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FREE (2m, m)-GROUPS
PROSTE (2m, m)-GRUPE

Dončo Dimovski, Snežana Ilić

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Abstract. In this paper we give a combinatorial description of (2m, m)–groups ($m \geqslant 2$), inspired by a characterization of (2m, m) – groups as algebras with one (2m, m)–operation, one nullary and m unary operations.

Izvleček. V članku kombinatorno podajamo proste (2m, m)–grupe ($m \geqslant 2$), izhajajoč iz opisa (2m, m)–grupe kot algebre s po eno (2m, m)– in O–operacijo ter m unarnimi operacijami.

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Doc. dr. Dončo Dimovski, PMF Univerziteta u Skopju, Gazi Baga b. b. 91000 SKOPJE

Doc. dr. Snežana Ilić, Filozofski fakultet Niš, Ćirila i Metodija 2, 18000 NIŠ

0. Introduction

(2m, m)–groups are considered and examined in more details in [2]. Here we recall the necessary definitions, notations and known results.

Let Q be a nonempty set. The elements $(a_1, a_2, \ldots, a_t) \in Q^t$ will be denoted by $a_1 a_2 \ldots a_t$ or shortly by a_t^1. Acording to this notation we can identify the Cartesian power Q^t with the subset $Q_t = \{a_1 a_2 \ldots a_t \mid a_j \in Q\}$ of the free semigroup Q^+ with basis Q. The free monoid with basis Q will be denoted by Q^*, and its unit will be denoted by **1**. We consider Q^+ as a subset of Q^*. The subset $\{1\}$ will be denoted by Q_0. We say that an element $u \in Q$ has **dimension** t, and write $\dim u = t$, if $u \in Q^t$. The set of nonnegative integers will be denoted by $\mathbb{N}$. and the set $\{1, 2, \ldots, t\}$ by $\mathbb{N}_t$.

A (2m, m)–**operation** on a nonempty set G is a map [ ]: $G^{2m} \to G^m$. We say that the pair (G, [ ]) is a (2m, m) **semigroup** if for each $i \in \mathbf{N}_m$, and each $\mathbf{xuvw} \in G^{3m}$, $\mathbf{xu}, \mathbf{uv} \in G^{2m}$

$$[\mathbf{x}[\mathbf{uv}]\mathbf{w}] = [\,[\mathbf{xu}]\,\mathbf{vw}] \qquad (0.1)$$

A (2m, m)–**group** is a (2m, m)–semigroup (G, [ ]) such that for each $\mathbf{a}, \mathbf{b} \in G^m$ the equations

$$[\,\mathbf{ax}\,] = \mathbf{b} = [\,\mathbf{ya}\,] \qquad (0.2)$$

have solutions **x** and **y** in G^m.

The existence of a (2m, m)–operation [ ] on a set G is equivalent to the existence of m 2m–operations, $[\ ]_1, [\ ]_2, \ldots [\ ]_m$ on G defined by:

for each $i \in \mathbf{N}_m$ $[\,\mathbf{a}\,]_i = b_i$ if and only if $[\,\mathbf{a}] = b_m^1$. (0.3)

To each (2m,m)–semigroup (G, [ ]) we associate a semigroup $(G^m, \cdot)$ where $\mathbf{x} \cdot \mathbf{y} = [\,\mathbf{xy}\,]$. It is shown in [2] that a (2m,m)–semigroup is a (2m,m)–group if and only if the associated semigroup $(G^m, \cdot)$ is a group. The unit of the associated group is of the form $\mathbf{e} = e^m = ee \ldots e \in G^m$, for a fixed $e \in G$, and for each $\mathbf{xy} \in G^m$, $[\mathbf{xey}] = \mathbf{xy}$. We say that e is the unit of the (2m,m)–group (G, [ ]).

The existence of the unit e with its properties, allows us to give the following characterization of (2m,m)–groups, as algebras with one (2m,m)–operation, one nullary and m unary operations.

Proposition 0.1. A (2m,m)–semigroup (G, [ ]) is a (2m,m)–group if and only if there exist $e \in G$ and maps $f_i : G \to G$, $i \in \mathbf{N}_m$, such that, for each $x \in G$, $\mathbf{x} \in G^m$, r, $k \in \mathbf{N}_m$:

(i) $[\mathbf{xe}] = [\mathbf{ex}] = \mathbf{x}$;

(ii) $[f_k(x) f_{k+1}(x) \ldots f_m(x) e^{m-1} x f_1(x) f_2(x) \ldots f_{k-1}(x)\,] = \mathbf{e} =$
$[e^{r-1} x f_1(x) f_2(x) \ldots f_m(x) e^{m-r}\,]$.

Moreover, for each $\mathbf{uv} \in G^{m-1}$, $x \in G$, $i \in \mathbf{N}_m$, and $\mathbf{e}' = e^{m-1}$:

(iii) f_m is injective;

(iv) $[\,\mathbf{u} x f_1(x) \ldots f_m(x) \mathbf{v}\,] = \mathbf{uev}$;

(v) $[f_{i+1}(x) \ldots f_m(x) \mathbf{e}' x f_1(x) \ldots f_{i-1}(x) e\,] = (f_s(f_i(x)), \ldots, f_m(f_i(x)))$.

