Q}=(Q;\mathcal{D};f)$ be a partial groupoid with the domain $\mathcal{D}$. Define two partial groupoids, $\underline{Q}^{\Delta}=(Q^{\Delta};\mathcal{D}^{\Delta};f^{\Delta})$ and $\underline{Q}^{\bullet}=(Q^{\bullet};\mathcal{D}^{\bullet};f^{\bullet})$, in the following way:

1) $Q^{\Delta} = Q\cup\{(i,a_1^n) \mid i\in\mathbb{N}_m,\ a_1^n\in Q^{(n)}\setminus\mathcal{D}\},\ \mathcal{D}^{\Delta} = Q^{(n)}$,

$a_1^n\in\mathcal{D} \Longrightarrow f^{\Delta}(a_1^n) = f(a_1^n)$,

$a_1^n\in Q^{(n)}\setminus\mathcal{D} \Longrightarrow f^{\Delta}(a_1^n) = (1,a_1^n)(2,a_1^n)\dots(m,a_1^n)$;

2) $Q^{\bullet} = Q\cup R,\ \mathcal{D}^{\bullet} = \mathcal{D}\cup\mathcal{E}$, where:

$R = \{(i;a,b) \mid i\in\mathbb{N}_m,\ a\in Q^{(k)},\ b\in Q^{(m)},$

$(\forall x\in Q^{(m)})[ax\notin\mathcal{D} \text{ or } (ax\in\mathcal{D} \text{ and } f(ax)\neq b)]\}$,

$\mathcal{E} = \{a(1;a,b)(2;a,b)\dots(m;a,b) \mid (i;a,b)\in R\}$,

$a_1^n\in\mathcal{D} \Longrightarrow f^{\bullet}(a_1^n) = f(a_1^n)$,

$f^{\bullet}(a(1;a,b)\dots(m;a,b)) = b$, for every $(i;a,b)\in R$.

It is easy to show that, if $\underline{Q}$ is a partial quasigroup, then $\underline{Q}^{\Delta}$ and $\underline{Q}^{\bullet}$ are partial quasigroups as well.

Now suppose that $\underline{Q}_1,\underline{Q}_2,\dots,\underline{Q}_\alpha,\underline{Q}_{\alpha+1},\dots$ is a sequence of partial groupoids such that $\underline{Q}_\alpha$ is a partial subgroupoid of $\underline{Q}_{\alpha+1}$. Setting

$$Q = \bigcup_{\alpha\geq 1} Q_\alpha,\quad \mathcal{D} = \bigcup_{\alpha\geq 1}\mathcal{D}_\alpha$$

and

$$f(a_1^n) = b_1^m \iff (\exists\alpha)(a_1^n\in\mathcal{D}_\alpha \ \&\ f_\alpha(a_1^n) = b_1^m),$$

we obtain a partial groupoid $(Q;\mathcal{D};f)=\underline{Q}$ where $\underline{Q}_\alpha$ is a partial subgroupoid of $\underline{Q}$ for every $\alpha\geq 1$. It is clear also that, if $\underline{Q}_\alpha$ is a partial quasigroup, then $\underline{Q}$ is a partial quasigroup too. (In general, $\underline{Q}$ may not be a quasigroup, even in the case when all of $\underline{Q}_\alpha$ are cancellative groupoids.)

Now suppose that $\underline{Q}$ is a given partial quasigroup and that the sequence of partial groupoids $\underline{Q}_0,\underline{Q}_1,\dots,\underline{Q}_\alpha,\underline{Q}_{\alpha+1},\dots$ is formed in the following way:

$$\underline{Q}_0 = \underline{Q},\ \underline{Q}_{2\alpha+1} = \underline{Q}^{\bullet}_{2\alpha},\ \underline{Q}_{2\alpha} = \underline{Q}^{\Delta}_{2\alpha-1}.$$

Then the union $S(\underline{Q})$ of the obtained sequence is a quasigroup.

A complete proof of this one can obtain easily by the assumption that $\underline{Q}$ is a partial quasigroup and by the definition of the functors Δ and $\bullet$. We note that a similar construction in the case of (usual) binary quasigroups is known. (See, for example, [2] ch. I.)