Proof. Suppose that (G, [ ]) is a (2m,m)–group. As we have already mentioned, in [2] it is shown that (i) is satisfied. Let $a \epsilon G$. Then, there exists a uniquely determined $a_1^m = a \epsilon G^m$, such that $[e'aa] = e$. We define $f_i: G \to G$ by $f_i(a) = a_i$. For shorter notations we denote by $F_k(a)$ and $F'_k(a)$ the elements $f_1(a) \ldots f_k(a) \epsilon G^k$ and $f_k(a) \ldots f_m(a) \epsilon G^{m-k+1}$, respectively. Then, the condition $[e'aF_m(a)] = e$, implies that for each $r \epsilon \mathbf{N}_m$, $[e^r[e'aF_m(a)]e^{m-r}] = e$, i.e. $[e^{r-1} aF_m(a)e^{m-r}] = e$. Now, for $r=1$, we have $[aF_m(a)e'] = e$, and so, $[F'_k(a)e'[aF_m(a)e']eF_{k-1}(a)] = [F'_k(a)e'aF_{k-1}(a)F'_k(a)eF_{k-1}(a)] = [F'_k(a)eF_{k-1}(a)]$. i.e. for each $k \epsilon \mathbf{N}_m$, $[F'_k(a)e'aF_{k-1}(a)] = e$. Hence, (ii) is satisfied.

Let $x, y \epsilon G$ and let $f_m(x) = f_m(y)$. Then, the condition (ii) for $k=m$, i. e. $[f_m(x)e'xF_{m-1}(x)] = e = [f_m(y)e'yF_{m-1}(y)]$ implies that $x=y$.

The condition (iv) follows directly from (i) and (ii).

Let $x \epsilon G$ and let $i \epsilon \mathbf{N}_m$. Then. $[F'_{i+1}(x)e'\, xF_{i-1}(x)e] = [F_{i+1}(x)e'xF_{i-1}(x)[f_i(x)F_m(f_i(x))e']e] = (f_1(f_i(x)), \ldots, f_m(f_i(x)))$

Conversely, let $(G, [\ \ ])$ be a (2m,m)–semigroup, with an $e \epsilon G$ and maps $f_i: G \to G$ satisfying the conditions (i) and (ii). Then, e is the unit in the semigroup $(G^m, \cdot)$, and the inverse for an element $x_1^m \epsilon G^m$ is $[F_m(x_m)e'F_m(x_{m-1})e' \ldots F_m(x_1)e'] \epsilon G^m$. ∎

The above characterization of (2m,m)–groups, was the inspiration for the combinatorial description of free (2m,m)–groups given here. It is analogous to the combinatorial description of free (m+1,m)–groups given in [3]

1. A combinatorial description of free (2m,m)–groups

Let A be a nonempty set, and $m \geqslant 2$, a positive integer. We will construct a free (2m,m)–group with basis A. For each $a \epsilon A$, let $D(a) = \{a(1), a(2), \ldots, a(m)\}$ be a new set, disjoint from A, such that $D(a) \cap D(b) = \phi$ for $a \neq b$. Let A' be the union of all the sets D(a), let e be a new element which is not in $A \cup A'$, and let $B = \{e\} \cup A \cup A'$. By induction on α we will define a sequence of sets $B(0), B(1), \ldots, B(\alpha), \ldots$. Let $B(0) = B$. Suppose that the set $B(\alpha)$ is defined. Then, $B(\alpha+1) = B(\alpha) \cup (C(\alpha) \times \mathbf{N}_m)$, where $C(\alpha) \subset B(\alpha)^+$ is defined by $C(\alpha) = \{x \mid x \epsilon B(\alpha)^{nm}, n \geqslant 2$. At the end, let $D = \bigcup_{\alpha \geqslant 0} B(\alpha)$.

Remark. Using, if necessary, different notations for the elements of B, we can achieve that for each α, $(C(\alpha) \times \mathbf{N}_m) \cap B = \phi$, and $(B(\beta+1) \setminus B(\beta)) \cap (B(\gamma+1) \setminus B(\gamma)) = \phi$ for $\beta \neq \gamma$. With this remark, $u \epsilon D$ if and only if $u \epsilon B(0)$, or $u=(\mathbf{x},i)$ for $i \epsilon \mathbf{N}_m$, and $x \epsilon D^{nm}$, $n \geqslant 2$.

Next, by induction, we define a map $|\ \ |: D \rightarrow \mathbf{N}$, called **length**, as follows: $|b|=1$ for $b \epsilon B$; and $|(x_1^{nm}, i)| = |x_1| + \ldots + |x_{nm}|$. By induction on the length, we define a map $\varphi: D \rightarrow D$, called **reduction**, as follows:

(A) Let $\varphi(u)=u$ for each $u \epsilon B$.

(B) Suppose that for each $u \epsilon D$ with $|u| < K$ $K \epsilon \mathbf{N}$, $\varphi(u)$ is well defined, and that:

$\varphi(u) \neq u$ if and only if $|\varphi(u)| < |u|$; and (1.1)

$\varphi^2(u)=\varphi(\varphi(u))=u$.

Remark. For simpler notation, we denote $\varphi((u,i))$ by $\varphi(u, i)$, and although $(v_1^m, i) \notin D$, we denote $\varphi(v_i)$ by $\varphi(v_1^m, i)$.

(C) Let $u=(x,i)$ with $|x| = K$. Then $\varphi(u)$ is defined by the first possible application of one of the following steps:

(1) If $\mathbf{x}=u_1^{nm}$, and $\varphi(u_j) \neq u_j$ for some $j \epsilon \mathbf{N}_{nm}$, then $\varphi(u)=\varphi(\varphi_1(\mathbf{x}),i)$, where $\varphi_1(\mathbf{z})=\varphi_1(z_1^t)$ is only a notation for $\varphi(z_1)\ldots\varphi(z_t)$.

(2) If $\mathbf{x}=\mathbf{v}c_1^m\,\mathbf{w}$, where $c_j=(\mathbf{z},j)$, and $\mathbf{v}$ has the smallest such dimension, then

$\varphi(u)=\varphi(\mathbf{vzw}, i)$.

(3) If $\mathbf{x}=\mathbf{vew}$, and $\mathbf{v}$ has the smallest such dimension, then

$\varphi(u)=\varphi(\mathbf{vw},i)$.

(4) If $\mathbf{x}=\mathbf{v}aa(1)\ldots a(m)\mathbf{w}$, $a \epsilon A$, and $\mathbf{v}$ has the smallest such dimension, then

$\varphi(u)=\varphi(\mathbf{vew}, i)$.

(5) If $\mathbf{x}=\mathbf{v}a(k)a(k+1)\ldots a(m)\mathbf{e}'aa(1)a(2)\ldots a(k-1)\mathbf{w}$, $a \epsilon A$, and $\mathbf{v}$ has the smallest such dimension, then

$\varphi(u)=\varphi(\mathbf{vew},i)$.

(6) If $\mathbf{x}=\mathbf{v}\mathbf{c}_r^m\,\mathbf{y}c_1^{r-1}\,\mathbf{z}$, $r \geqslant 2$, $c_j=(\mathbf{w}j)$, $\varphi(\mathbf{wy}, j)=\mathbf{e}$ for each $j \epsilon \mathbf{N}_m$, $\mathbf{v}$ has the smallest such dimension, and $\mathbf{y}$ has the smallest such dimension for this $\mathbf{v}$, then

$\varphi(u)=\varphi(\mathbf{vez}, i)$.

(7) $\varphi(u)=u$.

The well definedness and the essential properties of the map φ are given by the following three propositions, which will be proven later.

Proposition 1.1. (a) The map φ is well defined, satisfies (1.1) and (1.2), and $\varphi(b)=b$ for each $b \in B$.

(b) $|\varphi(u)| \leqslant |u|$ for each $u \in D$. (1.3)

Proposition 1.2. For each $u=(vxw, i) \in D$,

(a) $\varphi(u)=\varphi(\varphi_1(vxw),i)$, (1.4)

(b) $\varphi(u)=\varphi(v\varphi(x)w,i)$. (1.5)

Proposition 1.3. Let $i \in \mathbf{N}_m$. Then:

(a) $\varphi(uc_1^m w,i)=\varphi(uvw,i)$ for $c_j=(v,j)$; (1.6)

(b) $\varphi(uev,i)=\varphi(uv,i)$; (1.7)

(c) $\varphi(uaa(1)...a(m)v,i)=\varphi(uev,i)$; (1.8)

(d) $\varphi(ua(k)...a(m)e'aa(1)...a(k-1),v,i)=\varphi(uev,i)$; (1.9)

(e) $\varphi(uc_r^m yc_1^{r-1} v,i)=\varphi(uev,i)$, (1.10)

for $c_j=(w,j)$, and $\varphi(wy, j)=e$ for each $j \in \mathbf{N}_m$;

(f) $\varphi(xu,i)=e$ implies $\varphi(ux,i)=e$; (1.11)

(g) $\varphi(uyv,i)=\varphi(uev,i)$, when $\varphi(y,j)=e$ for each $j \in \mathbf{N}_m$. (1.12)

Let $Q=\varphi(D)$ and $[\ \ ] : Q^{2m} \to Q^m$ be defined, for each $i \in \mathbf{N}_m$, by:

$[u]_i = \varphi(u, i)$. (1.13)

The following theorem is the main result of this paper.

Theorem 1.4. $(Q, [\ \ ])$ is a free (2m,m)–group with basis A.

Proof. (a) The proof that $(Q, [\ \ ])$ is a (2m,m)–semigroup, follows directly from the definition of $[\ \ ]$ and the conditions (1.5) and (1.6). Moreover, the condition (1.7) implies that e is the unit of $(Q,[\ \ ])$.

(b) We will show that there exist maps $f_j : Q \to Q$, $j \in \mathbf{N}_m$, satisfying the condition (ii) from Proposition 0.1. We define the maps f_j by induction on the length as follows:

$f_j(a)=a$ for each $a \in A$;

$f_j(a(r))=[a(r+1)...a(m)e'aa(1)...a(r-1)e]_j$, for each $r \in \mathbf{N}_m$;

$f_j(e)=e$; and

$f_j(x,r)=[(x,r+1)...(x,m)G(x)(x,1)...(x,r-1)e]_j$, where $G(x)=G(x_1^{nm})=$
$= F_m(x_{nm})e'F_m(x_{nm-1})e'...F_m(x_1)e'$.