If B is a given set and if we put $\mathcal{D}=\emptyset$, then we obtain a partial quasigroup $(B;\emptyset;f)=\underline{B}$. The quasigroup which in this case one obtaines by B is the free quasigroup with a basis B.

It is natural to ask the question for existence of quasigroups with a given carrier Q. By the construction of $Q^{\bullet}$ and Q^{Δ} it is clear that: if Q is an infinite set, then Q is equivalent with the both sets $Q^{\bullet}$ and Q^{Δ}. Therefore, if $(Q;\mathcal{D};f)=\underline{Q}$ is a partial quasigroup and $S(\underline{Q})$ is the quasigroup obtained above, then Q and $S(\underline{Q})$ has the same cardinal number. This implies the following result:

THEOREM 3.1. Every infinite set is a carrier of a quasigroup. ▯

Note that if one starts by a partial groupoid $\underline{Q}=(Q;\mathcal{D};f)$ and forms the sequence of partial groupoids $(\underline{Q}_p \mid p\geq 0)$ such that $\underline{Q}_0=\underline{Q}$, $\underline{Q}_{p+1}=\underline{Q}_p^{\Delta}$, then one obtains that the union $S(\underline{Q})$ of this sequence is a groupoid which is a free extension of $\underline{Q}$. (Here, it is not necessary to assume that $n>m$.) In particular, if we assume that $\mathcal{D}=\emptyset$, then we obtain that $S(\underline{Q})$ is the free groupoid with a basis Q.

Now let Q be a nonempty set and let Φ be the family defined by

$$\Phi = \{(Q;\mathcal{D};f) \mid (Q;\mathcal{D};f) \text{ is a partial quasigroup}\}.$$

It is natural to define a partial ordering $\leq$ in Φ by:

$$(Q;\mathcal{D};f) \leq (Q;\mathcal{D}';f') \text{ iff } \mathcal{D}\subseteq\mathcal{D}' \text{ and } f \text{ is a restriction of } f'.$$

It is clear that the conditions of Zorn's lemma are satisfied. Therefore:

PROPOSITION 3.2. Every partial quasigroup on a set Q is a partial subgroupoid of a maximal partial quasigroup on Q. ▯

It is also clear that:

PROPOSITION 3.3. Every cancellative groupoid on Q is a maximal partial quasigroup on Q. ▯

PROPOSITION 3.4. A partial quasigroup $(Q;\mathcal{D};f)$ is maximal on Q iff for every $x\in Q^{(n)}\setminus\mathcal{D}$, $y\in Q^{(m)}$, there exist $a\in Q^{(k)}$, $u,v\in Q^{(m)}$ such that $x=au$, $av\in\mathcal{D}$, $f(av)=y$. □

4. FINITE FULLY COMMUTATIVE (n,m)-QUASIGROUPS

In this section we will assume that the set Q is finite with $q+1$ elements, i.e. that $Q=\{0,1,2,\dots,q\}$ and also that n,m,k are given positive integers such that $n-m = k\geq 1$ and $m\geq 2$.

Note that the elements of the set $Q^{(r)}$ can be thaught of as monotone sequences (of r members) of the elements of Q, i.e. that

$$Q^{(r)} = \{a_1a_2\dots a_r \mid a_\nu\in Q,\ 0\leq a_1\leq\dots\leq a_r\leq q\}.$$

Therefore (see, for example, [1], III.1.6, p. 137):

PROPOSITION 4.1. If $|Q|=q+1$, then $|Q^{(r)}| = \binom{q+r}{r}$. □

The first question which commes naturally is the existence of (n,m)-quasigroups with the carrier Q.

By obvious reason we consider first the case $q=1$, i.e. $Q=\{0,1\}$.