Above, $F_t(x)=f_1(x)f_2(x)...f_t(x)$, and with this notation, $F_m(x,r)=[(x,r+1)...(x,m)$ $G(x)(x,1)...(x,r-1)e]$. (1.14)

As a consequence of the inductive hypothesis and the conditions (1.12) and (1.7), we have:

$\varphi(xG(x),i)=e$ for each $i\in\mathbf{N}_m$. (1.15)

Next, the definition of [ ], and the conditions (1.11) and (1.14), imply that $\varphi((x,r)$ $F_m(x,r)e',i)=e$, for each $i\in\mathbf{N}_m$, i.e. the maps f_j satisfy the condition (ii) from Proposition 0.1.

(c) Since $\varphi(b)=b$ for each $b\in B$, and $A\subset B$, it follows that $A\subset Q$. Let $u=(x_1^{nm}, r)\in Q$ with $u_j\in \langle A\rangle$, where $\langle A\rangle$ denotes the (2m,m) – subgroup of (Q, [ ]), generated by A. Then, $[x_1^{nm}]=z_1^m\in\langle A\rangle^m$, and so, $u=\varphi(u)=\varphi(x_1^{nm},r) = [x_1^{nm}]_r = z_r \in \langle A\rangle$. Hence, $Q= =\langle A\rangle$.

(d) Let (G,[ ]') be a (2m,m)–group and $\lambda:A\to G$ a given map. We define a map $h:D\to G$, by induction, as follows:

$h(a)=\lambda(a)$, for each $a\in A$;

$h(e)=s$, where s is the unit in (G,[ ]');

$h(a(r))=g_r(h(a))$, for each $r\in\mathbf{N}_m$, where $g_1, g_2, ..., g_m: G\to G$ are the maps with the property $[s^{m-1}xg_1(x)...g_m(x)]'=s^m$ for each $x\in G$; and

$h(x_1^{nm},r)=[h(x_1)h(x_2)...h(x_{nm})]'_r$.

If $(x_1^{pm}, i)=(y_1^{qm}, j)$ in D, then $i=j$, $p=q$ and $x_r = y_r$ for each $r\in\mathbf{N}_m$. Hence, h is well defined.

By induction on the length, using the properties of the map φ and the properties of (2m,m)–groups, it can be shown that for each $(x_1^{pm}, i)\in D$ with $x_r\in Q$,

$h(\varphi(x_1^{pm}, i)) = [h(x_1)\, h(x_2) \, ... \, h(x_{pm})\,]_i$. (1.16)

So, the restriction ξ of h on Q is a (2m,m)–homomorphism from (Q, [ ]) into (G, [ ]'), such that $\xi(a)=\lambda(a)$ for $a\in A$.

The steps (a) through (d) show that (Q, [ ]) is a free (2m,m)–group with basis A. ■

2. Proofs of Propositions 1.1, 1.2 and 1.3

Proof of Proposition 1.1. (a) The right hand side of (1) to (6) in the definition of φ, is of the form $\varphi(v)$, where $|v|<K$ (see (B) from the definition of φ), and so, the inductive hypothesis implies that $\varphi(u)$ is well defined, satisfies (1.1) and (1.2), and $\varphi(b)=b$ for each $b\epsilon B$. (b) follows from (a). ■

Proof of Proposition 1.2. (a) Let $vxw=u_1^{nm}$. If $\varphi(u_r)=u_r$ for each $r\epsilon N_{nm}$, then the proposition is obvious. If there is some $r\epsilon N_{nm}$, such that $\varphi(u_r) \neq u_r$, then (a) follows from (1). The condition (b) follows from (a) and the condition (1.2). ■

Proof of Proposition 1.3. The proof is long, so, it will be divided into six lemmas. The proofs of the lemmas are by induction on the length. If, on the element in consideration in the first five lemmas, we can apply (1), the conclusion follows from Proposition 1.2 and the inductive hypothesis. So, in their proofs we assume that (1) is not applicable on the element in consideration.

Lemma. 2.1. If $\mathbf{x=uc_1^m w}$, for $c_j=(\mathbf{v},j)$ and $\varphi(c_j)=c_j$, then $\varphi(\mathbf{x},i)=\varphi(\mathbf{uvw},i)$.

Proof. Since, by assumption, $\varphi_1(\mathbf{x})=\mathbf{x}$, it follows that (2) is applicable on $(\mathbf{x},i)$.

(i) If **u** has the smallest such dimension, then the conclusion follows directly from (2).