Let $(Q;f)$ be an (n,m)-quasigroup. Then $\sigma\colon x\mapsto f(0^kx)$ is a permutation of $Q^{(m)}$, and $f(0^{k-1}1^{m+1})\neq f(0^{m+k-i}1^i)$, for every $i\in\mathbb{N}_m$. This implies that $f(0^{k-1}1^{m+1}) = f(0^{m+k})$. Similarly, if $k\geq 2$, we have:

$$f(0^{k-2}1^{m+2}) = f(0^{m+k-1}1),\quad f(0^{k-3}1^{m+3}) = f(0^{m+k-2}1^2),$$

and more generally:

$$f(0^{k-i-1}1^{m+1+i}) = f(0^{m+k-j}1^j),$$

where $i\equiv j \pmod{m+1}$, $0\leq i\leq k-1$, $0\leq j\leq m$.

Conversely, let $\sigma\colon x\mapsto\sigma(x)$ be a permutation of $Q^{(m)}$, and let an (n,m)-operation $f\colon Q^{(n)}\to Q^{(m)}$ be defined by:

$$f(0^kx) = \sigma(x) \text{ for every } x\in Q^{(m)}$$
$$f(0^{k-i-1}1^{m+1+i}) = \sigma(0^{m-j}1^j),$$

where i and j are as above. Then $(Q;f)$ is an (n,m)-quasigroup.

Thus, we have showed the following:

PROPOSITION 4.2. If $Q=\{0,1\}$, then there exist $(m+1)!$ (n,m)-quasigroups on Q. □

In the case $q \geq 2$, we have the following:

PROPOSITION 4.3. If $2 \leq q \leq m$, then there does not exist an (n,m)-quasigroup with $q+1$ elements.

Proof. Assume that $Q=\{0,1,2,...,q\}$, and that $(Q;\mathcal{D};f)$ is a partial (n,m)-quasigroup such that $0^k x \in \mathcal{D}$ for every $x \in Q^{(m)}$, $u=0^{k-1}1^{m-q+1}2.3...q \in \mathcal{D}$. Then $v=0^{k-1}1\,2^{m-q+2}3...q \notin \mathcal{D}$. Namely, if $v \in \mathcal{D}$ we would have $f(u) \neq f(0^k x)$, $f(v) \neq f(0^k x)$ for every $x \in Q^{(m)} \setminus \{0^m\}$, and this would imply $f(u)=f(0^m)=f(v)$, which is impossible, for $u=0^{k-1}1y$, $v=0^{k-1}1z$, and $y \neq z$. □

Thus, if $(Q;f)$ is an (n,m)-quasigroup with $q+1$ elements where $m \geq 2$, $q > 1$, it must be $q > m$.

EXAMPLE 4.4. Define a $(4,3)$ operation on the set $Q=\{0,1,2,3,4\}$ as follows:

0) $f(0x) = x$, for every $x \in Q^{(3)}$

1.1) $f(1^2ij) = 0^2k$, $f(1i^2j) = 0k^2$, $f(1ij^2) = k^3$,

where $\{i,j,k\}=\{2,3,4\}$, $i < j$;

1.2) $f(1^3i) = 0jk$, $f(1^2i^2) = j^2k$, $f(1i^3) = jk^2$,

where $\{i,j,k\} = \{2,3,4\}$, $j < k$;

1.3) $f(1\,2\,3\,4) = 0^3$, $f(1^4) = 234$;

2.1) $f(2^3i) = 01j$, $f(2^2i^2) = i^2j$, $f(2i^3) = 1j^2$,

where $i \neq j$;

2.2) $f(2^234) = 0^21$, $f(23^24) = 01^2$, $f(234^2) = 1^3$

$f(2^4) = 134$;

3) $f(3^34) = 012$, $f(3^34^2) = 1^22$, $f(34^3) = 12^2$

$f(3^4) = 124$;

4) $f(4^4) = 123$.

It is easy to show that $(Q;f)$ is a $(4,3)$-quasigroup.

More generally, it can be shown that:

PROPOSITION 4.5. If $Q=\{0,1,2,...,q\}$, $q \geq 3$, then there exists a $(q,q-1)$-quasigroup. □

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PMF, p.f. 162
91000 Skopje
Yugoslavia

Mašinski fakultet
91000 Skopje
Yugoslavia