(ii) If **u** does not have the smallest such dimension, then $\mathbf{u=u'd_1^m u''}$, for $d_r=(\mathbf{z},r)$. Then, (2) and the inductive hypothesis imply that $\varphi(\mathbf{x},i)=\varphi(\mathbf{u'zu''c_1^m w},i)=\varphi(\mathbf{u'zu''vw},i)=$ $=\varphi(\mathbf{uvw},i)$. ■

Remark. If (2) is applicable on $(\mathbf{x},i)\epsilon D$, we will write $\varphi_2(\mathbf{x}) \neq \mathbf{x}$, and $(\varphi_2(\mathbf{x}),i)$ will denote the element obtained from $(\mathbf{x},i)$ by one application of step (2).

Lemma 2.2. If $\mathbf{x=uev}$, then $\varphi(\mathbf{x},i)=\varphi(\mathbf{uv},i)$.

Proof. (a) If $\varphi_2(\mathbf{x}) \neq \mathbf{x}$, then $\varphi_2(\mathbf{u}) \neq \mathbf{u}$ or $\varphi_2(\mathbf{v}) \neq \mathbf{v}$, and the conclusion follows from Lemma 2.1 and the inductive hypothesis.

(b) Let $\varphi_2(\mathbf{x})=\mathbf{x}$. Then (3) is applicable on $(\mathbf{x},i)$.

(b.1) If **u** has the smallest such dimension, then the conclusion follows directly from (3).

(b.2) If **u** does not have the smallest such dimension, then two cases are possible:

(b.2.1) $\mathbf{u}=\mathbf{u'e}^r$, $r\in N_{m-1}$. Then, $\varphi(\mathbf{x},i)=\varphi(\mathbf{u'e}^r\mathbf{v},i)=\varphi(\mathbf{uv},i)$.

(b.2.2) $\mathbf{u}=\mathbf{u'eu''}$. Then $\varphi(\mathbf{x},i)=\varphi(\mathbf{u'u''ev},i)=\varphi(\mathbf{u'u''v},i) = \varphi(\mathbf{u'eu''v},i)=\varphi(\mathbf{uv},i)$. □

Remark. If (3) is applicable on $(\mathbf{x},i)\in D$, we will write $\varphi_3(\mathbf{x}) \neq \mathbf{x}$, and $(\varphi_3(\mathbf{x}),i)$ will denote the element obtained from $(\mathbf{x},i)$ by one application of step (3).

Lemma. 2.3. If $\mathbf{x}=\mathbf{u}aa(1)\ldots a(m)\mathbf{v}$, $a\in A$, then $\varphi(\mathbf{x},i)=\varphi(\mathbf{uev},i)$.

Proof. (a) If $\varphi_j(\mathbf{x}) \neq \mathbf{x}$ for some $j=\{2,3\}$, then $\varphi_j(\mathbf{u}) \neq \mathbf{u}$ or $\varphi_j(\mathbf{v}) \neq \mathbf{v}$, and the conclusion follows from Lemma 2. (j–1) and the inductive hypothesis.

(b) Let $\varphi_j(\mathbf{x})=\mathbf{x}$ for $j=2,3$. Then (4) is applicable on $(\mathbf{x},i)$.

(b.1) If **u** has the smallest such dimension, then the conclusion follows directly from (4).

(b.2) If **u** does not have the smallest such dimension, then $\mathbf{u}=\mathbf{u'}bb(1)\ldots b(m)\mathbf{u''}$, and the conclusion follows from (4) and the inductive hypothesis. □

Remark. If (4) is applicable on $(\mathbf{x},i)\in D$, we will write $\varphi_4(\mathbf{x}) \neq \mathbf{x}$, and $(\varphi_4(\mathbf{x}),i)$ will denote the element obtained from $(\mathbf{x},i)$ by one application of step (4).

Lemma 2.4. If $\mathbf{x}=\mathbf{u}a(k)\ldots a(m)\mathbf{e'}aa(1)\ldots a(k-1)\mathbf{v}$, $a\in A$, then $\varphi(\mathbf{x},i)=\varphi(\mathbf{uev},i)$.

Proof. (a) If $\varphi_j(\mathbf{x}) \neq \mathbf{x}$ for some $j\in\{2,3\}$, then $\varphi_j(\mathbf{u}) \neq \mathbf{u}$ or $\varphi_j(\mathbf{v}) \neq \mathbf{v}$, and if $\varphi_4(\mathbf{x}) \neq \mathbf{x}$, then

$\varphi_4(\mathbf{u}) \neq \mathbf{u}$, or $\varphi_4(\mathbf{v}) \neq \mathbf{v}$, or $\varphi_4(\mathbf{u}a(k)\ldots a(m)) \neq \mathbf{u}a(k)\ldots a(m)$, i.e. $\mathbf{u}=\mathbf{u'}aa(1)\ldots a(k-1)$, or $(\varphi_4(aa(1)\ldots a(k-1)\mathbf{v}) \neq aa(1)\ldots a(k-1)\mathbf{v}$, i.e. $\mathbf{v}=a(k)\ldots a(m)\mathbf{v'}$, So, the conclusion follows from Lemma 2. (j–1), the inductive hypothesis and Lemma 2.2.

(b) Let $\varphi_j(\mathbf{x})=\mathbf{x}$ for $2 \leqslant j \leqslant 4$. Then (5) is applicable on $(\mathbf{x},i)$.

(b.1) If **u** has the smallest such dimension, then the conclusion follows directly from (5)

(b.2) If **u** does not have the smallest such dimension, then two cases are possible:

(b.2.1) If $\mathbf{u}=\mathbf{u'}a(r)\ldots a(k-1)$, $r \geqslant 2$, then by (5) and Lemma 2.2 $\varphi(\mathbf{x},i)=\varphi(\mathbf{u'e}a(r)\ldots a(k-1)\mathbf{v},i)=\varphi(\mathbf{u'}a(r)\ldots a(k-1)\mathbf{v},i) = \varphi(\mathbf{uev},i)$.

(b.2.2) If $\mathbf{u}=\mathbf{u'}b(t)\ldots b(m)\mathbf{e'}bb(1)\ldots b(t-1)\mathbf{u''}$, then, the conclusion follows from (5) and the inductive hypothesis. □

Remark. If (5) is applicable on $(x,i)\epsilon D$, we will write $\varphi_5(x) \neq x$, and $(\varphi_5(x),i)$ will denote the element obtained from (x,i) by one application of step (5).

Lemma 2.5. (I) If $x=uc_r^m yc_1^{r-1} v$, where $c_j=(w,j)$, $\varphi(c_j)=c_j$, $r\geqslant 2$, and $\varphi(wy,t)=e$ for each $t\epsilon \mathbb{N}_m$, then $\varphi(x,i)=\varphi(uev,i)$.

(II) If $x=uyv$, and $\varphi(y,t)=e$ for each $t\epsilon \mathbb{N}_m$, then $\varphi(x,i)=\varphi(uev,i)$

(III) If $x=uv$ and $\varphi(uv,i)=e$, then $\varphi(vu,i)=e$.

Proof. (I) (a) If $\varphi_j(x) \neq x$ for some $2 \leqslant j \leqslant 5$, then the conclusion follows from Lemma 2. (j−1), Lemma 2.2, the inductive hypothesis, and the fact that $\varphi(wy,t)=e=\varphi(yw,t)$ for each $t\epsilon \mathbb{N}_m$, which follows from Lemma 2. (j−1) and Lemma 2.5. (III), inductively.

(b) Let $\varphi_j(x)=x$ for $2 \leqslant j \leqslant 5$. Then (6) is applicable on (x,i).

(b.1) If u has the smallest such dimension, and y has the smallest such dimension for the given u, then the conclusion follows directly from (6).

(b.2) If u has the smallest such dimension, and y does not have the smallest such dimension for the given u, i.e. $y=y'c_1^{r-1}y''$, and $\varphi(wy',t)=e$ for each $t\epsilon \mathbb{N}_m$, then Lemma 2.5 (II) and (III) inductively and Lemma 2.2, imply that for each $t\epsilon \mathbb{N}_m$, $e=\varphi(wy,t)=\varphi(ec_1^{r-1}y'',t)= =\varphi(y''ec_1^{r-1},t)=\varphi(ey''c_1^{r-1},t)$, and so, by (6) and Lemma 2.5 (II) inductively, $\varphi(x,i)=\varphi(uev,i)$.

Remark. If (6) is applicable on $(x,i)\epsilon D$, we will write $\varphi_6(x) \neq x$, and $(\varphi_6(x),i)$ will denote the element obtained from (x,i) by one application of step (6).

(b.3) If u does not have the smallest such dimension, then the following four cases are possible.

(b.3.1) $\varphi_6(u) \neq u$. Then the conclusion follows directly from (6) and the inductive hypothesis. If $\varphi_6(u)=u$, then since $\varphi_2(uc_r^m)=uc_r^m$, it follows that $\varphi_6(uc_r^m)=uc_r^m$.

(b.3.2) $\varphi_6(uc_r^m)=uc_r^m$ and $\varphi_6(uc_r^m y) \neq uc_r^m y$. Then two cases are possible.

(b.3.2. i) $u=u'c_k^{r-1}$, $2 \leqslant k < r$, $y=y'c_1^{k-1}y''$, and $\varphi(wy',j)=e$ for each $j\epsilon \mathbb{N}_m$. Then Lemma 2.5 (II), (III) inductively, and Lemma 2.2, imply that for each $t\epsilon \mathbb{N}_m$, $e=\varphi(wy,t)= =\varphi(ec_1^{k-1}y'',t) = \varphi(y''ec_1^{k-1},t)$, and so, by (6), Lemma 2.5 (II) inductively, and Lemma 2.2, $\varphi(x,i)=\varphi(u'ey''c_1^{r-1}v,i)=\varphi(u'ec_k^{r-1}v,i)=\varphi(u'c_k^{r-1}ev,i)=\varphi(uev,i)$.

(b.3.2.ii) $u=u'd_p^m u''$, $y=y'd_1^{p-1}y''$, $p\geqslant 2$, $d_q=(z,q)$ and $\varphi(zu''c_r^m y',t) = e$ for each

$t \in \mathbf{N}_m$. Then, Lemma 2.5 (III) inductively, implies that $e=\varphi(c^m_r\ y'zu'', t)$ and $e=\varphi(wy, t) = =\varphi(wy'd^{p-1}_1\ y'', t)=\varphi(y''wy'd^{p-1}_1, t)$ for each $t \in \mathbf{N}_m$, and so, (6), Lemma 2.2, Lemma 2.5 (II) inductively, and Lemma 2.1, imply that $\varphi(xi)=\varphi(u'\ ey''c^{r-1}_1\ v, i)=\varphi(u'y''c^{r-1}_1\ ev, i) = = \varphi(u'y''c^{r-1}_1\ c^m_r\ y'zu''v,i) = \varphi(u'y''wy'zu''v,i) = \varphi(u'\ y''\ wy'\ d^m_1\ u''v, i) = \varphi(u'ed^m_p\ u''v,i)= =\varphi(u'd^m_p\ u''ev,i) = \varphi(uev, i)$.

(b.3.3) $\varphi_6(uc^m_r\ y)=uc^m_r\ y$ and $\varphi_6(uc^m_r\ yc^{r-1}_1) \neq uc^m_r\ yc^{r-1}_1$. Then two cases are possible.

(b.3.3.i) $u=u'c^{r-1}_k$, $2 \leqslant k < r$. Then the conclusion follows directly from (6) and Lemma 2.2.

(b.3.3.ii) $u=u'c^m_k\ u''$, $2 \leqslant k < r$ and $\varphi(wu''c^m_r y, t) = e$ for each $t \in \mathbf{N}_m$. Then, Lemma 2.5 (III) inductively, implies that $e=\varphi(wy, t) = \varphi(yw, t)$ and $e = \varphi(wu''c^m_r\ y, t)= =\varphi(ywu''c^m_r, t)$ for each $t \in \mathbf{N}_m$, and so, Lemma 2.5 (II), (III) inductively implies that $e= =\varphi(eu''c^m_r, t) = \varphi(c^m_r\ eu'',t)$ for each $t \in \mathbf{N}_m$. Next, (6), Lemma 2.2 and Lemma 2.5 (II) inductively, imply that $\varphi(x,i)=\varphi(u'ec^{r-1}_k\ v, i)=\varphi(u'c^{r-1}_k ev,) =$
$= \varphi(u'c^{r-1}_k\ c^m_r\ eu''v,i)=\varphi(u'c^m_k\ u''ev,i)=\varphi(uev,i)$.

(b.3.4) $\varphi_6(uc^m_r\ yc^{r-1}_1)=uc^m_r\ yc^{r-1}_1$, and $\varphi_6(uc^m_r\ yc^{r-1}_1\ v) \neq uc^m_r\ yc^{r-1}_1 v$. Then two cases are possible.

(b.3.4.i) $u=u'c^{r-1}_k$, $2 \leqslant k < r$, $v=v'\ c^{k-1}_1\ v''$. Then $e=\varphi(wyc^{r-1}_1\ v',t)$ for each $t \in \mathbf{N}_m$. Since $\varphi(wy,t)=e$ for each $t \in \mathbf{N}_m$, Lemma 2.5 (II), (III) inductively, and Lemma 2.2 imply that $e=\varphi(ec^{r-1}_1\ v',t)=\varphi(c^{r-1}_k\ ev'c^{k-1}_1, t)$ for each $t \in \mathbf{N}_m$. Next, (6) and Lemma 2.5 (II) inductively, imply that $\varphi(x,i)=\varphi(u'ev'',i) = \varphi(u'c^{r-1}_k ev'c^{k-1}_1 v'',i)=\varphi(uev,i)$.

(b.3.4.ii) $x=u'd^m_p\ u''c^m_r yc^{r-1}_1\ v'\ d^{p-1}_1\ v''$, $p \geqslant 2$, $d_q=(z,q)$ and $\varphi(zu''c^m_r\ yc^{r-1}_1 v',t)=e$ for each $t \in \mathbf{N}_m$. Then the inductive hypothesis and Lemma 2.5 (III) inductively, imply that $e=\varphi(zu''ev',t)=\varphi(d^m_p\ u''ev'd^{p-1}_1, t)$ for each $t \in \mathbf{N}_m$, and so, (6) and Lemma 2.5 (II) inductively, imply that $\varphi(x,i)=\varphi(u'ev'',i)=\varphi(u'd^m_p\ u''ev'd^{p-1}_1 v'',i)=\varphi(uev,i)$.

(II) Since $\varphi_1(x)=x$ and $e=\varphi(y,t)$ for each $t \in \mathbf{N}_m$, it follows that $\varphi_j(y) \neq y$ for some $j \in \{2,3,4,5,6\}$. Then, Lemma 2. (j–1) implies that $e=\varphi(\varphi_j(y),t)$ for each $t \in \mathbf{N}_m$, and the conclusion follows from Lemma 2. (j–1) and the inductive hypothesis. Here, for j=6, Lemma 2. (j–1) stands for Lemma 2.5 (I).

(III) (a) If $\varphi_j(v) \neq v$ for some $j\in\{2,3,4,5,6\}$, then Lemma 2. (j–1) and $e=\varphi(x,i)$ imply that $e=\varphi(u\varphi_j(v),i)$, and so, by Lemma 2. (j–1) and the inductive hypothesis, $e=\varphi(\varphi_j(v)u,i)=$ $=\varphi(vu,i)$. Here, for j=6, Lemma 2. (j–1) stands for Lemma 2.5 (I).

(b) Let $\varphi_j(v)=v$ for each $j\in\mathbf{N}_6$.

(b.1) If $\varphi_2(x) \neq x$, then $x=c_1^m y$, $c_q=(w,q)$, and by Lemma 2.1, $e=\varphi(wy,i)$. By Lemma 2.5 (I) inductively, $e=\varphi(c_2^m yc_1,i)=\varphi(vu,i)$.

(b.2) Let $\varphi_2(x)=x$. If $\varphi_3(x) \neq x$, then $\varphi_3(v)=v$ implies that $x=ey$, and so, by Lemma 2.2, $e=\varphi(y,i)$ which contradicts the assumption of (b).

(b.3) Let $\varphi_j(x)=x$ for $j\in\{1,2,3\}$. If $\varphi_4(x) \neq x$, then $x=aa(1)...a(m)y$, and Lemma 2.3 implies that $e=\varphi(ey,i)$. Since, by (b), $\varphi_j(y)=y$ for each $j\in\{2,3,4,5,6\}$, it follows that $y=e'$. Hence, $\varphi(a(1)...a(m)e'a,i)=e$.

(b.4) Let $\varphi_j(x)=x$ for each $j\in\mathbf{N}_4$. If $\varphi_5(x) \neq x$, then $x=a(k)...a(m)e'aa(1)...a(k-1)y$, and Lemma 2.4 implies that $e=\varphi(ey,i)$. Since, by (b), $\varphi_j(y)=y$ for each $j\in\{2,3,4,5,6\}$, it follows that $x=a(k)...a(m)e'aa(1)...a(k-1)$. Hence, $e=\varphi(vu,i)$.

(b.5) Let $\varphi_j(x)=x$ for each $j\in\mathbf{N}_5$. Then $\varphi_6(x) \neq x$, which implies that $x=c_r^m yc_1^{r-1}z$ for $c_q=(w,q)$ and $\varphi(wy,t)=e$ for each $t\in\mathbf{N}_m$. Then, Lemma 2.5 (I) inductively, implies that $e=\varphi(ez,i)$, and so, $\varphi(ez,i) \neq (ez,i)$. Since, by (b), $\varphi_j(z)=z$ for each $j\in\{2,3,4,5,6\}$, it follows that $x=c_r^m yc_1^{r-1}$, and by Lemma 2.5. (I) inductively, $\varphi(vu,i)=\varphi(c_{r+1}^m yc_1^r, i)=e$. □

Lemma 2.6. Let $u=(x,i)$, $c_q=(w,q)$, $u,c_q\in D$, $q\in\mathbf{N}_m$. Then:

(I) If $x=vc_1^m z$, then $\varphi(u)=\varphi(vwz,i)$.

(II) If $x=vc_k^m yc_1^{k-1}z$, and $\varphi(wy,t)=e$ for each $t\in\mathbf{N}_m$, then $\varphi(u)=\varphi(vez,i)$.

Proof. If $\varphi(c_q)=c_q$, the conclusion follows from Lemma 2.1 for (I), and Lemma 2.5. for (II). If $\varphi(c_q) \neq c_q$ for some $q\in\mathbf{N}_m$, then $\varphi(c_q) \neq c_q$ for each $q\in\mathbf{N}_m$. Hence, $\varphi_j(w) \neq w$ for some $j\in\mathbf{N}_6$.

(I) If $\varphi_1(w) \neq w$, then by Proposition 1.2 and the induction, $\varphi(u) = \varphi(v\varphi_1(c_1^m)z,i) =$ $= \varphi(v\varphi(\varphi_1(w),1)...\varphi(\varphi_1(w),m)z,i) = \varphi(v\varphi_1(w)z,i) = \varphi(vwz,i)$.

Let $\varphi_1(w)=w$. Then $\varphi_j(w) \neq w$ for some $j\in\mathbf{N}_6\setminus\{1\}$, and so, by Proposition 1.2 (b), Lemma 2. (j–1) and the induction, $\varphi(u) = \varphi(v\varphi_1(c_1^m)z,i) = \varphi(v\varphi(\varphi_j(w),1)...\varphi(\varphi_j(w),m)z,i) =$

$= \varphi(v\varphi_j(w)z,i) = \varphi((v\varphi_j(w)z,i) = \varphi(vwz,i)$.

(II) The proof is analogous to the proof of (I), where we use the additional fact that

$e=\varphi(wy,t)=\varphi(\varphi_j(w)y,t)$ for each $t\in\mathbf{N}_m$ and for each $j\in\mathbf{N}_6$. □ ■

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Povzetek

Naš cilj je zgraditi prosto (2m, m)–grupo na podlagi opisa (2m, m)–grupe G = (G; [ ]) z binarno grupo $(G^m, \circ)$, kjer je operacija $\circ$ definirana s predpisom: $x \circ y = [x\ y]$.
Prvi del članka je posvečen opisu navedene konstrukcije.
V drugem delu je zgrajena prosta (2m, m)–grupa, ki jo poraja neprazna množica A.
V tretjem delu posredujemo dokaze nekaterih trditev, uporabljenih pri navedeni konstrukciji.

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